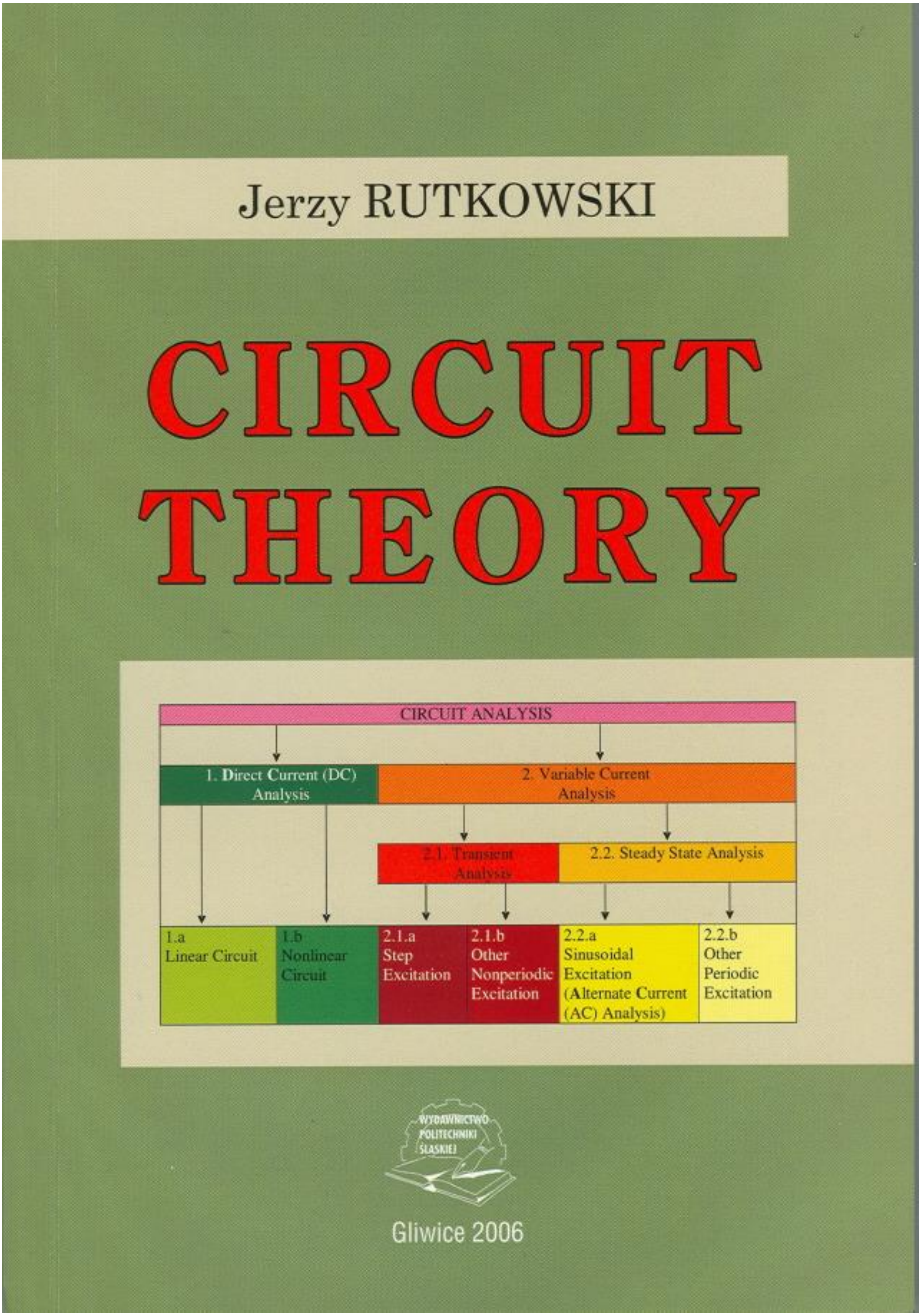

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LIST OF CONTENT

PREFACE

The aim of this book is to provide introductory, yet comprehensive, treatment of circuit analysis and design, to lay down some important and necessary foundations for subsequent use in later engineering courses, such as Signal Theory, Electronics Fundamentals and others.

Since this book is designated primarily for the first or the second year introductory courses, the presentation is geared to students who are being exposed to the basic concepts of electric circuits for the first time. However, it is assumed that students have possessed some elementary knowledge of physics and have some understanding of freshman calculus, such as differential-integral calculus and vector-matrix formulation and solution of linear systems of equations. Other more complex mathematical topics, necessary to describe the considered circuit theory problems, such as i) Laplace transform and singularity functions, ii) algebraic manipulation of complex numbers, iii) solution of nonlinear systems of equations, are raised in a limited and self-contained manner, and are not required as prerequisite background. The first two are the subjects of appendices, the third is developed in the chapter in which it is needed. The book does not contain proofs of theorems, as they can be found in commonly available books dealing with the same subject. Resistance from expanding the length of the book to the extremes sometimes found in current practice was Author's motivation. On the other hand all the theorems and definitions are illustrated by many practical examples. It should be emphasized, that while presenting basic components of electric circuits and introducing different techniques of circuit analysis, particular attention is given to the practical aspects and the physical interpretation of results.

The main aim of the book is to provide students with essential tools of analysis of circuits together with many important concepts underlying the theory of electronic circuits. Care has been taken to fashion the selection and order of content to be of use to the electrical engineering baccalaureate students, but also to students of other engineering disciplines, as the analysis and design of electric circuits is a critical skill for all engineers. Nowadays, English is the binding language in engineering world and the book provides complete vocabulary of terms and concepts used in the Circuit Theory. They are collected in the Appendix C glossary, together with Polish equivalents. This makes the book a very useful educational aid, addressed not only to English speaking students, but also to Polish speaking students having some minor fluency in written English – indispensable for today's engineers.

The book consists of five basic parts – chapters and three appendices. The general order of the content has been selected so that students may learn as many of the techniques of circuit analysis and design as possible in the simplest context. These logically divide into i) real numbers domain - dc analysis), ii) time-domain and Laplace transform domain - transient analysis), iii) phasor or frequency domain - ac analysis. These analyses are discussed first for circuits with lumped constants, next the transmission line transient and ac analyses are considered.

In the brief introductory Chapter 1, the electric variables used to describe circuit elements are revised and problems of circuit analysis and design are classified.

The second Chapter is intended to provide a thorough treatment of circuit analysis based on direct current (dc) circuits. First, linear circuits are discussed. Then, nonlinear resistive circuits and their network analogy (magnetic circuits) are studied. Many important definitions and fundamental principles are given. Various computational techniques are presented with numerous practical examples, such that the student is expected to be conversant with the principles of circuits before entering the next Chapter 3.

The third Chapter and the fourth Chapter are intended to provide a thorough treatment of transient analysis and alternate current (ac) analysis, respectively. Some of the concepts taught in Chapter 1 are revised and extended to more useful and general practical application in time domain and in frequency domain.

The fifth Chapter is intended to provide a thorough treatment of circuits with distributed in place (not lumped) constants. Transmission line is discussed, first its transient response to aperiodic input, then, steady-state sinusoidal response.

It is recommended to organize the material here into a two-semester introductory course, with 30 hours in Semester 1 and 30-45 hours in Semester 2, and to proceed chapter by chapter. Appendix A on Laplace transform and singularity functions should be reviewed before studying Chapter 3, Appendix B on complex numbers should be reviewed before studying Chapter 4, which relies heavily on complex and phasor algebra.

The major results of the theory may appear quite subtle or even abstract, and to make them easy to comprehend numerous practical problems have been provided. The problems are organized into: i) examples and ii) drill problems. Each section of each chapter has numerous step-by-step solved examples and ends with drill problems which are designed to range over all topics of the section and they are generally simple. They can be well used as the formative assessment test or final examination test problems. In all there are near 80 exercises and near 300 drill problems.

1. INTRODUCTION to CIRCUIT THEORY

1.1 CIRCUIT VARIABLES - BASIC TERMS and DEFINITIONS

Our physical world may be interpreted in terms of matter and energy, both of which exist in a variety of forms.

Matter has been defined as anything that occupies space and possesses mass.

Energy is the ability to do work.

In the 18th Century, Benjamin Franklin introduced the term charge and Charles Coulomb his law and terms: electricity, electric or electrostatic field.

Charge is the fundamental unit of matter responsible for electric phenomena.

There are two kinds of charge, positive and negative, Q denotes a positive and **fixed charge**, while q or $q(t)$ denotes a positive and **time-varying charge**.

Capital letters are used to denote constant (in time) variables, while small letters are used to denote time-varying variables.

Coulomb [C] is the unit of charge, the accumulated charge on $6.24145 \cdot 10^{18}$ electrons equals 1 [C].

Electricity are physical phenomena arising from the existence of interaction of charges.

Electric field is a region in space wherein a charge, a test charge Q, experiences an electric force $\mathbf{F}_e$ [N].

Electric field between two fixed **unlike charges** is presented in Fig. 1.1.1. Path along which a test charge Q moves when attracted by one charge and repelled by the other is called the electric line of force. Since a basic phenomenon of charge is that **like charges** repel and **unlike charges** attract, then, the direction of lines of force is always from the **positive charge** to the **negative charge**.

Electric field is uniquely defined in its every point by **electric field intensity**.

Electric field intensity is defined as the electric force per unit charge at a particular point of space.

$$\mathbf{K} = \mathbf{F}_e / Q \tag{1.1.1}$$

Its unit is [N/C]=[V/m].

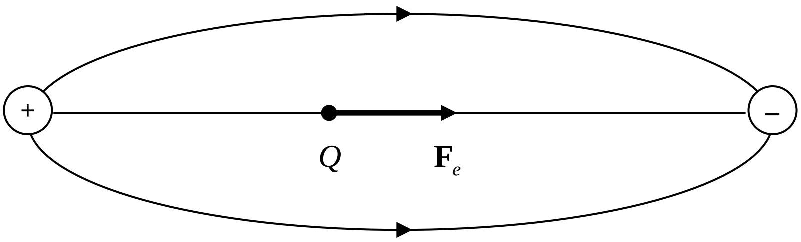

Fig. 1.1.1 Electric field between two unlike charges with three electric lines of force denoted

Next, **work** required to move a test charge Q from point A to point B, as shown in Fig. 1.1.2, will be considered.

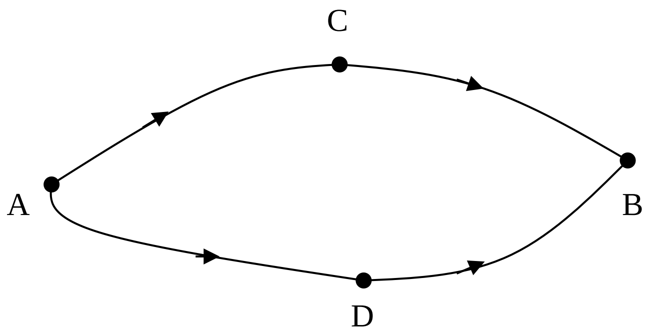

Fig. 1.1.2 Two paths between points A and B located in an electric field

$$W_{\mathrm{AB}} = \int_{\mathrm{A}}^{\mathrm{B}} \mathbf{F}_e \, \mathbf{dl} = Q \int_{\mathrm{A}}^{\mathrm{B}} \mathbf{K} \, \mathbf{dl} \tag{1.1.2}$$

Joule [J] is the unit of this work. The work performed along a closed path (loop) ACBDA is equal zero.

$$W_{\mathrm{ACBDA}} = 0 \tag{1.1.3}$$

Then, work performed along the path ACB is equal to the work performed along the path ADB. In other words, only location of terminal points designates the work performed, not the path shape.

A work required to move a unit charge Q in an electric field is defined as a **voltage**.

$$U_{\mathrm{AB}} = W_{\mathrm{AB}} / Q = \int_{\mathrm{A}}^{\mathrm{B}} \mathbf{K} \, \mathbf{dl} \tag{1.1.4}$$

In the MKS system of units, a voltage of 1[J/C] is defined to be a **volt** [V].

If, in an electric field, the reference point P is chosen, then, voltage between this point (node) and the other one A is called a **potential** or **node voltage** and will be denoted as

$$V_{\mathrm{A}} = U_{\mathrm{AP}} = \int_{\mathrm{A}}^{\mathrm{P}} \mathbf{K}\,\mathbf{dl} \tag{1.1.5}$$

Consider a work performed along a closed path PABP, as shown in Fig. 1.1.3.

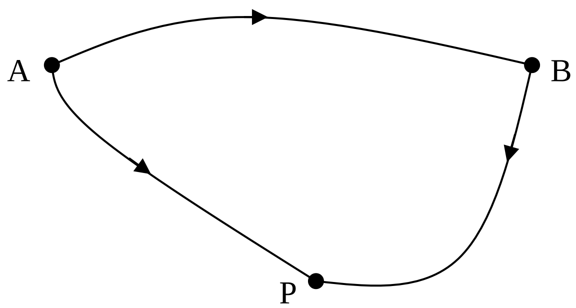

Fig. 1.1.3 Closed path crossing points A, B and P located in an electric field

As

$$W_{\mathrm{PABP}} = W_{\mathrm{AB}} + W_{\mathrm{BP}} - W_{\mathrm{AP}} = 0$$

then,

$$W_{\mathrm{AB}} = W_{\mathrm{AP}} - W_{\mathrm{BP}}$$

and finally:

$$U_{\mathrm{AB}} = V_{\mathrm{A}} - V_{\mathrm{B}} \tag{1.1.6}$$

Thus, a voltage between (across) A and B, or in other words a **voltage drop** from A to B is also called a **potential difference**.

To define a flow of electric charge across any area, such as a cross-section of a wire, term of electric current, or simply **current** is introduced.

A net flow of a charge past a given point, per unit time is defined as **electric current**. In the MKS system, the unit of current is an **ampere** [A]=[C/s].

There are two important current types:

- direct current (dc),
- alternating current (ac).

If a force that moves a charge along a wire is constant, then, the rate of charge transferred is constant and the **direct current** (dc) can be defined:

$$I = \Delta Q / \Delta t \tag{1.1.7a}$$

If a rate of flow of charge is varying in time, then, the **instantaneous current** can be defined:

$$i(t) = i = dq / dt \tag{1.1.7b}$$

Periodic current is the special case. In this case, the instantaneous value of a waveform changes periodically, through negative and positive values. Sinusoidal current, so called **alternating current** (ac) is the most important case.

Finally, **electric power** and **electric energy** delivered to/supplied by a single element or whole (sub)circuit will be discussed.

Power is the time rate of expending or absorbing energy:

$$p = dw / dt \, , \tag{1.1.8}$$

dw is the unit energy in joules and *dt* is the unit time in seconds. Then, *p* is the **instantaneous power** measured in **watts** [W]=[J/s]. A power associated with a current flow through an element/subcircuit is:

$$p = \frac{dw}{dq} \cdot \frac{dq}{dt} = ui \tag{1.1.9a}$$

As can be seen, the instantaneous power absorbed/supplied by element/subcircuit is simply a product of a voltage across this element/subcircuit and a current flowing through the element/subcircuit.

For the dc case:

$$P = U\, I \tag{1.1.9b}$$

From (1.1.8), the unit energy:

$$dw = p\, dt \tag{1.1.10}$$

Then, the total energy absorbed/supplied within a time interval from $t_0 = 0$ to arbitrary time instant *t* is:

$$w = \int_0^t p\, dt \tag{1.1.11a}$$

For the particular $t = T$, the total energy absorbed/supplied is:

$$W_T = \int_0^T p\, dt \tag{1.1.11b}$$

Electric energy absorbed by an element/subcircuit is dissipated as a heat. Such **thermal energy** w_{th}, in **calories** [cal], can be converted from electric energy:

$$w_{\text{th}} = 0.239\, w \tag{1.1.12}$$

Drill problems 1.1

1. A constant current of 2 A flows through an element. The energy to move the current for 1 second is 10 joules. Find the voltage across the element.
2. Find the energy required to move 2 coulombs of charge through 4 volts.
3. A constant current of I=10 A is delivered to an element for 5 seconds. Find the energy required to maintain a voltage of 10 V.
4. Voltage of energy absorbing element is constant, $u = U = 10\,\mathrm{V}$ and its current rises linearly from 0 to 2 mA within period of 2 s, and then, remains constant. Find the absorbed energy during the period of 5 s.
5. When fully charged, a car 12 V battery stored charge is 56 A·h. How many times car can be started if each attempt lasts 10 s and draws 30 A of current from the battery ?

