Two questions on semigroup laws

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Abstract

B. H. Neumann recently proved some implication for semigroup laws in groups. This may help in solution of a problem posed by G. M. Bergman in 1981.

Let $G$ be a group, and $S \subseteq G$ be a subsemigroup generating $G$. It is clear that if $S$ is commutative, then $G$ is commutative. The following question is equivalent to the one posed by G. M. Bergman [2], [3].

**Question 1** Let $S$ generating $G$ satisfy a law. Must $G$ satisfy the same law?

For some laws the answer is positive [9], [5], [8], [1], however in general the question is open and in opinion of S. V. Ivanov and E. Rips it has a negative answer. All semigroups we consider are cancellative.

**Question 2** Let a semigroup law $a = b$ implies a semigroup law $u = v$ in groups. Does the same implication hold in semigroups?

To show implication of laws in semigroups we can use only so-called positive endomorphisms, which map generators to positive words. It is shown in [8] (an example at the end of this paper), that all implications for positive laws of length $\leq 5$ which hold in groups, also are valid for semigroups. The fact that the law $x^2y^2x = yx^3y$ implies $xy = yx$ in semigroups (and hence in groups) is proved in [5, p.132].

We show the equivalence of the above Questions.

It is shown in [10], that the law $x^{s+t}y^2x^t = yx^{s+2t}y$, $gcd(s, t) = 1$, implies $xy^2x = yx^2y$ in groups (which is equivalent to $[x, y, x] = 1$ [12]). So if
there exists a semigroup satisfying \( x^{s+t}y^2x^t = yx^{s+2t}y \), \( \gcd(s,t) = 1 \), but not \( xy^2x = yx^2y \), the desired counterexample for Question 1 would be found.

Let \( a = a(x_1, ..., x_n) \), \( b = b(x_1, ..., x_n) \) be positive words. A semigroup law \( a = b \) is called balanced if every \( x_i \) has the same exponent sum in \( a \) and \( b \). The law is trivial if \( ab^{-1} = 1 \) in \( F \). The law is called cancelled if the first (and the last) letters in \( a \) and \( b \) are different.

**Notation**

Let \( F \) be a free group and \( F \ni f \) be a free semigroup, both generated by \( x_1, x_2, x_3, ... \). Words in \( F \) are called positive. We denote:

- \( \text{End}^+ \) – the set of positive endomorphisms which map \( x_i \) to positive words,
- \( N_w \) – a normal \( \text{End}^+ \)-invariant closure of a word \( w \) in \( F \),
- \( \text{End} \) – the set of all endomorphisms of the free group \( F \),
- \( V_w \) – a fully invariant subgroup generated by a word \( w \) in \( F \),
- \( (u,v)^\# \) – the smallest cancellative congruence in \( F \) providing the law \( u = v \).

A relatively free cancellative semigroup, defined by the law \( u = v \) is isomorphic to \( F / (u,v)^\# \) [8].

We note that if \( N_w \) contains a positive word, say \( x^2yz^4 \), then it contains \( x^7 \) and hence \( x^{-1} \in x^6N_w \) implies \( F = F \mod N_w \).

**Remark 1** Since each semigroup with a non-balanced law is a group, we have to consider only balanced non-trivial semigroup laws. Each such a law implies a binary balanced and cancelled law \( A(x,y) = B(x,y) \) [6].

**Questions and Results**

To formulate the above Questions in terms of normal subgroups we need

**Lemma 1** A semigroup law \( u = v \) implies \( a = b \) in semigroups if and only if \( N_{ab^{-1}} \subseteq N_{uv^{-1}} \). The implication holds in groups if and only if \( V_{ab^{-1}} \subseteq V_{uv^{-1}} \).

**Proof** The law \( u = v \) implies \( a = b \) in semigroups if and only if corresponding smallest congruences satisfy \( (a,b)^\# \subseteq (u,v)^\# \). If we map \( F \to F/N \), then \( F \) is mapped onto \( F/N^\# \), where \( N^\# \) is a cancellative congruence in \( F \) defined as: \( N^\# = \{(s,t); st^{-1} \in N \cap FF^{-1}\} \). It is proved in [7], Thm. 2, that \( N := N_{uv^{-1}} \) is a smallest normal subgroup such that \( N^\# = (u,v)^\# \). So we have

\[
(u,v)^\# = \{(s,t); st^{-1} \in N_{uv^{-1}} \cap FF^{-1}\}.
\]
Since $\mathcal{F}/(u, v)^\#$ is embeddable into a group $F/N_{uv^{-1}}$, and $N_{uv^{-1}}$ is the smallest normal subgroup with this property, it follows by [4], 12.3, that

$$N_{uv^{-1}} = gpm(st^{-1}; \ (s, t) \in (u, v)^\#).$$

(2)

Hence by (1), (2): $(a, b)^\# \subseteq (u, v)^\#$ if and only if $N_{ab^{-1}} \subseteq N_{uv^{-1}}$, which gives the first statement of the Lemma. The second statement is known [11]. □

In terms of normal subgroups the above Questions are:

**Question 1′** Does $N_{a^{-1}b^{-1}} = V_{ab^{-1}}$ hold for each semigroup law $a = b$?

**Question 2′** Does $V_{ab^{-1}} \subseteq V_{uv^{-1}}$ imply $N_{ab^{-1}} \subseteq N_{uv^{-1}}$ for semigroup laws $a = b$ and $u = v$?

We shall prove that for each semigroup law $a = b$ there exists a semigroup law $u = v$ such that the fully invariant closure of $ab^{-1}$ coincides with the $\text{End}^+$-invariant normal closure of $uv^{-1}$. This will imply the equivalence of the Questions.

**Theorem** For every $n$-variable semigroup law $a = b$ there exists an $n+1$-variable semigroup law $u = v$ such that $V_{ab^{-1}} = N_{uv^{-1}}$.

**Corollary** The Questions 1 and 2 are equivalent.

**Proof** We have to show that for each semigroup law $a = b$ the equality holds: $N_{ab^{-1}} = V_{ab^{-1}}$. Take $u = v$ as in the Theorem, then $V_{ab^{-1}} \cong N_{uv^{-1}}$. By taking the fully invariant closure we get $V_{ab^{-1}} = V_{uv^{-1}}$. If Question 2 has a positive answer then we have $N_{ab^{-1}} = N_{uv^{-1}} \cong V_{ab^{-1}}$, as required. □

**Lemmas and Proof of the Theorem**

**Lemma 2** Let $A(x, y) = B(x, y)$ be a balanced and cancelled semigroup law and the first letter in $A(x, y)$ is $x$. Then there exist $a_i = a_i(x, y), \ b_i = b_i(x, y) \in \mathcal{F}$, $i = 1, 2$, such that

(i) $x^{-1}y = a_1b_1^{-1} \cdot (A^{-1}B)^{b_1^{-1}},$ \quad (ii) $xy^{-1} = a_2^{-1}b_2 \cdot (AB^{-1})^{b_2}, \ \varepsilon = \pm 1,$

(iii) $F = \mathcal{F}F^{-1}N_{AB^{-1}} = \mathcal{F}^{-1}FN_{AB^{-1}}.$

**Proof** Since the law $A = B$ is cancelled, it can be written as $x \cdot a_1 = y \cdot b_1$, which gives $A^{-1}B = a_1^{-1}x^{-1}yb_1$ and hence (i). The law $A = B$ (or $B = A$) can be written as $a_2 \cdot x = b_2 \cdot y$. In the first case $AB^{-1} = a_2xy^{-1}b_2$ gives $xy^{-1} = a_2^{-1}b_2 \cdot (AB^{-1})^{b_2}$. If $B = a_2 \cdot x, \ A = b_2 \cdot y$, then $xy^{-1} = a_2^{-1}b_2 \cdot (AB^{-1})^{-b_2}$, which gives (ii).
Since \((A^{-1}B) = (AB^{-1})B^{-1} \in N_{ab^{-1}}\), we get from \((i)\), that \(x^{-1}y \in \mathcal{F}\mathcal{F}^{-1} \mod N_{ab^{-1}}\), which holds under every substitution elements from \(\mathcal{F}\) for \(x\) and \(y\). Since every word in \(F\) is a product of words in \(\mathcal{F} \cup \mathcal{F}^{-1}\), we get \(F = \mathcal{F}\mathcal{F}^{-1}N_{ab^{-1}}\). Similarly, from \((ii)\) we get \(F = \mathcal{F}^{-1}\mathcal{F}N_{ab^{-1}}\). □

The following Lemma is well known in terms of a group of fractions and Ore conditions.