The power absorbed by a circuit element is shown. At what time is the net energy absorbed a maximum, at what time is the net energy supplied a maximum, at what time the net energy is zero ? Is the total net energy (for the whole period of time) absorbed or supplied?

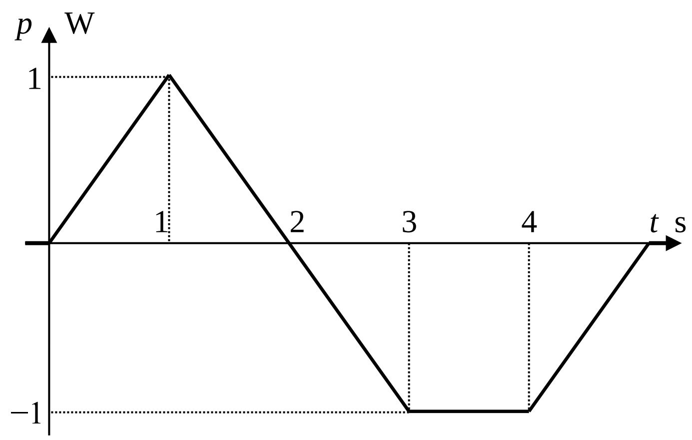

Fig. P.1.1.6

6. An element absorbs energy as shown. If the current entering its terminal is $i = 10t$ mA, find the element voltage at $t = 1$ ms and $t = 5$ ms.

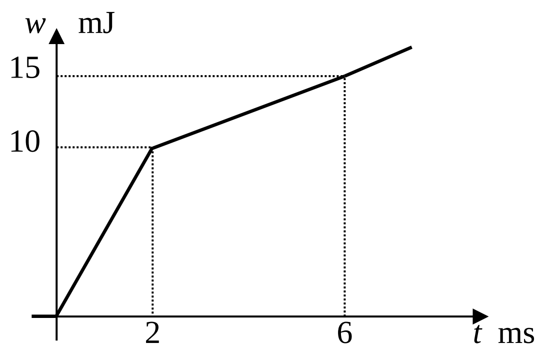

Fig. P.1.1.7

7. A small 1.5-volt alkaline (AA) battery has a nominal life of 150 joules. For how many minutes will it power a calculator that draws a 2 mA current ?

8. A CD player uses four AA batteries in series to provide 6 V to the player circuit. Each battery stores 50 watt-seconds of energy. If the player is drawing a constant 10 mA from the battery pack, how long will the player operate at nominal power ?
9. A circuit element with a constant voltage of 4 V across it dissipates 80 J of energy in 2 minutes. What is the current through the element ?

1.2 CLASSIFICATION of CIRCUIT THEORY PROBLEMS

In general, all **Circuit Theory** problems fall into two categories:

- analog circuit synthesis,
- analog circuit analysis.

Problems related with **analog circuit analysis** will be discussed. To start a circuit analysis (simulation), its model should be designated by a design engineer. Problem of circuit modeling, very important from a practical point of view, will be not discussed. A circuit model is built of ideal elements, or simply elements, such as resistors, capacitors, coils, etc. - practical elements are modeled by means of ideal elements. Before proceeding to circuit analysis problems, the following basic terms have be introduced:

Circuit parameter or **circuit constant**, denoted by P: a constant describing an element, such as resistance R, capacitance C, inductance L, etc.

Circuit **input signal** or **circuit excitation**, denoted by X: a source of signal, voltage or current source.

Circuit **output signal** or **circuit response** Y: a circuit variable, such as voltage, current, gain, etc.

Problem of circuit analysis can be expressed by means of block diagram, as presented in Fig. 1.2.1, for one-dimensional (**S**ingle **I**nput **S**ingle **O**utput – **SISO**) case.

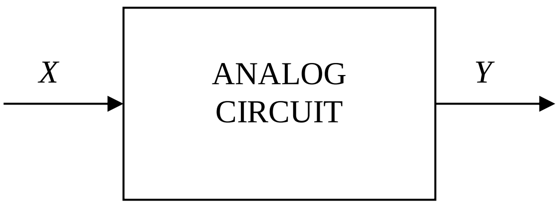

Fig. 1.2.1 Block diagram of SISO circuit

For multi-dimensional (**M**ultiple **I**nput **M**ultiple **O**utput – **MIMO**) case, **X** and **Y** are vectors.

According to a character of elements, circuits can be classified into:

- **linear circuits**,
- **nonlinear circuits**,

or

- **circuits with lumped constants**,
- **circuits with distributed constants**.

The meaning of the above terms will be explained in next chapters.

Then, two different problems of circuit analysis can be formulated:

PROBLEM 1. (Classical analysis)

Given: all circuit constants P_i; $i = 1,...,L$ and input signal(s) X_i; $i = 1,...,M$. Find: circuit response(s) Y_i; $i = 1,...,N$.

PROBLEM 2. (Parameter identification)

Given: $L_1 < L$ circuit constants (parameters) and/or $M_1 \le M$ input signals and $N_1 \le N$ responses (measurements). Find: $L_2 = L - L_1$ unknown circuit constants and/or $M_2 = M - M_1$ input signals and $N_2 = N - N_1$ other responses, $L_2 + M_2 = N_1$.

Both problems can be modeled by a system of algebraic equations. For P1 and linear circuit, the system consists of linear equations. For P2, some constants became variables and the system is nonlinear, even for linear circuit.

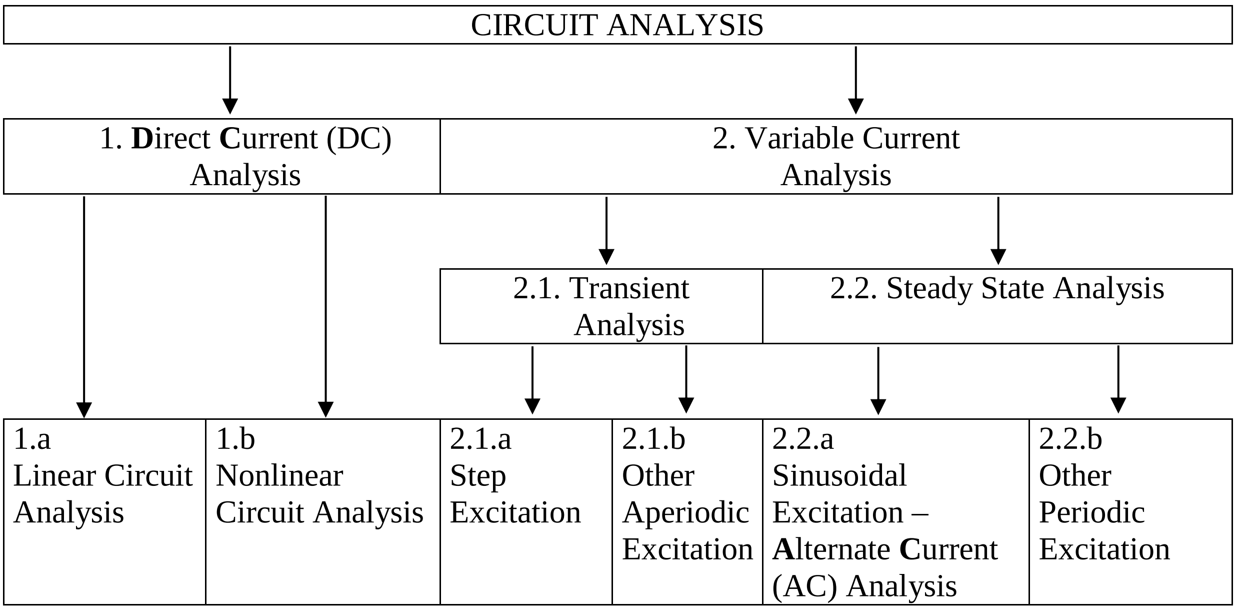

Fig. 1.2.2 Classification of analyses

Classification of analyses, subject to a character of excitation is presented in Fig. 1.2.2. All these analyses, except a steady-state analysis in arbitrary periodic excitation case, will be discussed in next chapters.

2. DC ANALYSIS

2.1 CIRCUIT ELEMENTS

CLASSIFICATION

An electric circuit or electric network is an interconnection of elements linked together in a closed path so that an electric current may continuously flow.

Generally, all elements can be classified into two categories:

- two-terminal elements,
- multi-terminal elements.

In further considerations, two-terminal elements are taken into account, while multi-terminal elements are discussed in details in Chapter 2.9.

A **general two-terminal element** is presented in Fig. 2.1.1.

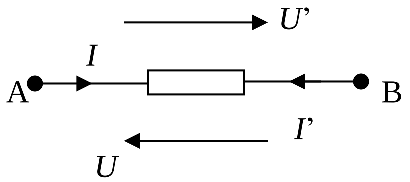

Fig. 2.1.1 General two-terminal element

By the convention, an element voltage is denoted by an arrow placed along an element.

> **Voltage arrowhead** points terminal of a higher potential if its value is positive, or terminal of a lower potential if its value is negative.

Quite often the **double subscript notation** is used. For the generalized element of Fig. 2.1.1:

$$U = U_{\mathrm{AB}},\ U' = U_{\mathrm{BA}} \tag{2.1.1}$$

> Actual flow of free electrons is from negative to positive terminal and this is termed the **electron flow**. The flow of current is conventionally represented as a flow of positive charges. **Current arrowhead** indicates direction of the **conventional flow** if the current value is positive, or direction opposite to the conventional flow if the current value is negative.

Example 2.1.1

The measured potentials (terminal voltages) of Fig. 2.1.1 general element are: $V_{\mathrm{A}} = 10\,\mathrm{V}$, $V_{\mathrm{B}} = 3\,\mathrm{V}$. Find its voltage and current.

÷

The element voltage is:

$U = U_{\mathrm{AB}} = 7\,\mathrm{V}$, or $U' = U_{\mathrm{BA}} = -7\,\mathrm{V}$.

The element current, both value and sign, are designated by the element $I - U$ relationship.

♦

Mutual position of voltage and current arrowheads together with their signs decide whether an element absorbs or supplies energy. Two possible positions of arrows are presented in Fig. 2.1.2.

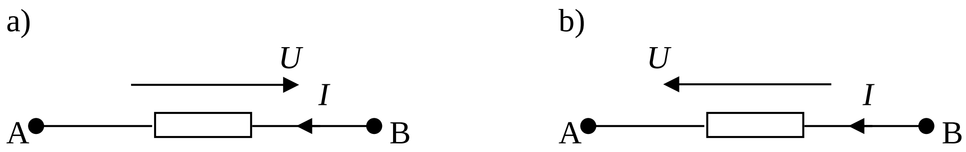

Fig. 2.1.2 Two possible mutual positions of voltage and current arrowheads

For the opposite position of Fig. 2.1.2 a, current and voltage are said to satisfy the **passive sign convention**, and
  an element absorbs power (energy) if $P = UI > 0$,
  an element supplies power (energy) if $P = UI < 0$.

For the same position of Fig. 2.1.2 b,
  an element absorbs power (energy) if, $P = UI < 0$,
  an element supplies power (energy) if $P = UI > 0$.

According to the direction of energy flow, elements can be classified into two categories:

- **passive elements**,
- **active elements**.

An element is said to be **passive** if the total energy delivered to it from the rest of a circuit is always nonnegative (zero or positive).

For a passive element, mutual position of voltage and current arrowheads has to be, by the convention (passive sign convention) opposite, as shown in Fig. 2.1.2 a. Then, the total energy delivered to passive element is:

$$w = \int_0^t ui\,dt \geq 0 \tag{2.1.2a}$$

For the dc case:

$$w = UI\,t \tag{2.1.2b}$$

An element is said to be **active** if the total energy delivered to it is not always nonnegative.

For an active element, mutual position of voltage and current arrowheads is arbitrary, however, same position is preferred. Then, for active element and same mutual position of arrowheads:

$$w = \int_0^t ui\,dt < 0 \,\text{or} \geq 0 \tag{2.1.3}$$

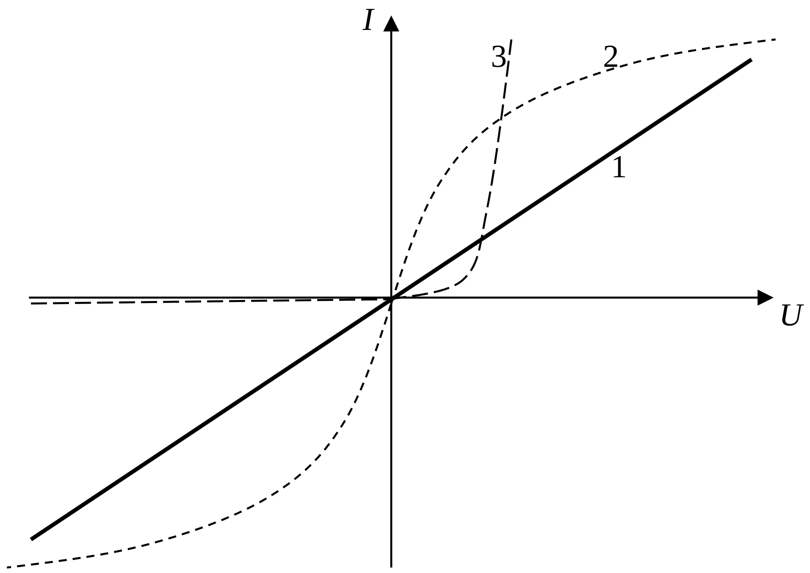

Fig. 2.1.3 Exemplary I−U relationships of passive elements

Passive or active two-terminal element is uniquely described by its $I-U$ relationship:

$$I = f(U) \text{ or } U = f^{-1}(I) = g(I) \tag{2.1.4}$$

This relationship can be given by the manufacturer or it can be measured. Taking into account character of $I-U$ relationship, elements can be classified into two categories:

- linear elements,
- nonlinear elements.

Fig. 2.1.3 presents exemplary $I-U$ relationships of passive elements:

1. **linear element**,
2. **bilateral nonlinear element**, $f(U) = -f(-U)$,
3. **unilateral nonlinear element**, $f(U) \neq -f(-U)$.

PASSIVE TWO-TERMINAL ELEMENTS

Linear elements, resistor and meters, voltmeter and ammeter, will be discussed.

Resistor

Linear resistor graphic symbol is presented in Fig. 2.1.4 (mutual position of arrowheads is always opposite).

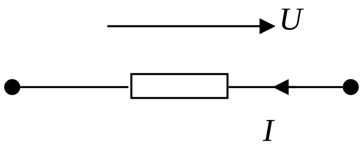

Fig. 2.1.4 Graphic symbol of a linear resistor

Its $I-U$ relationship (Fig. 2.1.3–1) is the well known **Ohm's law**, satisfied also for instantaneous values, in brackets.

$$U = RI \quad (u(t) = Ri(t)) \tag{2.1.5a}$$

$$I = GU \quad (i(t) = Gu(t)) \tag{2.1.5b}$$

R and *G* are constants of proportionality, $R = 1/G$.

R is called **resistance**, its unit is **ohm** [Ω],

G is called **conductance**, its unit is **siemens** [S].

A resistor power absorbed is

$$P = I^2R = U^2G \geq 0 \quad (p = i^2R = u^2G) \tag{2.1.6}$$

Voltmeter

A voltmeter graphic symbol is presented in Fig. 2.1.5 (mutual position of arrowheads is always opposite).

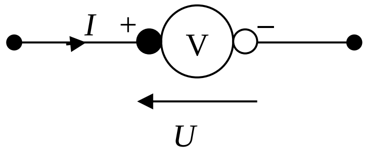

Fig. 2.1.5 Graphic symbol of voltmeter

Its $I-U$ relationship is presented in Fig. 2.1.6, for an ideal voltmeter (horizontal axis) and a practical voltmeter (dashed line).

For an ideal voltmeter: $G_\mathrm{V} = 0$ $(R_\mathrm{V} = \infty)$, $I = 0$

For a practical voltmeter: $G_\mathrm{V} > 0$, $I > 0$ and voltmeter is represented by the resistor

$$I = G_\mathrm{V}U\,. \tag{2.1.7}$$

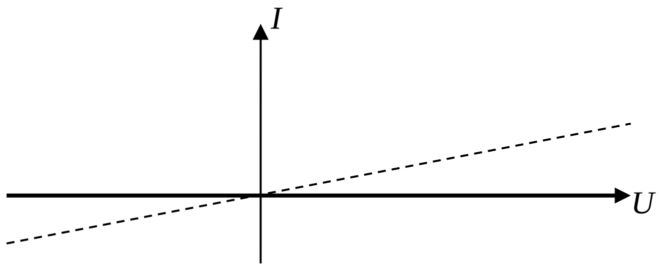

Fig. 2.1.6 I–U relationship of ideal and practical voltmeter

Ammeter

An ammeter graphic symbol is presented in Fig. 2.1.7 (mutual position of arrowheads is always opposite).

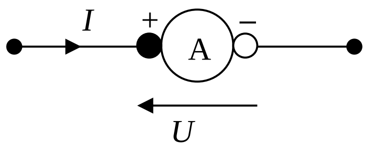

Fig. 2.1.7 Graphic symbol of ammeter

Its $I-U$ relationship is presented in Fig. 2.1.8, for ideal ammeter (vertical axis) and practical voltmeter (dashed line).

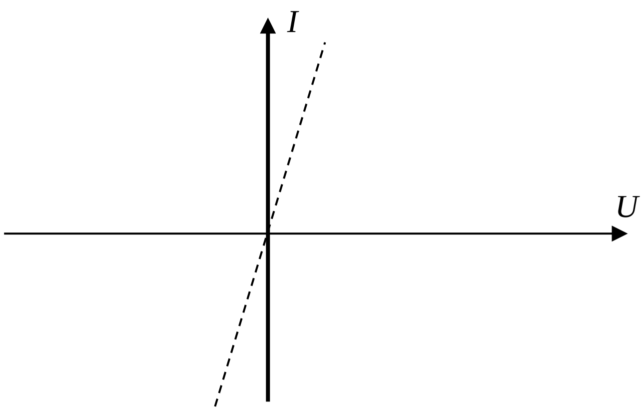

Fig. 2.1.8 *I–U* relationship of ideal and practical ammeter

For an ideal ammeter: $R_{\mathrm{A}} = 0$ $(G_{\mathrm{A}} = \infty)$, $U = 0$ and ammeter is the short-circuited branch.
For a practical voltmeter: $R_{\mathrm{A}} > 0$, $U > 0$ and ammeter is represented by the resistor

$$U = R_{\mathrm{A}} I \tag{2.1.8}$$

ACTIVE TWO-TERMINAL ELEMENTS

Linear ideal DC sources, a **voltage source**, so called **electromotive force** (emf), and a **current source**. will be discussed in this Chapter, while practical sources will be discussed in Chapter 2.5.

Voltage source

An ideal voltage source graphic symbol is presented in Fig. 2.1.9 (mutual position of arrowheads is arbitrary).

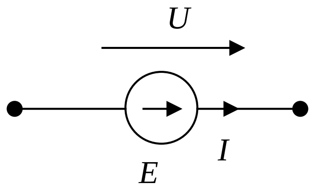

Fig. 2.1.9 Graphic symbol of ideal voltage source

Its $I-U$ relationship is:

$$U = E \tag{2.1.9}$$

as presented in Fig. 2.1.10, and the power supplied/absorbed is:

$$P = E\,I \geq 0 \text{ or } \leq 0\,. \tag{2.1.10}$$

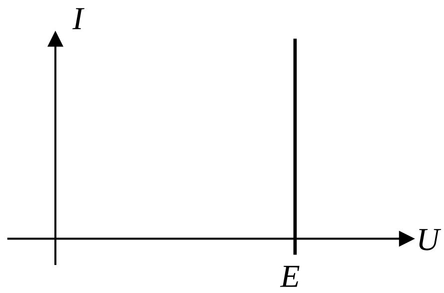

Fig. 2.1.10 *I*−*U* relationship of ideal voltage source

Current source

An ideal current source graphic symbol is presented in Fig. 2.1.11 (mutual position of arrowheads is arbitrary).

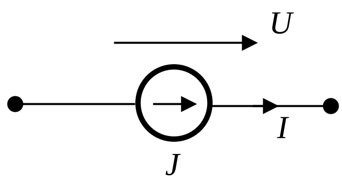

Fig. 2.1.11 Graphic symbol of ideal current source

Its $I-U$ relationship is:

$$I = J \tag{2.1.11}$$

as presented in Fig. 2.1.12.

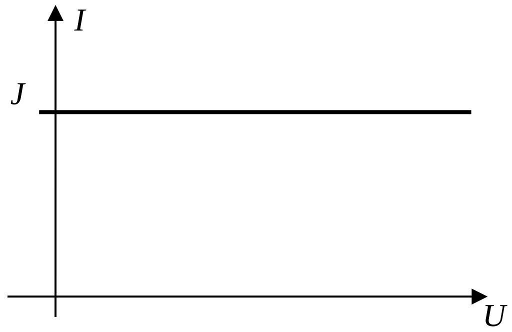

Fig. 2.1.12 I–U relationship of ideal current source

A current source power is

$$P = U\,J \geq 0 \text{ or } \leq 0 \tag{2.1.12}$$

<u>Drill problems 2.1</u>

1. For the given currents that flow through 10 Ω resistor, calculate the total energy absorbed by the resistor.

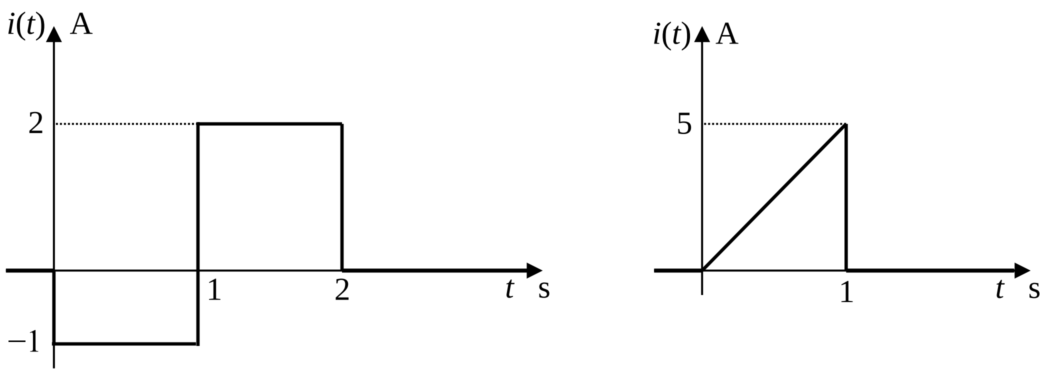

Fig. P.2.1.1

2. The above given currents flows through 10 V emf. Assuming the same position of arrowheads, find the total energy supplied.
3. A heater (resistor) rated 200 V and 100 W is connected to 100V dc supply. Find energy absorbed in 5 hours. If electric energy costs 10 cents/kW·h, find the cost of heating during the entire 5 hours ?

4. A circuit element labeled with passive sign convention has the current and voltage waveforms as graphed. Sketch the instantaneous power absorbed and the total energy absorbed over the interval <0, 4> s.

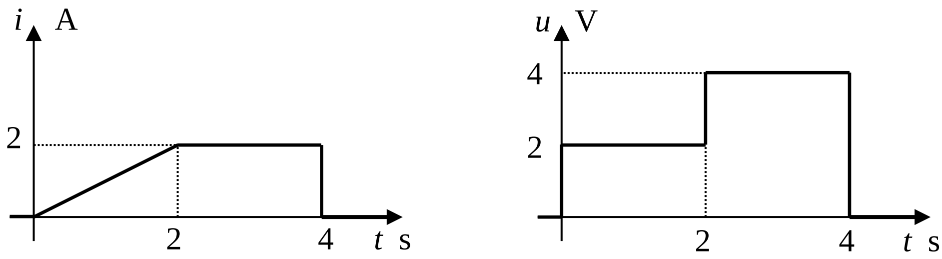

Fig. P.2.1.4

5. A certain element with i and u that satisfy the passive sign convention is described by the relationship $u = 2|i|$. For the current shown, sketch the power p. Is this element active or passive ?

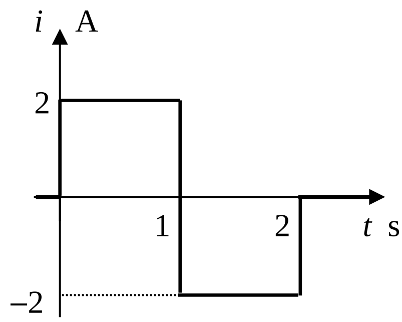

Fig. P.2.1.5

6. Repeat Problem 2.1.5 for $u = i^2 \operatorname{sgn}(i)$.

7. An automobile battery is charged with a constant current of 2 A for 5 hours. The terminal voltage is $u = 10 + 0.5t$ for $t > 0$, where t is in hours. Sketch $w(t)$ and find the total energy delivered to the battery during the entire 5 hours. If electric energy costs 10 cents/kW·h, find the total cost of charging the battery.

8. If the voltage across an element is 10 V and the current i entering the positive terminal is as shown, find the power delivered to the element at $t = 4$ s and the total energy delivered between 0 and 5 s.

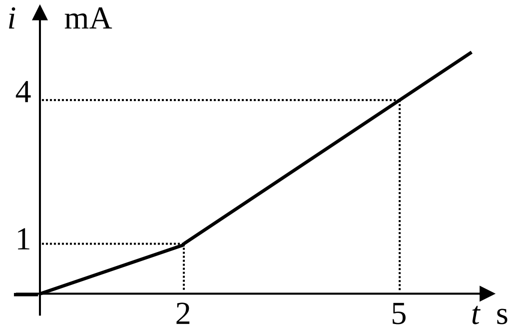

Fig. P.2.1.8

9. If the function applied in Problem 2.1.8 is the voltage u in volts and the current entering the positive terminal is 2 mA, find the power delivered to the element at $t = 4$ s and the total energy delivered between 0 and 15 s.

10. The current entering the positive terminal of a 10-volt battery rises linearly from 2 to 10 mA between $t = 0$ and $t = 10$ minutes. How much charge passes through the battery during the first 5 minutes ? What is the power absorbed at $t = 5$ minutes and $t = 10$ minutes ? What is the energy supplied during the first 5 minutes and during the entire 10 minutes ?

11. An electric range has a dc of 10 A entering the positive terminal at a voltage drop of 100V dc. The range is operated for 4 hours. Find the charge, in coulombs, that passes through the range. Find the total power and the total energy absorbed.

12. The energy w absorbed by a two-terminal device is shown. If the voltage across the device is $u = 10\cos(\pi t)$ V, where t in ms, find the current entering the positive terminal at $t = 1$, 2, 3 ms (current and voltage satisfy the passive sign convention).

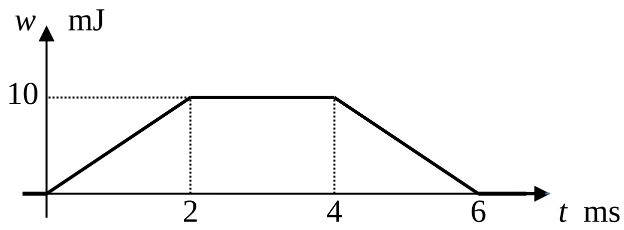

Fig. P.2.1.12

13. Sketch the power absorbed or delivered in Problem 2.1.12.

14. Repeat calculations in Problems 2.1.12 and 2.1.13, if the passive sign convention is not satisfied.

15. The voltage across an element is a constant 15 V. The current leaving the positive terminal is $i = 10 - 10\sin(2\pi t)$ A. Find the instantaneous power p and sketch it in the interval from 0 to 1 s. Is this element active or passive ? Calculate the energy received or delivered by the element in the interval from 0.5 to 1 s.

2.2 CIRCUIT DIAGRAM and KIRCHHOFF's LAWS

Before starting a circuit analysis, its model has to be created and expressed in a form of diagram. This term and other related terms will be defined at first.

CIRCUIT DIAGRAM

A drawing that shows schematically the interconnection of circuit elements, represented by their graphic symbols, is called a **circuit diagram**.

A circuit structure (element interconnections) can be expressed by a **circuit graph**. Such graph is built of branches connected in nodes.

A connection point between two or more elements/branches is called a circuit/graph **node**. Number of circuit/graph nodes is denoted as n.

A circuit/graph **branch** is defined as an element or string of elements connected between two nodes. Number of circuit/graph branches is denoted as b.

Then, terms of circuit/graph loop, mesh and cutset can be introduced.

Two or more branches that form a closed path is called a **loop**.

Cutset is a closed line around one or more nodes, crossing two or more branches, each branch only once.

Planar circuit is a circuit whose graph can be drawn on a plane surface so that no branch cross. Then, plane is divided by the circuit graph into distinct areas, windowpane areas.

The closed boundary of each windowpane area, a loop that does not contain any other loop within it, is called a **mesh**.

An exemplary circuit is used to illustrate these terms.

Example 2.2.1

Diagram of an exemplary circuit, built of five resistors, ammeter, voltmeter and two ideal sources, is presented in Fig. 2.2.1. The circuit graph is presented in Fig. 2.2.2.

The graph is built of $b = 8$ branches, numbered from 1 to 8, connected in $n = 5$ nodes, denoted by letters A, B, C, D, E.

Some finite number of loops can be found, e.g. loops built of the following branches:

I: 2,3,4;
II: 3,6,5; (2.2.1)
III: 2,6,5,4.

Loops I and II are independent loops, while loop III is sum of I and II.