**Lemma 3** Let \(a = b\) be a nontrivial semigroup law, and \(g_1, g_2, \ldots, g_n\) be elements in \(F\). Then there exist elements \(s_1, s_2, \ldots, s_n\) and \(d\) in \(\mathcal{F}\) such that \(g_i = s_id^{-1} \mod N_{ab^{-1}}\).

**Proof** By [6], the law \(a = b\) implies balanced and cancelled binary law \(A = B\).

Since \(N_{ab^{-1}} \subseteq N_{ab^{-1}}\), the inclusions in Lemma 2 are valid \(\mod N_{ab^{-1}}\). Then by \((iii)\) we have modulo \(N_{ab^{-1}}\): \(g_i = a_ib_i^{-1}\) for some \(a_i, b_i \in \mathcal{F}\). For \(n = 2\), \(g_1 = a_1b_1^{-1}\), \(g_2 = a_2b_2^{-1}\). Also by \((iii)\), there exist \(c, d\) such that \(b_2^{-1}b_1 = cd^{-1}\). We introduce \(r := b_1d = b_2c\), then \(g_1 = a_1b_1^{-1} = a_1dd^{-1}b_1^{-1} = a_1dr^{-1} =: sr^{-1}\), \(g_2 = a_2b_2^{-1} = a_2cc^{-1}b_2^{-1} = a_2cr^{-1} =: tr^{-1}\), \(s, t, r \in \mathcal{F}\). So, by repeating this step we can write \(g_1, \ldots, g_n\) with a "common denominator" \(\mod N_{ab^{-1}}\) as required. □

To compare \(End^+\)-invariant and \(End\)-invariant closures of words we make an observation that by positive endomorphisms we can map \(xy^{-1}\) into any word \(g \in F \mod N_{ab^{-1}}\) if write \(g = st^{-1}\) and map \(x\) to \(s\), and \(y\) to \(t\).

**Lemma 4** There exists an automorphism \(\alpha \in Aut F\) such that for any \(w \in F\), \(N_{w^\alpha}\) is fully invariant \(\mod N_{ab^{-1}}\), for any nontrivial \(ab^{-1} \in \mathcal{F}\mathcal{F}^{-1}\). That is \(V_w \subseteq N_{w^\alpha}N_{ab^{-1}}\).

**Proof** Let \(w = w(x_1, \ldots, x_n)\). We take \(\alpha \in Aut F\) which maps \(x_i \rightarrow x_ix_{i+1}^{-1}, i = 1, \ldots, n\) and leaves \(x_i, i > n\), fixed. It is enough to show that for any \(g_1, \ldots, g_n\) in \(F\), \(w(g_1, \ldots, g_n) \in N_{w^\alpha}N_{ab^{-1}}\). By Lemma 3, we write \(g_i = s_id^{-1} \mod N_{ab^{-1}}\) and define \(\nu \in End^+\) by \(x_{i}^{\nu} = s_i\), \(i \leq n\), and \(x_{n+1}^{\nu} = d\).

Then modulo \(N_{ab^{-1}}\) we have \((x_ix_{i+1}^{-1})^{\nu} = g_i\) and \(w(g_1, \ldots, g_n) = w(x_1x_{1+1}^{-1}, \ldots, x_{n}x_{n+1}^{-1})^{\nu} = (w(x_1, \ldots, x_n)^{\nu})^{\nu} \in N_{w^\alpha} \subseteq N_{w^\alpha}\), as required. □

**Corollary 1** For a nontrivial semigroup law \(a = b\) the equality holds

\[V_{ab^{-1}} = N_{(ab^{-1})^\alpha}.\]
Proof We have $ab^{-1} \in N^{a^{-1}}_{(ab^{-1})\alpha}$. Since $\alpha^{-1}$ is in $End^+$, then $N^{a^{-1}}_{(ab^{-1})\alpha} \subseteq N_{(ab^{-1})\alpha}$ and hence $ab^{-1} \in N_{(ab^{-1})\alpha}$, which gives

$$N_{ab^{-1}} \subseteq N_{(ab^{-1})\alpha}.$$  

By Lemma 4 for $w := ab^{-1}$, by (3), and since $End^+ \subseteq End$, we have:

$$V_{ab^{-1}} \subseteq N_{(ab^{-1})\alpha} N_{ab^{-1}} = N_{(ab^{-1})\alpha} \subseteq V_{ab^{-1}},$$

which implies $V_{ab^{-1}} = N_{(ab^{-1})\alpha}$. □

We denote by $\delta$ the endomorphism which maps $x_{n+1} \rightarrow 1$ and leaves other generators fixed, then $\delta \in End^+$. As above, $\alpha \in Aut F$ maps $x_i \rightarrow x_i x_{n+1}^{-1}$, $i = 1, \ldots, n$ and leaves $x_i$, $i > n$, fixed.