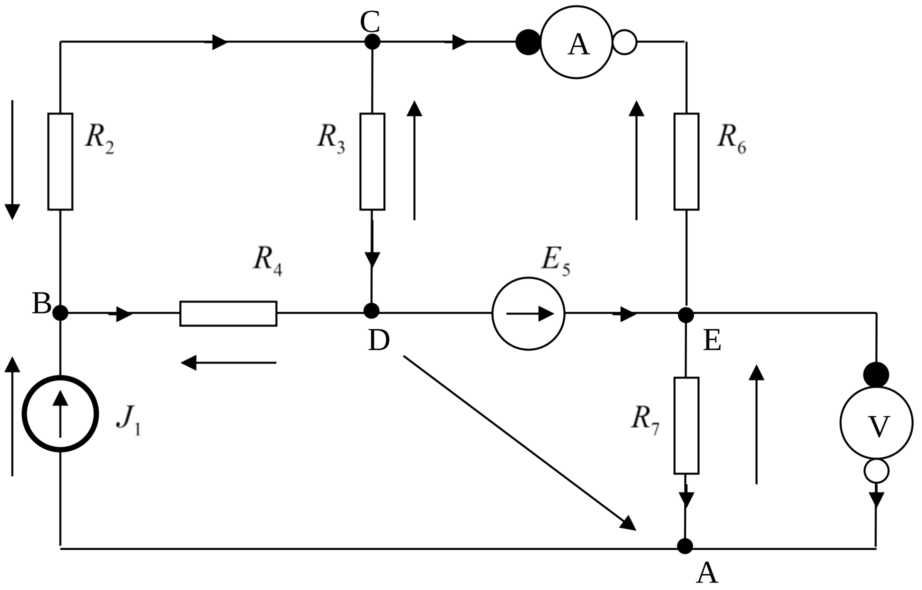

Fig. 2.2.1 Diagram of an exemplary circuit (Example 2.2.1)

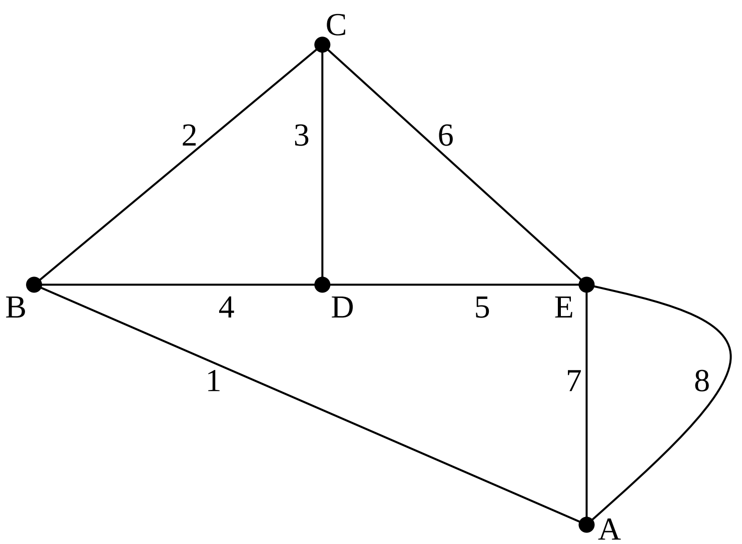

Fig. 2.2.2 Graph of exemplary circuit (Example 2.2.1)

Some finite number of cutsets can be found, e.g. cutsets around the following nodes:

1: E (crossing branches 5,6,7,8);
2: A (crossing branches 1,8,7); (2.2.2)
3: AE (crossing branches 1,5,6).

Cutsets 1 and 2 are independent cutsets, while cutset 3 is sum of 1 and 2.

♦

KIRCHHOFF'S LAWS

Kirchhoff's Current Law (KCL)

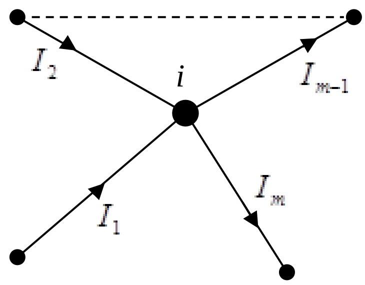

Fig. 2.2.3 Cutset around node i

Consider the cutset around a single node *i*, crossing branches $1,2,...,m$, as presented in Fig. 2.2.3. Charge can not accumulate at the node. Then,

$$\sum_{\bullet i} \Delta Q = 0\,, \tag{2.2.3}$$

where $\sum\limits_{\bullet i}$ denotes the algebraic sum of charges/currents entering or leaving the node *i*, by the convention:

+ , if current arrowhead is directed to the node/cutset, (2.2.4)
– , if current arrowhead is directed from the node/cutset.

After dividing (2.2.3) by Δt, the KCL can be formulated.

The algebraic sum of currents entering or leaving arbitrary node *i* equals zero.

$$\sum_{\bullet i} I = 0\,. \tag{2.2.5}$$

The above KCL can be generalized into arbitrary cutset *i* crossing branches $1,2,...,m$, as presented in Fig. 2.2.4.

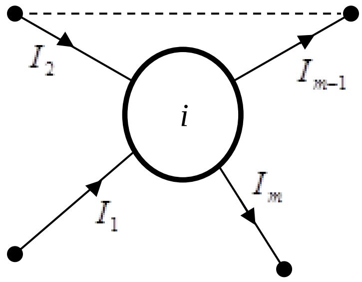

Fig. 2.2.4 Arbitrary cutset i crossing branches 1,2,…,*m*

The algebraic sum of currents entering or leaving arbitrary cutset *i* equals zero, (2.2.5).

For the given circuit of *b* branches and *n* nodes, total of

$$t = n - 1 \tag{2.2.6}$$

independent KCL equations can be formulated, e.g. for all cutsets around individual nodes except the reference one.

Example 2.2.1 cont.

For the three cutsets of (2.2.2), KCL equations are:

$$\begin{aligned} &1:\ I_5 + I_6 - I_7 - I_8 = 0\,; \\ &2:\ -I_1 + I_8 + I_7 = 0\,; \\ &3:\ -I_1 + I_5 + I_6 = 0\,. \end{aligned} \tag{2.2.7}$$

Equation (2.2.7-3) is the algebraic sum of (2.2.7-1) and (2.2.7-2).

The total number of independent KCL equations is $t = 4$, and they can be equations of any four from the circuit five nodes.

♦

Kirchhoff's Voltage Law (KVL)

Consider the loop *i*, built of branches $1,2,...,m$, as presented in Fig. 2.2.5. A work performed along a loop is equal zero. Then, KVC can be formulated.

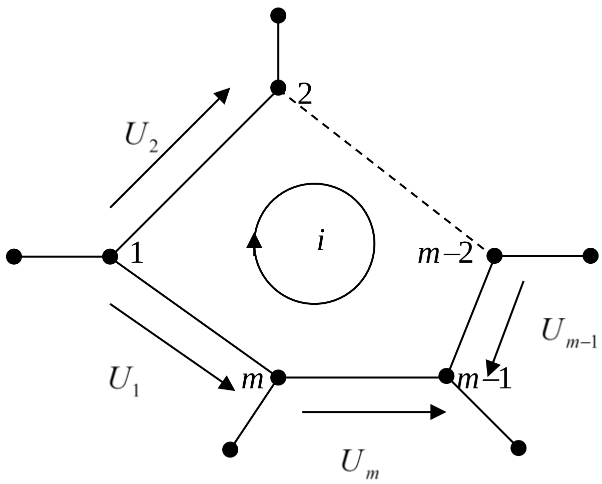

Fig. 2.2.5 Loop i built of m branches

The algebraic sum of voltages around arbitrary loop *i* equals zero,

$$\sum_{\text{O}i} U = 0 \,, \tag{2.2.8}$$

where $\sum_{\text{O}i}$ denotes the algebraic sum of voltages around the loop *i*,

by the convention:

+ , if, voltage arrowhead has clockwise direction, (2.2.9)
– , if voltage arrowhead has anticlockwise direction.

The above KVL can be generalized into arbitrary closed path *i*, as presented in Fig. 2.2.6.

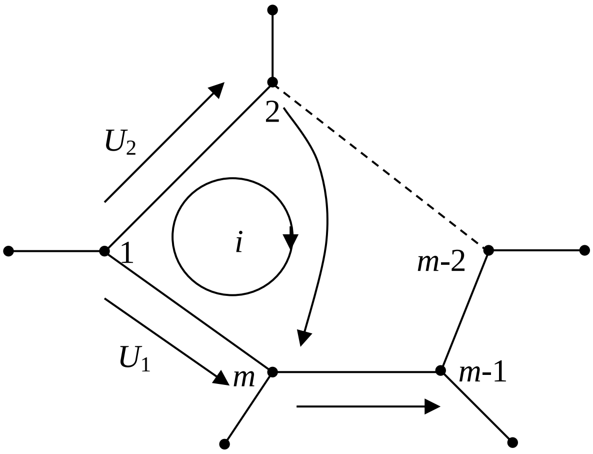

Fig. 2.2.6 Closed path i

The algebraic sum of voltages around arbitrary closed path *i* equals zero, (2.2.8).

For the given circuit of *b* branches and *n* nodes, total of	
$l = b - n + 1$	(2.2.10)
independent KVL equations can be formulated, e.g. for all meshes.	

<u>Example 2.2.1 cont.</u>

For the three loops of (2.2.1), KVL equations are:

$$\begin{aligned} &\text{I:} \quad -U_2 - U_3 + U_4 = 0, \\ &\text{II:} \quad U_3 - U_{\mathrm{A}} - U_6 - E_5 = 0, \\ &\text{III:} \quad -U_2 - U_{\mathrm{A}} - U_6 - E_5 + U_4 = 0. \end{aligned} \tag{2.2.11}$$

Equation (2.2.11-III) is the algebraic sum of equations (2.2.11-I) and (2.2.11-II).

The total number of independent KVL equations is $l = 4$ and they can be equations of all meshes:

I: 2,3,4;
II: 3,6,5; (2.2.12)
III: 1,4,5,7;
IV: 7,8.

♦

<u>Drill problems 2.2</u>

1. Calculate voltage U_{AB}.

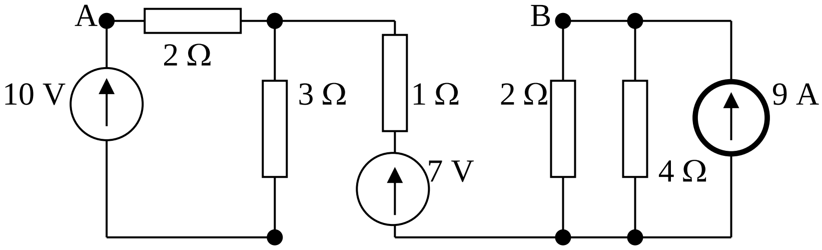

Fig. P.2.2.1

2. Find indication of an ideal ammeter.

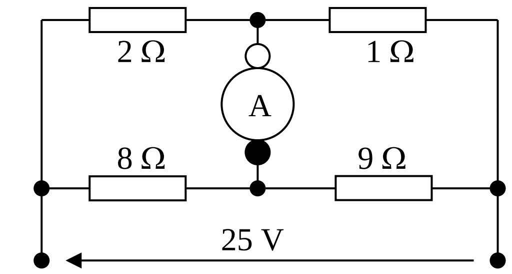

Fig. P.2.2.2

3. What should be the value of R so that current $I = 0.5$ A ?

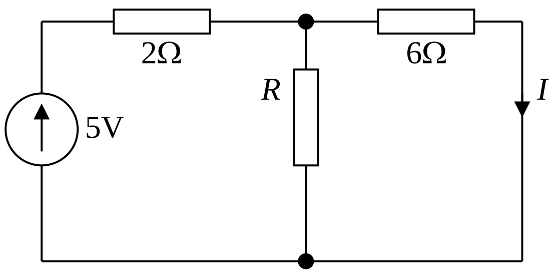

Fig. P.2.2.3

4. Find E_x, if the power supplied by $E = 10$ V is 10 W and $R = 5\Omega,\ J = 1$A .

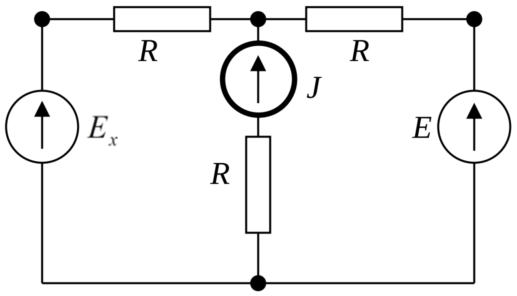

Fig. P.2.2.4

5. Find the value of R so that the power delivered by the source is 48 W.

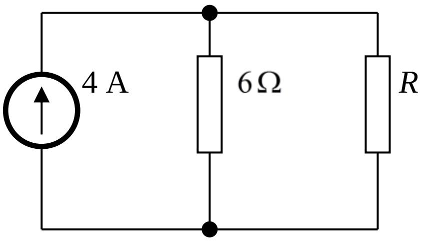

Fig. P.2.2.5

6. Suppose the indicated voltage is 12 V. Find R.

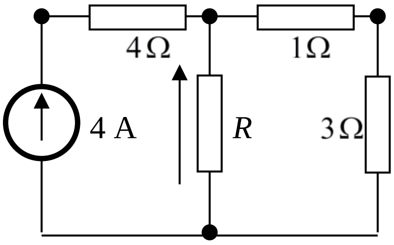

Fig. P.2.2.6

7. Find the power absorbed (supplied ?) by the current source.

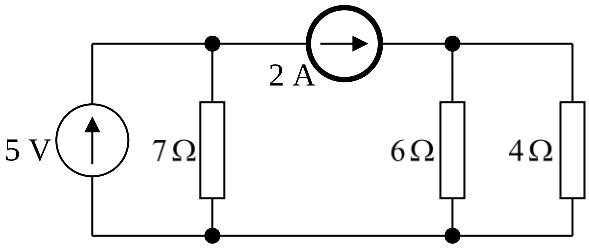

Fig. P.2.2.7

8. Find the supply voltage of a ladder network shown, so that $I = 2\,\mathrm{A}$. Assume $R = 5\,\Omega$.

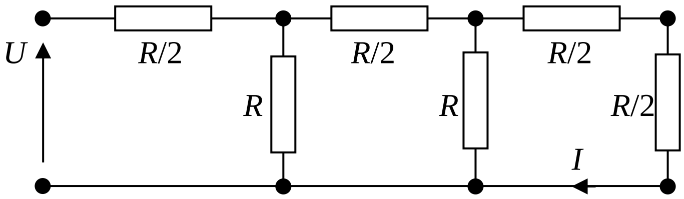

Fig. P.2.2.8

9. Find R if $I = 2\,\mathrm{A}$. Then, find all currents and voltages.

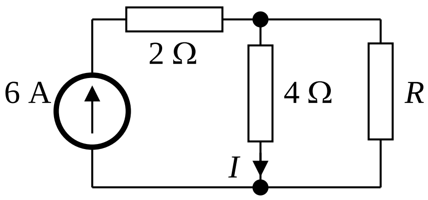

Fig. P.2.2.9

10. By what factor the 6 A current source in Problem 2.2.9 circuit should be increased to double the power it supplies, by what factor it should be increased to double the current I.

2.3 ANALYSIS of COMPLEX CIRCUITS

Resistor equations together with $t = n - 1$ KCL equations and $l = b - n + 1$ KVL equations allow to formulate the well defined system of circuit equations. Two approaches will be presented: the generalized Kirchhoff's analysis and the commonly used nodal analysis.

GENERALIZED KIRCHHOFF'S ANALYSIS

- **Branch Current Analysis**

Algorithm 2.3.1 – Branch Current Analysis

1. Assume unknown currents in each branch, $I_1,...,I_b$. Indicate their directions by arrows (direction is arbitrary).
2. For each element indicate, by an arrow, its voltage drop (passive elements) or rise (active elements) that a particular current causes in passing the element.
3. Select the reference node (selection of the node is arbitrary). Write down KCL equations at all other $t = n - 1$ nodes.
4. Write down KVL equations for all $l = b - n + 1$ meshes.
5. Express resistor voltages by their currents, by means of Ohm's law.
6. Solve the set of *b* equations with resistor currents and voltages of ideal current sources, if present, as unknown variables.
7. Find circuit responses, if not found already.

Example 2.3.1

Diagram of an exemplary circuit, built of $b = 6$ branches connected in $n = 4$ nodes, is presented in Fig. 2.3.1.

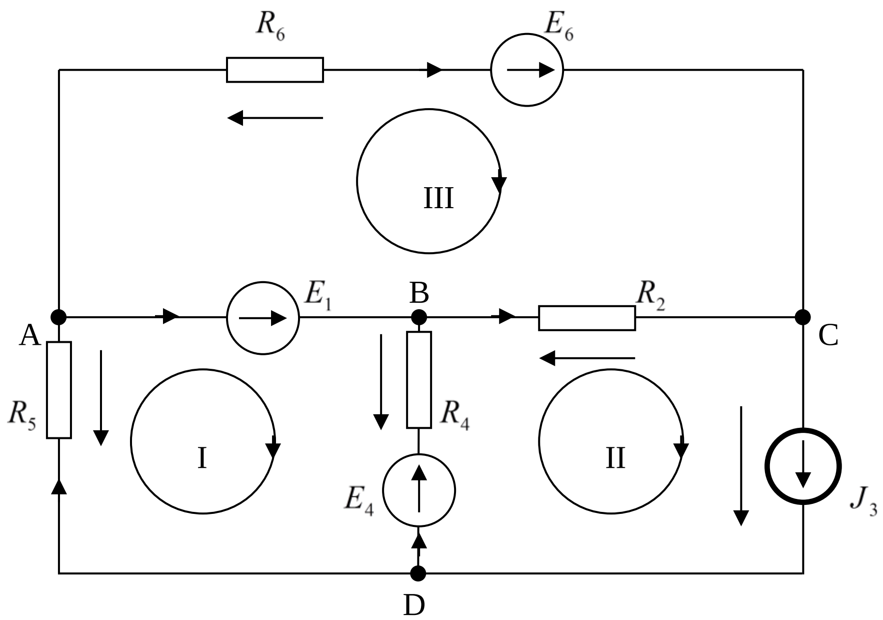

Fig. 2.3.1 Diagram of exemplary circuit (Example 2.3.1)

Node D is selected as the reference one. Then, $t = 3$ KCL equations (2.3.1a) and $l = 3$ KVL equations (2.3.2a) can be formulated.

$$
\begin{array}{llllllll}
\text{A:} & -I_1 & & & & +I_5 & -I_6 & = 0 \\
\text{B:} & +I_1 & -I_2 & & +I_4 & & & = 0 \\
\text{C:} & & +I_2 & -J_3 & & & +I_6 & = 0
\end{array}
\qquad (2.3.1a)
$$

$$
\begin{array}{llllllll}
\text{I:} & +E_1 & & & +U_4 - E_4 & -U_5 & & = 0 \\
\text{II:} & & -U_2 & +U_3 & -U_4 + E_4 & & & = 0 \\
\text{III:} & -E_1 & +U_2 & & & & +E_6 - U_6 & = 0
\end{array}
\qquad (2.3.2a)
$$

Resistor voltages can be expressed by their currents:

$U_i = R_i I_i$; $i = 2,4,5,6$

Then, KVL equations have the following form:

$$
\begin{array}{llllllll}
\text{I:} & +E_1 & & & +R_4 I_4 - E_4 & -R_5 I_5 & & = 0 \\
\text{II:} & & -R_2 I_2 & +U_3 & -R_4 I_4 + E_4 & & & = 0 \\
\text{III:} & -E_1 & +R_2 I_2 & & & & +E_6 - R_6 I_6 & = 0
\end{array}
\qquad (2.3.2b)
$$

Equations (2.3.1a) and (2.3.2b) form a system of $b = 6$ linear equations with six unknowns, currents: I_1, I_2, I_4, I_5, I_6 and voltage U_3.

♦

- **Branch Voltage Analysis**

Algorithm 2.3.2 – Branch Voltage Analysis

1. Assume unknown currents in each branch, $I_1,\ldots,I_b$. Indicate their directions by arrows (direction is arbitrary).
2. For each element indicate, by an arrow, its voltage drop (passive elements) or rise (active elements) that a particular current causes in passing the element.
3. Select the reference node (selection of the node is arbitrary). Write down KCL equations at all other $t = n - 1$ nodes.
4. Write down KVL equations for all $l = b - n + 1$ meshes.
5. Express resistor currents by their voltages, by means of Ohm's law.
6. Solve the set of *b* equations with resistor voltages and currents of ideal voltage sources, if present, as unknown variables.
7. Find circuit responses, if not found already.

Example 2.3.1 cont.

Resistor currents can be expressed by their voltages:

$$I_i = G_i U_i \,;\; i = 2,4,5,6$$

Then, KCL equations have the following form:

$$\begin{array}{llllllll} \text{A:} & -I_1 & & & & +G_5U_5 & -G_6U_6 & =0 \\ \text{B:} & +I_1 & -G_2U_2 & & +G_4U_4 & & & =0 \\ \text{C:} & & +G_2U_2 & -J_3 & & & +G_6U_6 & =0 \end{array} \qquad (2.3.1b)$$

Equations (2.3.1b) and (2.3.2a) form a system of $b = 6$ linear equations with six unknowns, voltages: U_2, U_3, U_4, U_5, U_6 and current I_1.

♦

NODE VOLTAGE (NODAL) ANALYSIS

In branch voltage analysis, in KCL equations, resistor currents are expressed by corresponding voltages. These voltages can be expressed by node voltages and that way a system of *t* equations with *t* unknown node voltages is obtained. These equations, the so called nodal equations, can be formulated straightforward from a circuit diagram. A general branch, connected between nodes *i* and *j*, is presented in Fig. 2.3.2.

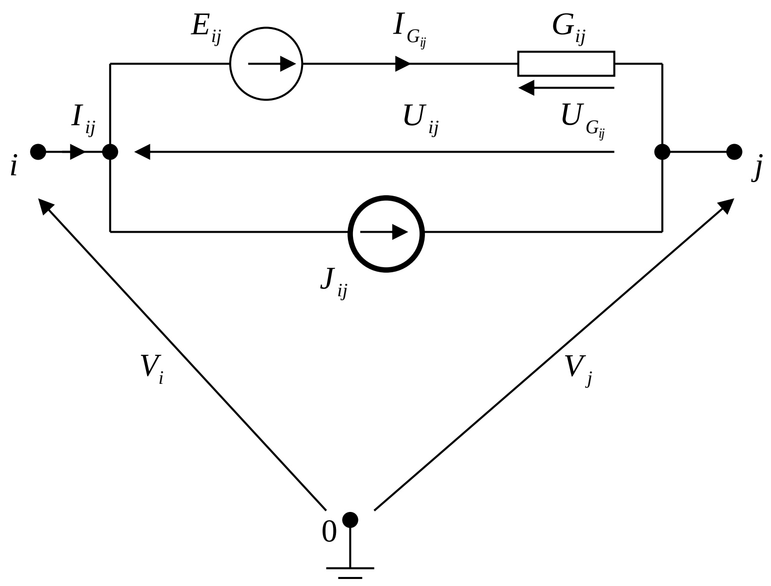

Fig. 2.3.2 General branch

From the branch KVL, the resistor voltage drop is

$$U_{G_{ij}} = U_{ij} + E_{ij} = V_i - V_j + E_{ij} \qquad (2.3.4)$$

Then, taking into account the branch KCL and the resistor Ohm's law, the branch current can be calculated,

$$I_{ij} = J_{ij} + I_{G_{ij}} = J_{ij} + (V_i - V_j + E_{ij})G_{ij} = V_i G_{ij} - V_j G_{ij} + I_{s\,ij}\,, \tag{2.3.5}$$

where,

$$I_{s\,ij} = J_{ij} + E_{ij} G_{ij} \tag{2.3.6}$$

is the total source current of the branch.

Next, the *i*-th node KCL can be formulated,

$$\sum_{\substack{j=0 \\ j\neq i}}^{t} I_{s\,ij} + \sum_{\substack{j=0 \\ j\neq i}}^{t} V_i G_{ij} - \sum_{\substack{j=0 \\ j\neq i}}^{t} V_j G_{ij} = 0 \tag{2.3.7}$$

This equation can be formulated for all nodes, $i = 1,...,t = n-1$, except the reference one, $i = 0$. Then, the final version of **nodal equations** can be written,

$$V_i G_{ii} - \sum_{\substack{j=1 \\ j\neq i}}^{t} V_j G_{ij} = I_{si} \tag{2.3.8}$$

where,

$$G_{ii} = \sum_{\substack{j=0 \\ j\neq i}}^{t} G_{ij} \tag{2.3.9a}$$

is the total conductance of the *i*-th node, sum of conductances of all branches incident with the *i*-th node,

$$G_{ij} = G_{ji} \tag{2.3.9b}$$

is the total conductance of branch(es) connected between nodes *i* and *j*,

$$I_{si} = -\sum_{\substack{j=0 \\ j\neq i}}^{t} I_{s\,ij} \tag{2.3.10}$$

is the total source current of the *i*-th node, sum of source currents of all branches incident with the *i*-th node. These currents are

$$I_{s\,ij} = \pm J_{ij} \pm E_{ij} G_{ij} \tag{2.3.11}$$

+ , if arrow of the *ij*-th branch source is directed to the *i*-th node,
– , if arrow of the *ij*-th branch source is directed from the *i*-th node.

Nodal equations in the matrix form are:

$$\mathbf{G}\,\mathbf{V} = \mathbf{I}_s \tag{2.3.12}$$

where,

$$\mathbf{G} = \begin{bmatrix} +G_{11} & \cdots & -G_{1i} & \cdots & -G_{1t} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ -G_{i1} & \cdots & +G_{ii} & \cdots & -G_{it} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ -G_{t1} & \cdots & -G_{ti} & \cdots & +G_{tt} \end{bmatrix} \tag{2.3.13}$$

is a circuit **conductance matrix**, and

$$\mathbf{V} = \begin{bmatrix} V_1 \\ \vdots \\ V_i \\ \vdots \\ V_t \end{bmatrix}, \quad \mathbf{I}_s = \begin{bmatrix} I_{s,1} \\ \vdots \\ I_{s,i} \\ \vdots \\ I_{s,t} \end{bmatrix} \tag{2.3.14}$$

are vectors of node voltages and node source currents.

Note:

If a branch resistance is zero (conductance is infinity), i.e. if ideal voltage source or ideal ammeter is the branch only element, then branch current can not be expressed by node voltages. This special case will be discussed further on.

Algorithm 2.3.3 – Nodal Analysis

1. Assume unknown currents in each branch, $I_1,...,I_b$. Indicate their directions by arrows (direction is arbitrary).
2. For each element indicate, by an arrow, its voltage drop (passive elements) or rise (active elements) that a particular current causes in passing the element.
3. Select the reference node (selection of the node is arbitrary). Write down nodal equations (2.3.8) for all other $t = n - 1$ nodes.
4. Solve the set of *t* equations, with *t* node voltages as unknown variables.
5. Find a circuit response(s), if not found already.

Example 2.3.1a

Diagram of an exemplary circuit, built of $b = 6$ branches connected in $n = 4$ nodes, is presented in Fig. 2.3.3. As can be observed, it is the circuit of Example 2.3.1 with resistor R_1 added, such that the circuit does not contain resistiveless branches. Designation of all branch currents is the task.

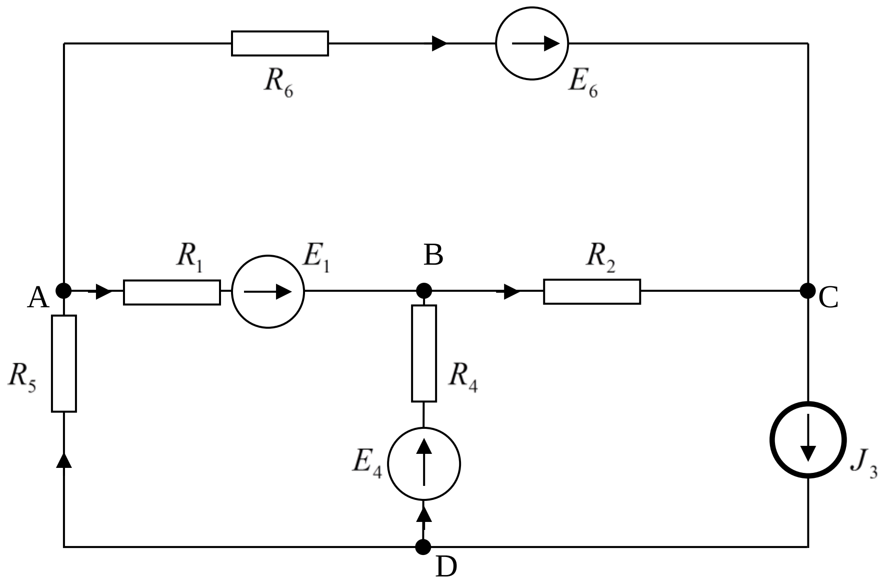

Fig. 2.3.3 Diagram of exemplary circuit (Example 2.3.1a)

Node D is selected as the reference one. Then, nodal equations are:

$$
\begin{array}{lllll}
\text{A:} & +V_{\mathrm{A}}(G_1+G_5+G_6) & -V_{\mathrm{B}}G_1 & -V_{\mathrm{C}}G_6 & =-E_1G_1-E_6G_6 \\
\text{B:} & -V_{\mathrm{A}}G_1 & +V_{\mathrm{B}}(G_1+G_2+G_4) & -V_{\mathrm{C}}G_2 & =+E_1G_1+E_4G_4 \\
\text{C:} & -V_{\mathrm{A}}G_6 & -V_{\mathrm{B}}G_2 & +V_{\mathrm{C}}(G_2+G_6) & =-J_3+E_6G_6
\end{array}
\tag{2.3.15}
$$

After solving this system, node voltages $V_{\mathrm{A}}, V_{\mathrm{B}}, V_{\mathrm{C}}$ are designated. Then, branch currents I_{ij} can be designated from (2.3.5), *i* and *j* are A, B or C.

$$
\begin{aligned}
I_1 &= I_{\mathrm{AB}} = (V_{\mathrm{A}} - V_{\mathrm{B}} + E_1)G_1\,, \\
I_2 &= I_{\mathrm{BC}} = (V_{\mathrm{B}} - V_{\mathrm{C}})G_2\,, \\
I_3 &= I_{\mathrm{CD}} = J_3\,, \\
I_4 &= I_{\mathrm{DB}} = (-V_{\mathrm{B}} + E_4)G_4\,, \\
I_5 &= I_{\mathrm{DA}} = -V_A G_5\,, \\
I_6 &= I_{\mathrm{AC}} = (V_{\mathrm{A}} - V_{\mathrm{C}} + E_6)G_6\,.
\end{aligned}
\tag{2.3.16}
$$

♦

Special (resistiveless branch) case

In case of resistiveless branch, $R_{ij} = 0 \equiv G_{ij} = \infty$, its voltage is known ($U_{ij} = E_{ij}$ or 0), however the current I_{ij} can not be expressed by node voltages V_i, V_j and Algorithm 2.3.3 have to be modified. Two modifications solve the problem.

Modification 1 of Algorithm 2.3.3

Do not consider the resistiveless branch at the left side of nodal equations (2.3.8). Set,

$$I_{ij} = I_{s,ij}$$

and add this current to source currents of the *i*-th node. That way, number of unknowns has been increased by one variable, the resistiveless branch current.

Supplement the system of nodal equations (2.3.8) with one trivial equation:

$$V_i = V_j \pm E_{ij}$$

Modification 2 of Algorithm 2.3.3

Select node *j* as the reference one. Then,

$$V_i = \pm E_{ij} \, .$$

Now, the *i*-th node voltage is not unknown and the *i*-th nodal equation can be disregarded. In case more resistiveless branches exist, the modification can be applied only if they all have one common node.

Example 2.3.1 – cont.

Branch 1, connected between nodes A and B, is the resistiveless branch (ideal voltage source).