**Lemma 5** Let $a = b$ be a nontrivial semigroup law, and $F_{n+1}$ be a free sub-semigroup generated by $x_1, \ldots, x_{n+1}$. Then for any positive word $p(x_1, \ldots, x_n)$, there exist positive words $u_i = u_i(x_1, \ldots, x_{n+1})$, $v_i = v_i(x_1, \ldots, x_{n+1})$, $i = 1, 2$, such that $p^\alpha = u_1 v_1^{-1} = u_2 v_2 \bmod (N_{ab^{-1}} \cap \ker \delta)$.

**Proof** We show first that for any words $c, q \in F_{n+1}$ the inclusion hold:

$$(*) \quad cx_{n+1}^{-1} \in F_{n+1}^{-1} F_{n+1} \bmod (N_{ab^{-1}} \cap \ker \delta),$$

$$(**) \quad x_{n+1} q \in F_{n+1} F_{n+1}^{-1} \bmod (N_{ab^{-1}} \cap \ker \delta).$$

The law $a = b$ implies balanced and cancelled binary law $A = B$, so it is enough to prove the inclusions for the law $A(x, y) = B(x, y)$.

If apply $\delta$ to the balanced equality $A(c, x_{n+1}) = B(c, x_{n+1})$, it becomes trivial, and hence the word $AB^{-1}(c, x_{n+1})$ is in $\ker \delta$. Similarly we get $A^{-1}B(x_{n+1}, q) \in \ker \delta$. We put now $c, x_{n+1}$ for $x, y$ in (ii) (Lemma 2) to get $(*)$, and then put $x_{n+1}, q$ in (i) (Lemma 2) to get $(**)$.

We continue the proof modulo $(N_{ab^{-1}} \cap \ker \delta)$. To show that:

$p(x_1 x_{n+1}^{-1}, \ldots, x_n x_{n+1}^{-1}) \in F_{n+1} F_{n+1}^{-1}$, and $p(x_1 x_{n+1}^{-1}, \ldots, x_n x_{n+1}^{-1}) \in F_{n+1}^{-1} F_{n+1}$, we use induction on the length $|p| = m$. Let $p(x_1, \ldots, x_n) = c_m c_{m-1} \cdots c_2 c_1$, $c_i \in \{x_1, \ldots, x_n\}$, then $p^\alpha = c_m x_{n+1}^{-1} c_{m-1} x_{n+1}^{-1} \cdots c_2 x_{n+1}^{-1} c_1 x_{n+1}^{-1}$. For $m = 1$, $p^\alpha = c x_{n+1}^{-1} \in F_{n+1} F_{n+1}^{-1}$ and by $(*)$, $p^\alpha = c x_{n+1}^{-1} \in F_{n+1}^{-1} F_{n+1}$.

Let $|p| = m$, then $p = c_m c_{m-1} \cdots c_2 c_1$ and by inductive assumption $p^\alpha = c_m x_{n+1}^{-1} \cdot q r^{-1}$. Then by $(**)$, there exist $s, t \in F_{n+1}$, such that $x_{n+1}^{-1} q = st^{-1}$ and hence $p^\alpha = c_m (x_{n+1}^{-1} q) r^{-1} = c_m (st^{-1}) r^{-1} = (c_m s)(rt)^{-1} \in F_{n+1} F_{n+1}^{-1}$. 5
Again for \(|p|=m\), we get by assumption \(p^n = r^{-1}s \cdot c_1 x_{n+1}^{-1} = r^{-1}(sc_1)x_{n+1}^{-1}\). By (\(\ast\)) for \(sc_1\) instead of \(c\), there exist \(t, u \in F_{n+1}\), such that \(sc_1 x_{n+1}^{-1} = t^{-1}u\). Then \(p^n = r^{-1}(sc_1)x_{n+1}^{-1} = r^{-1}t^{-1}u = (tr)^{-1}u \in F_{n+1}F_{n+1}\) as required. □

**Proof of the Theorem**

We have to show that for every nontrivial \(n\)-variable semigroup law \(a = b\) there exists an \(n + 1\)-variable semigroup law \(u = v\) such that \(V_{ab^{-1}} = N_{uv^{-1}}\).