Modification 1:

In nodal equations (2.3.15) conductance G_1 should be deleted and current I_1 should be algebraically added to source currents of two nodes (A and B) of the branch. The following system of nodal equations is obtained:

$$\begin{array}{llllll} \text{A:} & +V_\mathrm{A}(G_5+G_6) & & -V_\mathrm{C}G_6 & =-I_1-E_6G_6 \\ \text{B:} & & +V_\mathrm{B}(G_2+G_4) & -V_\mathrm{C}G_2 & =+I_1+E_4G_4 \\ \text{C:} & -V_\mathrm{A}G_6 & -V_\mathrm{B}G_2 & +V_\mathrm{C}(G_2+G_6) & =-J_3+E_6G_6 \end{array} \tag{2.3.16a}$$

This system is supplemented by the trivial equation:

$$V_\mathrm{B} = V_\mathrm{A} + E_1 \tag{2.3.16b}$$

Modification 2:

Node A is selected as the reference one. Then, $V_\mathrm{B} = E_1$ and a brand new system of two nodal equations with two unknowns is formulated:

$$\begin{array}{lllll} \text{C:} & -E_1G_2 & +V_\mathrm{C}(G_2+G_6) & & =-J_3+E_6G_6 \\ \text{D:} & -E_1G_4 & & +V_\mathrm{D}(G_4+G_5) & =+J_3-E_4G_4 \end{array} \tag{2.3.17}$$

♦

Drill Problems 2.3

1. Find the formula for U.

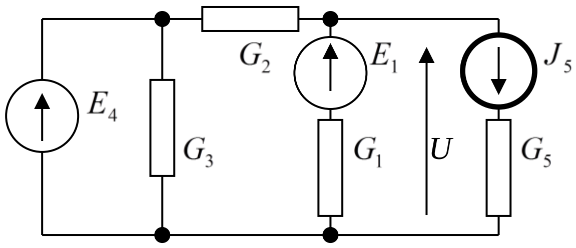

Fig. P.2.3.1

2. Find the formula for U.

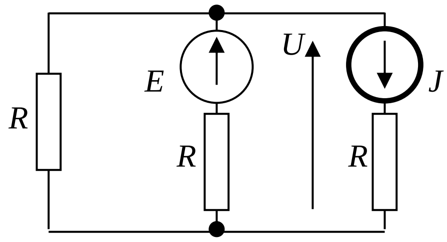

Fig. P.2.3.2

3. Find the value of R so that $I = 1$ A

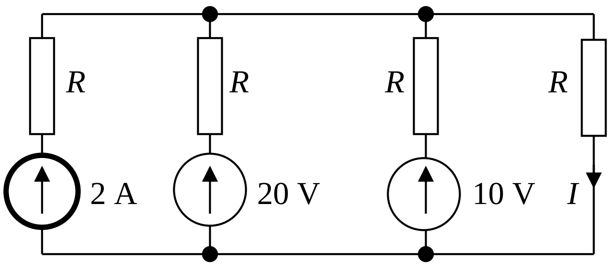

Fig. P.2.3.3

4. Draw a planar graph of a circuit shown. How many independent KVL equations can be formulated ?

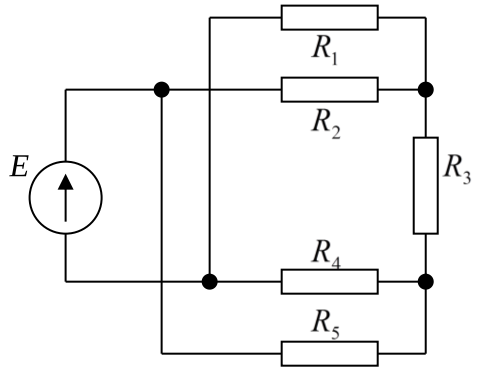

Fig. P.2.3.4

5. Find U in the circuit shown, using nodal analysis. Assume C to be the reference node.

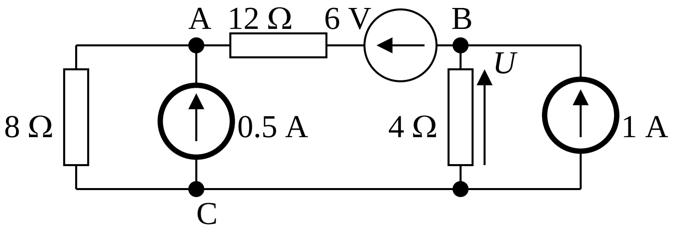

Fig. P.2.3.5

6. Find U in the circuit shown.

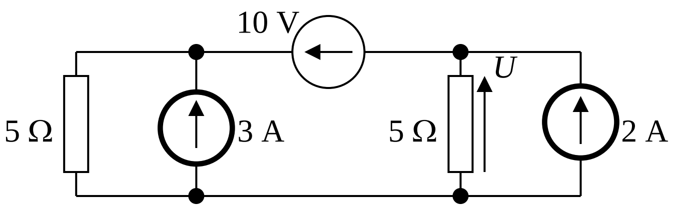

Fig. P.2.3.6

7. Find the voltage between node A and ground. All resistances are in ohms.

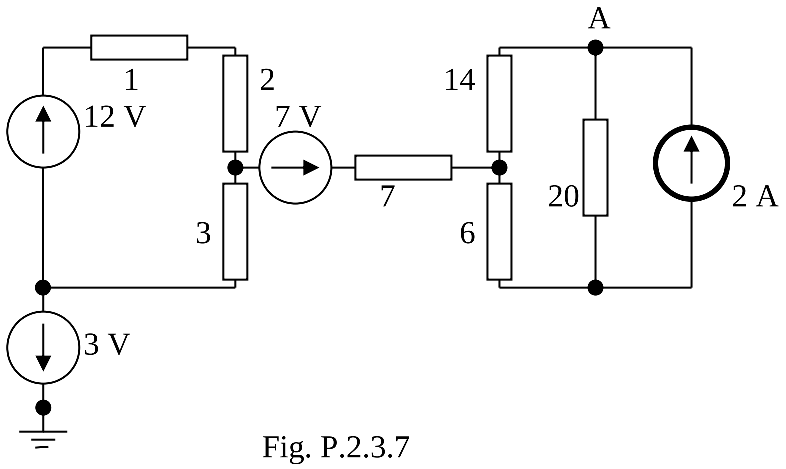

Fig. P.2.3.7

2.4 ENERGY and POWER CONSERVATION PRINCIPLE

The law of energy conservation must be obeyed in any electric circuit. For this reason,

the algebraic sum of power in a circuit, at any instant of time, must be zero:

$$\sum p = 0 \tag{2.4.1}$$

for DC case

$$\sum P = 0 \tag{2.4.1a}$$

where $\sum$ means algebraic addition over all elements,

+ , if power is supplied by an element,

– , if power is absorbed by an element.

In other words, the total power supplied to the circuit must balance the total power absorbed.

Power balance can be used to check correctness of results of the performed circuit analysis.

<u>Example 2.4.1</u>

For the circuit of Fig. 2.4.1 and the following values of parameters:

$J = 2\,\mathrm{A},\ R = 5\,\Omega,\ E = 25\,\mathrm{V}$

find all powers and check the power balance.

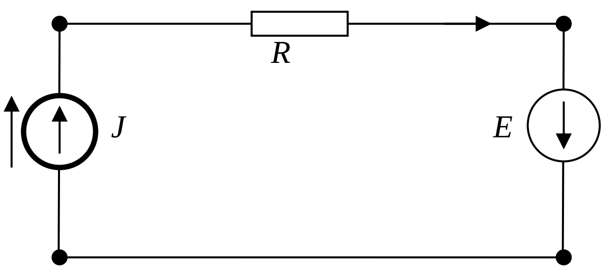

Fig. 2.4.1 Circuit for Example 2.4.1

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From KVL and Ohm's law:

$$U_J = U_R - E = JR - E = -15\,\mathrm{V} \tag{2.4.2}$$

Then, taking into account the mutual position of arrowheads of the mesh current $I = J$ and element voltages:

$P_J = JU_J = -30\,\mathrm{W}$, power absorbed,

$P_R = J^2 R = +20\,\mathrm{W}$, power absorbed,

$P_E = JE = +50\,\mathrm{W}$, power supplied.

The total power absorbed, $P_{\text{abs}} = -P_J + P_R = 50\,\text{W}$, equals the power supplied (delivered) by the voltage source.

♦

Drill problems 2.4

1. Find all powers and check the power balance.

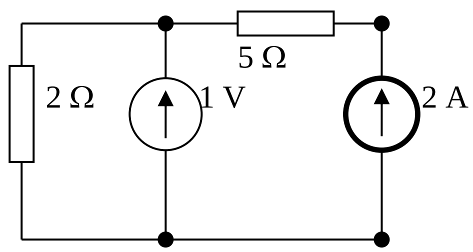

Fig. P.2.4.1

2. Calculate power dissipated on the most loaded resistor. Formulate the power balance.

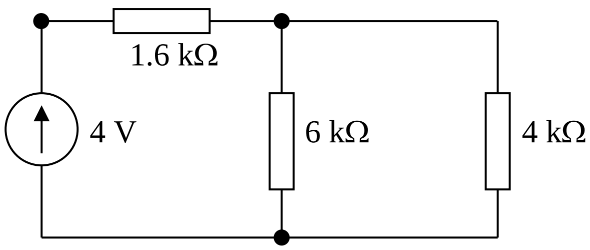

Fig. P.2.4.2

3. An ideal current source J and an ideal voltage source E are connected back to back ("+" with "+"). If $J = 2$ mA, what would E be so that 72 J/h was being supplied by the current source to the voltage source ?

4. Find the emf voltage E_x so that the current $I = 1\,\text{A}$. Other parameters are: $R = 5\,\Omega,\ E = 10\,\text{V},\ J = 1\,\text{A}$. Check the power balance.

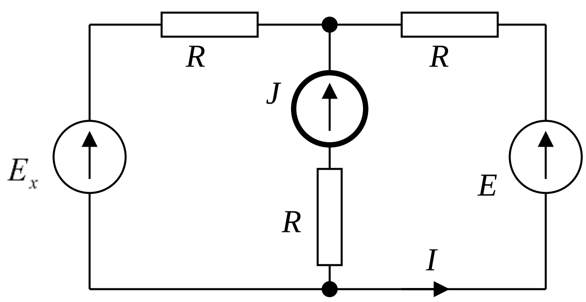

Fig. P.2.4.4

5. Calculate all dissipated and absorbed powers in Problem 2.3.2 circuit. Check the power balance, if $R = 5\,\Omega,\ E = 10\,\text{V},\ J = 1\,\text{A}$.

6. Calculate all dissipated and absorbed powers in Problem 2.3.3 circuit. Check the power balance.
7. Calculate all dissipated and absorbed powers in circuits of Problems 2.3.5 and 2.3.6, check the power balance.
8. If in Problem 2.4.2 circuit the maximum rated power of all resistors is 1 W, what is the maximum acceptable emf value ?

2.5 TWO-TERMINAL SUBCIRCUIT, THEVENIN's/NORTON's THEOREM

PASSIVE TWO-TERMINAL SUBCIRCUIT

A passive two-terminal subcircuit, a subcircuit built of resistors, connected with the rest of a circuit in two terminal nodes is presented in Fig. 2.5.1.

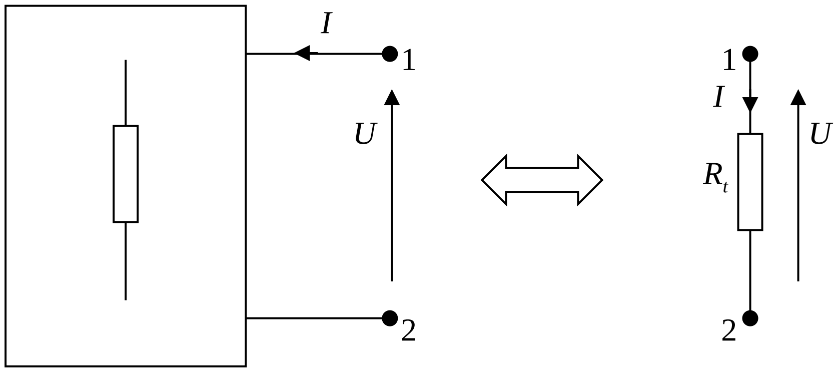

Fig. 2.5.1 Two-terminal subcircuit and its total (equivalent) resistance

It can be easily proved, that

> for any linear passive two-terminal subcircuit its **equivalent** or **total resistance** R_t can be found. The subcircuit is characterized by the Ohm's law:
>
> $$U = R_t I,\ I = G_t U \tag{2.5.1}$$

Series and parallel connection of resistors are the special cases.

<u>Series connection of resistors, Voltage divider</u>

Consider arrangement of resistors so that the same current passes through each resistor, the so called series connection, as depicted in Fig. 2.5.2.

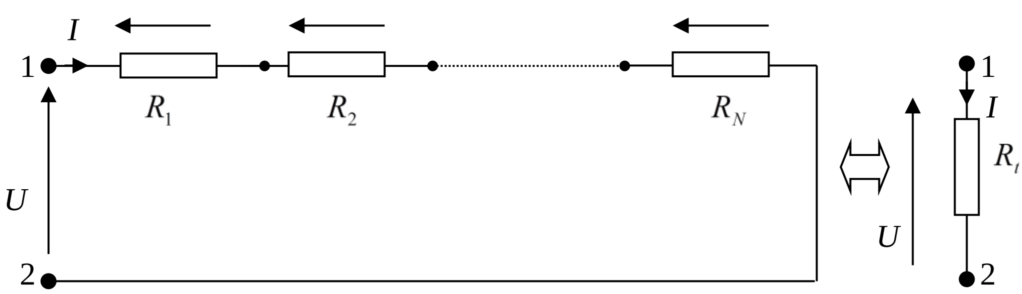

Fig. 2.5.2 Series connection of resistors

From KVL and Ohm's law,

$$U = \sum_{i=1}^{N} U_i = I \sum_{i=1}^{N} R_i = IR_t \qquad (2.5.2)$$

Then, the total resistance is

$$R_t = \sum_{i=1}^{N} R_i \; . \qquad (2.5.3)$$

Circuit of series resistors divides the input voltage by the ratio of the resistance R_i to the total resistance,

$$U_i = U \frac{R_i}{R_t} \qquad (2.5.4)$$

<u>Two-resistor voltage divider</u>

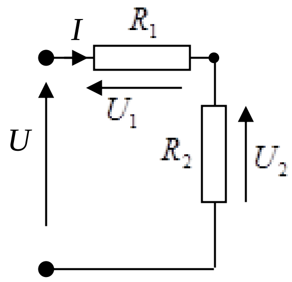

Fig. 2.5.2a Series connection of two resistors

The total resistance and resistor voltages are as follows:

$$R_t = R_1 + R_2 \qquad (2.5.5)$$

$$U_1 = U \frac{R_1}{R_1 + R_2} \qquad (2.5.6a)$$

$$U_2 = U \frac{R_2}{R_1 + R_2} \qquad (2.5.6b)$$

<u>Parallel connection of resistors, Current divider</u>

Consider arrangement of resistors so that each resistor has the same voltage, the so called parallel connection, as depicted in Fig. 2.5.3.

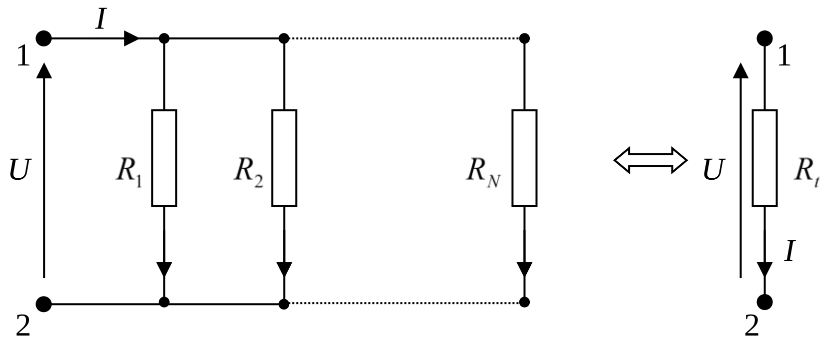

Fig. 2.5.3 Parallel connection of resistors

From KCL and Ohm's law,

$$I = \sum_{i=1}^{N} I_i = U \sum_{i=1}^{N} G_i = UG_t \tag{2.5.7}$$

Then, the total conductance is

$$G_t = \sum_{i=1}^{N} G_i \; . \tag{2.5.8}$$

Circuit of parallel resistors divides the input current by the ratio of the conductance G_i to the total conductance,

$$I_i = I \frac{G_i}{G_t} \tag{2.5.9}$$

<u>Two-resistor current divider</u>

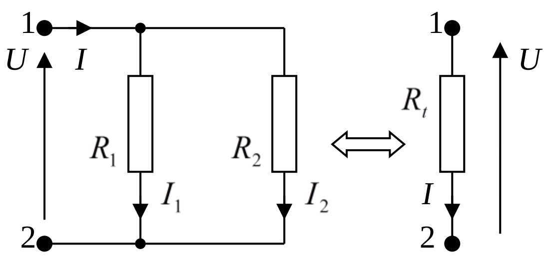

Fig. 2.5.3a Parallel connection of two resistors

The total resistance and resistor currents are as follows:

$$R_t = \frac{R_1 R_2}{R_1 + R_2} \tag{2.5.10}$$

$$I_1 = I \frac{R_2}{R_1 + R_2} \tag{2.5.11a}$$

$$I_2 = I \frac{R_1}{R_1 + R_2} \tag{2.5.11b}$$

For the given resistive two-terminal subcircuit the total resistance can be: a) calculated or b) measured.

a) Subcircuit diagram should be given and the total resistance can be normally found by parallel and series connections of resistors. In some cases, wye-delta or delta-wye conversion of three resistors is necessary. This exceptional cases are not discussed.

b) Subcircuit diagram may not be given and the total resistance can be measured by means of the external source. Fig. 2.5.4 presents two possible measurement circuits.

In case of ideal meters, their readings are the subcircuit current and voltage. Then,

$$R_t = U / I = U_\mathrm{V} / I_\mathrm{A} \qquad (2.5.12)$$

In case of practical meters, their resistances R_A or R_V have to be taken into account, and then, only current or only voltage is measured correctly. Thus, respectively:

$$U = U_\mathrm{V} - I_\mathrm{A} R_\mathrm{A}\,,\; I = I_\mathrm{A} \qquad (2.5.13a)$$

or

$$I = I_\mathrm{A} - U_\mathrm{V} / R_\mathrm{V}\,,\; U = U_\mathrm{V} \qquad (2.5.13b)$$

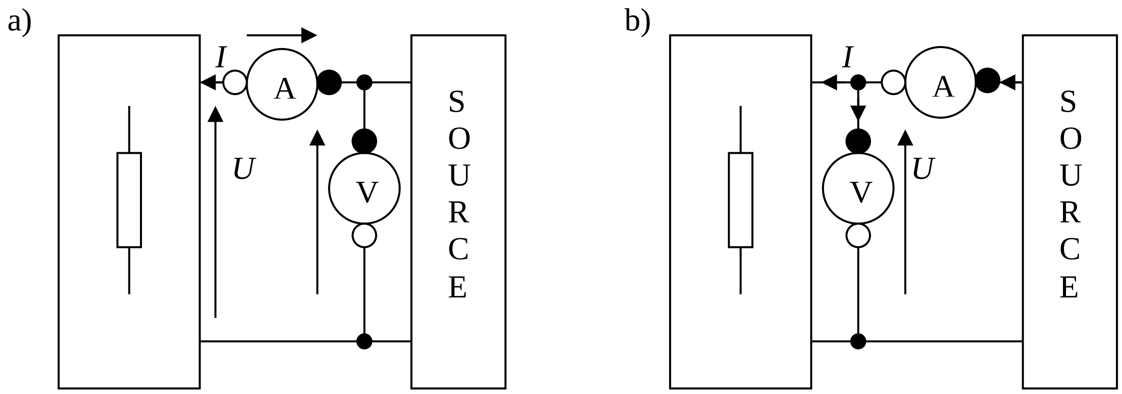

Fig. 2.5.4 Total resistance measurement circuits: a) correct current, b) voltage measurement

Example 2.5.1

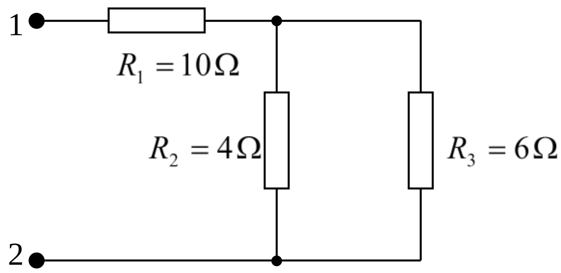

Fig. 2.5.5 Passive two-terminal subcircuit (Example 2.5.1)

Find the equivalent resistance of the subcircuit presented in Fig. 2.5.5.

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At first, the total resistance of parallel resistors is found. Then, this resistance connected in series with (added to) R_1 gives the following total resistance of the whole subcircuit:

$$R_t = R_1 + \frac{R_2 R_3}{R_2 + R_3} = 10 + 2.4 = 12.4\,\Omega \tag{2.5.14}$$

♦

ACTIVE TWO-TERMINAL SUBCIRCUIT

An active linear two-terminal subcircuit, a practical linear source or subcircuit built of resistors and linear source(s) is presented in Fig. 2.5.6. It can be proved that such subcircuit is characterized by the $I-U$ relationship depicted in Fig. 2.5.7. This relationship can by described by any of the following two equations:

$$U = E_o - R_t I \tag{2.5.15}$$

$$I = J_s - G_t U \tag{2.5.16}$$

where

$$G_t = \frac{1}{R_t} = \frac{\Delta I}{\Delta U} = \frac{J_s}{E_o}$$

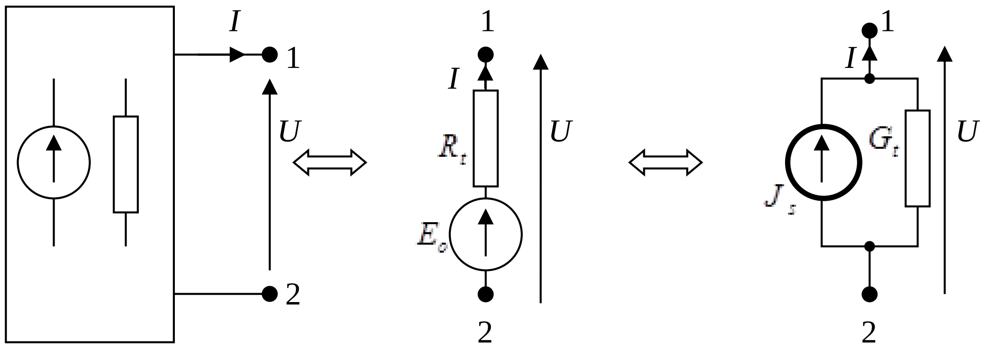

Fig. 2.5.6 Active two-terminal subcircuit and its equivalent diagrams

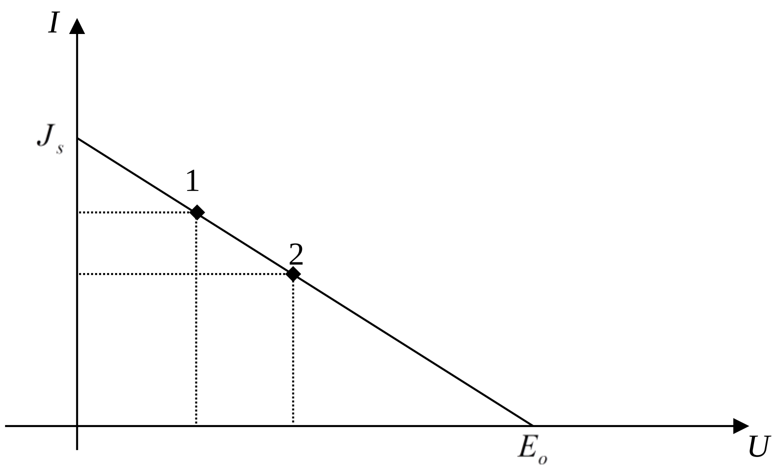

Fig. 2.5.7 $I-U$ relationship of active linear two-terminal circuit

These equations are KVL and KCL equations respectively. Then, the corresponding equivalent circuits can be built, as presented in Fig. 2.5.6. Two theorems can be formulated.

Thevenin's theorem

Any active linear two-terminal subcircuit can be replaced by the equivalent circuit that consists of a series connection of an ideal voltage source E_o and a resistance R_t, where:

E_o is the subcircuit open-circuit voltage, $E_o = U|_{I=0}$, (2.5.18a)

R_t is the subcircuit equivalent (internal) resistance.

Norton's theorem

Any active linear two-terminal subcircuit can be replaced by the equivalent circuit that consists of a parallel connection of an ideal current source J_s and a conductance G_t, where:

J_s is the subcircuit short-circuit current, $J_s = I|_{U=0}$ (2.5.18b)

G_t is the subcircuit equivalent (internal) conductance.

Applications of Thevenin's/Norton's theorem:

1. Linear dc circuit analysis: replacement of a complex two-terminal subcircuit by Thevenin equivalent or Norton equivalent circuit, what simplifies calculations.
2. Designation of the maximum power transfer condition.
3. Nonlinear dc circuit analysis: replacement of a linear part by the Thevenin or Norton equivalent.
4. Transient analysis of the 1st order circuit: replacement of a resistive part by the Thevenin or Norton equivalent.

Parameters of Thevenin and Norton equivalent circuit can be: a) calculated or b) measured.

a) It is assumed that subcircuit diagram is given.
   - To find the open-circuit voltage, subcircuit analysis is performed.
   - To find the equivalent resistance, subcircuit source(s) is(are) deactivated at first. Deactivation (zeroing) of a voltage source means shorting of its terminals, deactivation of a current source means opening of its terminals. Then, the equivalent resistance can be found, the same way as for a passive subcircuit,

     $$R_t = -(U/I)_{|E_o=0} \qquad (2.5.19)$$

b) A circuit is loaded by two different resistances and its current and voltage are measured, as presented in Fig. 2.4.8 (correct current measurement is applied).

   For two different values of the load resistance, R_l^1 and R_l^2, the current and the voltage are measured. That way, coordinates of two points on $I-U$ line are given, as presented in Fig. 2.5.7. These coordinates are set into equation (2.5.15) or (2.5.16), to form a system of two equations. Then, E_o and R_t or J_s and G_t are designated, respectively.

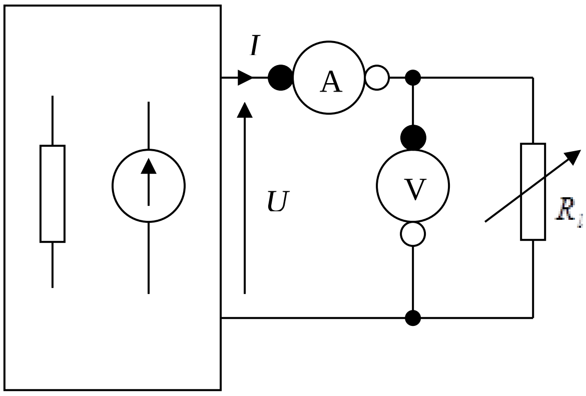

Fig. 2.5.8 Measurement circuit for two-terminal element

In the special case: R_l^1 can be an open-circuit and R_l^2 can be a short-circuit.

For a practical ammeter, its internal resistance R_A has to be taken into account. Then, the subcircuit voltage U has to be corrected, the same way as for a passive subcircuit, following equation (2.5.13a).

Example 2.5.2

The subcircuit built of two practical sources and one resistor is presented in Fig. 2.5.9 - $R_1 = 3\Omega, R_2 = 4\Omega, R_3 = 7\Omega, E_1 = 5\mathrm{V}, J_3 = 2\mathrm{A}$ Find: a) Thevenin equivalent, b) Norton equivalent.

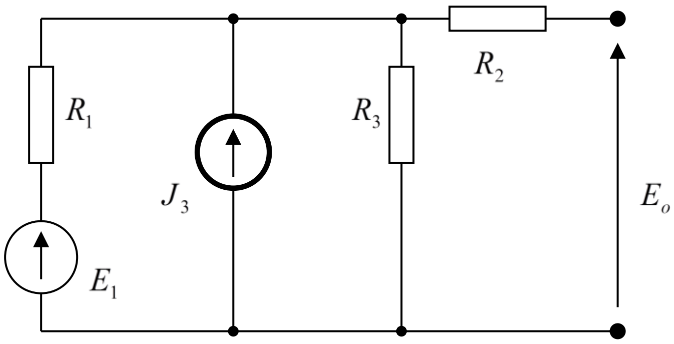

Fig. 2.5.9 Example 2.5.2 subcircuit

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a)

To find E_o, first, the Norton equivalent (J_3, R_3) can be converted into the Thevenin equivalent (E_3, R_3), following equations (2.5.20).

$$E_o = J_s R_t \tag{2.5.20a}$$

$$R_t = 1/G_t \tag{2.5.20b}$$

Then, the source $E_3 = 14\,\mathrm{V}$, $R_3 = 7\,\Omega$ replaces the source J_3, R_3, as presented in Fig. 2.5.9a.

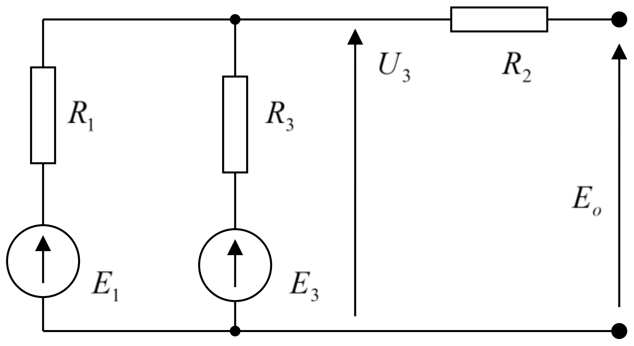

Fig. 2.5.9a Example 2.5.2 subcircuit after Norton-Thevenin transformation

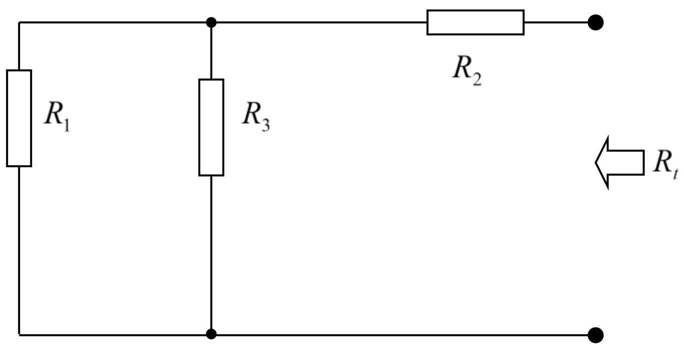

Fig. 2.5.9b Example 2.5.2 subcircuit with sources zeroed

Finally, the subcircuit open-circuit voltage is:

$$E_o = U_3 = E_3 + \frac{E_1 - E_3}{R_1 + R_3} R_3 = 7.7\,\mathrm{V} \tag{2.5.21a}$$

To find R_t, both sources have to be zeroed, the passive subcircuit presented in Fig. 2.5.9b is obtained. Then, the equivalent resistance is

$$R_t = R_2 + \frac{R_1 R_3}{R_1 + R_3} = 6.1\,\Omega. \tag{2.5.21b}$$

b)
For the calculated parameters of Thevenin equivalent circuit (2.5.21), parameters of Norton equivalent circuit can be calculated from the following equations:

$$J_s = E_o / R_t. \tag{2.5.22a}$$

$$G_t = 1 / R_t, \tag{2.5.22b}$$

Then, $J_s = 1.26\,\mathrm{A}$, $G_t = 0.16\,\mathrm{S}$.

♦

Practical sources

An $I-U$ relationships of practical sources, a voltage source and a current source, are shown in Fig. 2.5.10.

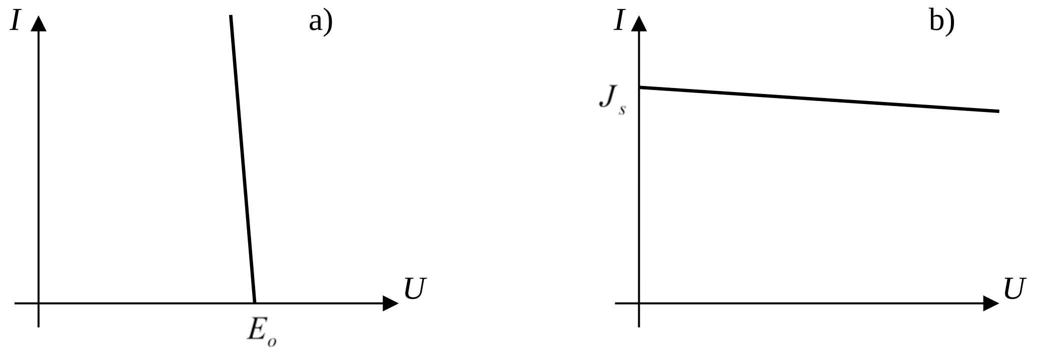

Fig. 2.5.10 *I*–*U* relationship of: a) practical voltage source, b) practical current source

Then, any of these sources can be described by any of equations (2.5.15) or (2.5.16) and therefore, can modeled by the Thevenin or Norton equivalent circuit. A source-load single loop circuits are presented in Fig. 2.5.11.