By Lemma 5 for the words \(a = a(x_1, \ldots, x_n)\) and \(b = b(x_1, \ldots, x_n)\) we get respectively: \(a^n = u_1 v_1^{-1} \mod (N_{ab^{-1}} \cap \text{Ker} \delta)\), and \(b^n = u_2^{-1}v_2 \mod (N_{ab^{-1}} \cap \text{Ker} \delta)\). Then \((ab^{-1})^a = u_1 v_1^{-1} v_2^{-1} u_2 = u_2^{-1}(u_2 u_1)(v_2 v_1)^{-1} u_2 \mod (N_{ab^{-1}} \cap \text{Ker} \delta)\). We denote \(u := u_2 u_1, v := v_2 v_1\), then

\[
(ab^{-1})^a = (uv^{-1}) u_2 \mod (N_{ab^{-1}} \cap \text{Ker} \delta) \tag{4}
\]

This implies:

\[
N_{(ab^{-1})^a} \subseteq N_{uv^{-1}} N_{ab^{-1}} \tag{5}
\]

and

\[
N_{uv^{-1}} \subseteq N_{(ab^{-1})^a} N_{ab^{-1}}. \tag{6}
\]

To prove the equality

\[
N_{(ab^{-1})^a} = N_{uv^{-1}}, \tag{7}
\]

we apply \(\delta\) to (4). Since \(\alpha \delta\) is the identity map on \(x_i\), \(i \leq n\), and \(\delta\) is in \(\text{End}^+\), we have that \(ab^{-1} = (ab^{-1})^\alpha \delta\) is conjugate to \((uv^{-1})^\delta \in N_{uv^{-1}}^\delta \subseteq N_{uv^{-1}}\). This implies \(N_{ab^{-1}} \subseteq N_{uv^{-1}}\) which, together with (5) gives \(N_{(ab^{-1})^a} \subseteq N_{uv^{-1}}\). Since by (3), \(N_{ab^{-1}} \subseteq N_{(ab^{-1})^a}\), it follows from (6), that \(N_{uv^{-1}} \subseteq N_{(ab^{-1})^a}\), and hence (7) holds.

Now, since by Corollary 1, \(V_{ab^{-1}} = N_{(ab^{-1})^a}\), we have by (7), the required equality \(V_{ab^{-1}} = N_{uv^{-1}}\). □

**Example of implications in semigroups** [8]

The law \((xy)^2 = (yx)^2\) implies \(xy^2 = y^2x\) for groups because we can apply the automorphism \(\alpha : x \rightarrow x, y \rightarrow x^{-1}y\). For semigroups we can not use this automorphism. To prove that \((xy)^2 = (yx)^2\) implies \(xy^2 = y^2x\) for semigroups we show first that \((xy)^2 = (yx)^2\) implies:
(i) \((yx)^2y = y(xy)^2\), (use the word \(y(xy)^2\)),
(ii) \(x((yx)^2y)^2 = ((yx)^2y)^2x\), (use \((i)^\alpha, x^\alpha = xyx^2, y^\alpha = y\)),
(iii) \(((yx)^2y)^2 = (yx)^4y^2\), (use \(((yx)^2y)((xy)^2y)\)),
(iv) \((yx)^4 = (xy)^4\).

Then for some word \(p\) we start with \(p \cdot xy^2\) and by using (i) - (iv) obtain \(p \cdot y^2x\), which by cancellation, implies required \(xy^2 = y^2x\).

Namely, for \(p = (xy)^4\) we have
\[
pxy^2 = (xy)^4xy^2 = x(yx)^2(xy)^2yy = x(xy)^2y(yx)^2y = \\
x((yx)^2y)^2 = ((yx)^2y)^2x = (yx)^4y^2x = (xy)^4y^2x = py^2x,
\]
which gives \(pxy^2 = py^2x\) and hence \(xy^2 = y^2x\) as required.

References


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