Now, the source voltage or current are not fixed at the values of $U = E_o$ or $I = J_s$, as for an ideal voltage or current source, respectively. For the given practical source, they are functions of the load resistance,

$$U = E_o \frac{R_l}{R_l + R_t} \tag{2.5.23a}$$

$$I = J_s \frac{R_t}{R_l + R_t} \tag{2.5.23b}$$

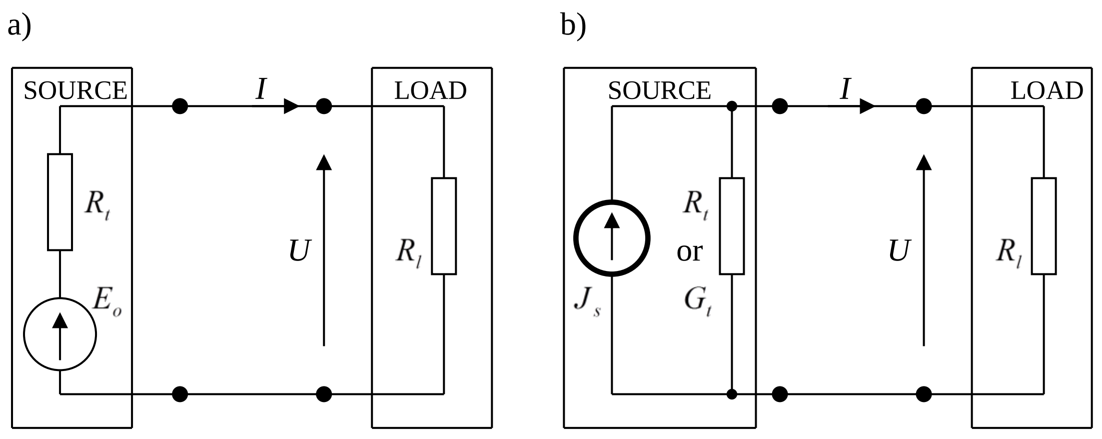

Fig. 2.5.11 Source-load single loop circuits

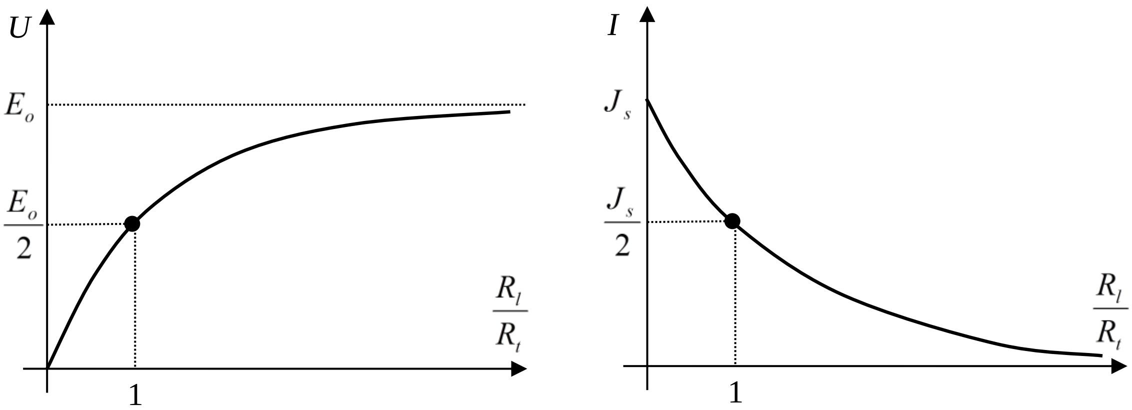

Fig. 2.5.12 Graphs of a practical voltage source voltage and current source current versus a load-to-source resistance ratio

Then, for a practical voltage source, to maintain the supply voltage at the fixed level of $U \cong E_o$, the load resistance has to be much greater than the source internal resistance,

$$R_l >> R_t, \tag{2.5.24a}$$

for a practical current source, to maintain the supply current at the fixed level of $I \cong J_s$, the load resistance has to be much less than the source internal resistance,

$$R_l << R_t. \tag{2.5.24b}$$

<u>Example 2.5.3</u>

Given an $E_o = 9\,\mathrm{V}$ battery, its internal resistance is . $R_t = 3\Omega$ Draw the $I - U$ relationship and the Thevenin and Norton equivalent circuits.

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The battery $I - U$ relationship and equivalent circuits are presented in Fig. 2.5.13.

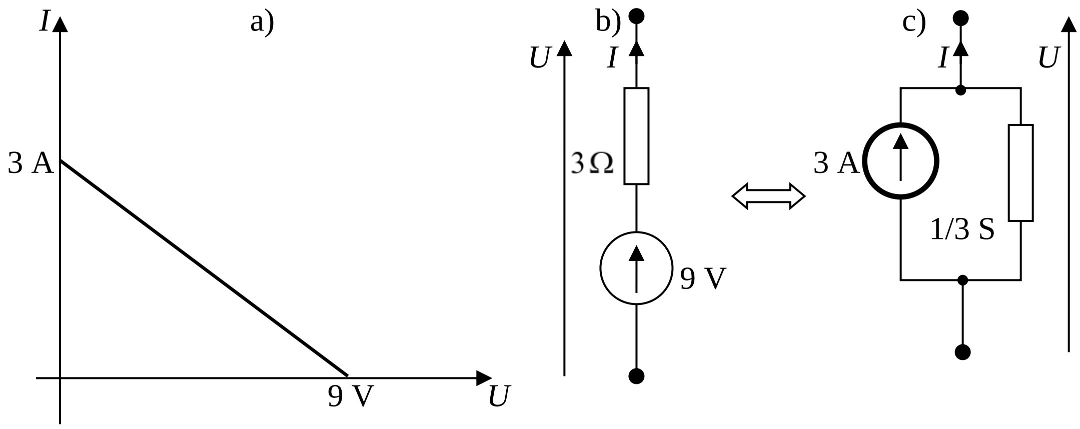

Fig. 2.5.13 Battery (Example 2.5.3): a) *I*−*U* relationship, b) Thevenin, c) Norton equivalent

♦

Example 2.5.3 cont.

Convert voltage source of Example 2.5.3 into 1 mA current source.

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The conversion can be done by series connection of a resistance R_s, its resistance being much greater than the load resistance R_l. The internal resistance of the modified source is

$$R_t^* = R_t + R_s \cong R_s, \qquad (2.5.25)$$

and its short-circuit current is

$$J_s^* = E_o / R_t^* \cong E_o / R_s \qquad (2.5.26)$$

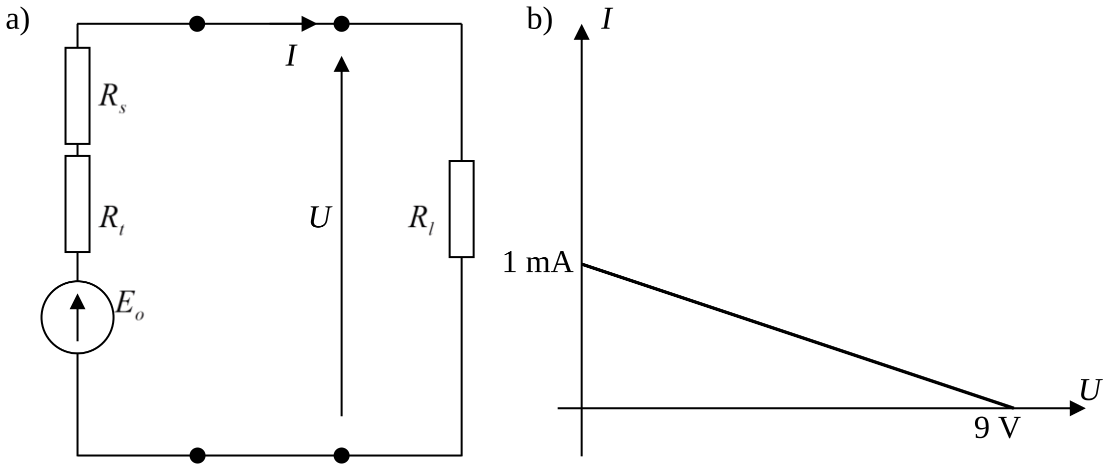

Fig. 2.5.14 Modified source (Example 2.4.3): a) source-load single loop circuit, b) $I-U$ relationship.

Then, for $J_s^* = 1\text{mA}$ and $E_o = 9\text{V}$ the required series resistance is $R_s = 9\,\text{k}\Omega$. The source-load single loop circuit is presented in Fig. 2.5.14a, the modified source $I-U$ relationship is presented in Fig. 2.5.14b.

♦

Drill problems 2.5

1. Two 1 W resistors: $R_1 = 100\,\Omega$, $R_2 = 50\,\Omega$ are connected in series. What maximum voltage can by safely supplied to such combination?
2. Two 1 W resistors: $R_1 = 100\,\Omega$, $R_2 = 50\,\Omega$ are connected in parallel. What maximum current can by safely supplied to such combination?
3. Determine resistance of a resistor that must be placed in series with R=100 Ω resistor supplied by 120 V, in order to limit its power dissipation to 90 W.
4. Two heaters (resistors) are each rated 1 kW and 220 V. What is the total dissipated power when they are connected in series across 220 V?

5. An electric meter of $R = 20\ \Omega$ resistance produces a maximum needle deflection with 10 mA flowing through its terminals. What resistance must be connected in series with the meter so that the maximum needle deflection occurs when series combination is connected to 150 V?

6. What resistance must be connected in parallel with the meter of the preceding problem so that the maximum needle deflection occurs when 100 mA current flows into the combination?

7. Two resistors, $R_1 = 9.2\,\mathrm{k\Omega}$ rated 1W and $R_2 = 5.1\,\mathrm{k\Omega}$ rated 0.5 W are connected in series. What maximum current can safely flow in the combination? What maximum voltage can by safely supplied?

8. Resistors of Fig. 2.5.5 circuit are rated 1 W. What maximum voltage can by safely supplied to such circuit?

9. Voltages and currents measured at terminals of a linear source at two different loads are: (2V, 6A); (6V, 2A). Find the current drawn by the $R = 6\,\Omega$ load.

10. Current entering the positive terminal of $E = 10\,\mathrm{V}$ battery (ideal source) raises linearly from 3 to 9 mA between *t*=0 and *t*=15 minutes. How much energy, in joules, is supplied to the battery during the entire period of time?

11. Given a 1.5 V AA battery (ideal source) with a nominal life of 150 J. For how many days will it power a calculator that draws 1 mA current?

12. A 12 V supply is used to charge 6 V battery of 0.8 Ω internal resistance. What series resistance is necessary to limit the charging current to 600 mA?

13. Two practical sources characterized by the following parameters: $E_o = 10\,\mathrm{V}, R_t = 2\,\Omega$ and $J_s = 5\,\mathrm{A}, G_t = 1/3\,\mathrm{S}$ are connected in parallel, "+" with "+". Find the open-circuit voltage and the short-circuit current of the obtained active circuit.

14. For the given $U - I$ relationship of a practical source that satisfies passive sign convention find Norton and Thevenin equivalents.

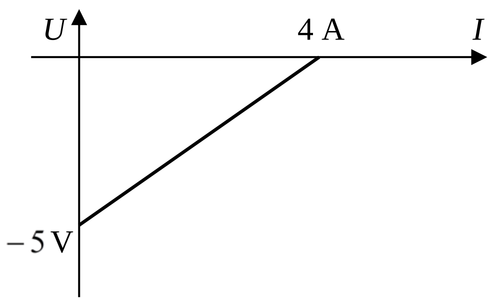

Fig. P.2.5.14

15. A source with open-circuit voltage of 50 V and short-circuit current of 25 A is connected to 2 Ω load. What resistance should be connected in series to limit the power absorbed by the load to 50 W ?

16. A 6 V battery has an internal resistance of 0.1 Ω. Find the load resistance which would reduce its terminal voltage to 5 V.

17. A linear circuit that satisfies passive sign convention is found experimentally to have the $I-U$ relationship shown. Find its Norton and Thevenin equivalents.

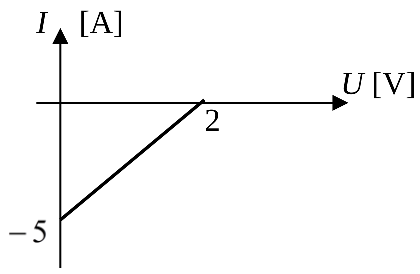

Fig. P.2.5.17

18. Find Thevenin and Norton equivalents.

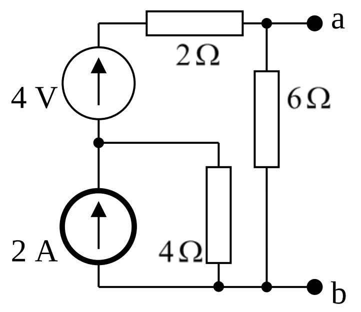

Fig. P.2.5.18

19. Find the Thevenin equivalents of the circuits shown.

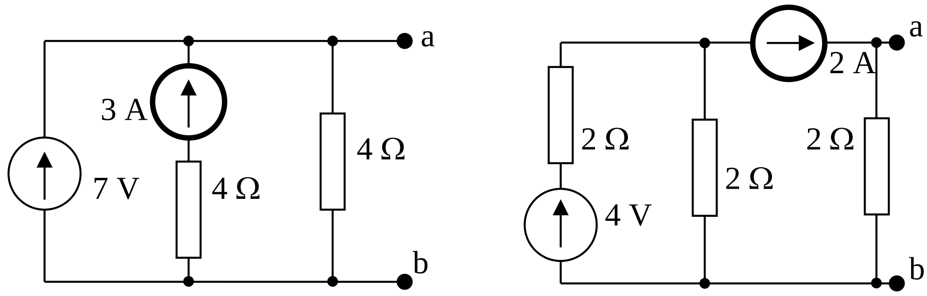

Fig. P.2.5.19

20. An alternator with o.c. (open-circuit) voltage of 20 V and s.c. current of 10 A dc is to be used to charge car battery with o.c. voltage of 12 V and internal resistance ranging from 1 to 5 Ω. What resistance should be connected in series to limit the charging current to 2 A ?

21. Suppose that voltage *u* of a car battery varies linearly from 14 to 12V as *t* varies from 0 to 10 min, and the constant current $I = 0.5$ A is entering the positive terminal. Find, a) the total energy supplied, b) the total charge delivered to the battery.

22. If a current $I = 0.5$ A is entering the positive terminal of $E_o = 12$ V, $R_t = 2\Omega$ battery, find the energy supplied to the battery in 2 h.

23. Find the voltage needed to charge the battery of Problem 2.5.25 with current of 0.5 A.

24. Find the equivalent resistance R_t if terminals a-b are: a) opened, b) shorted.

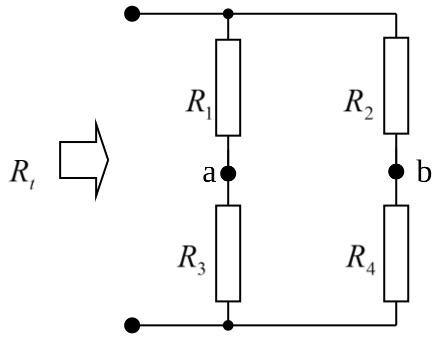

Fig. P.2.5.24

25. Use a series of Norton-Thevenin and series/parallel transformations to reduce a circuit shown into the single-loop circuit. Then, find current *I*.

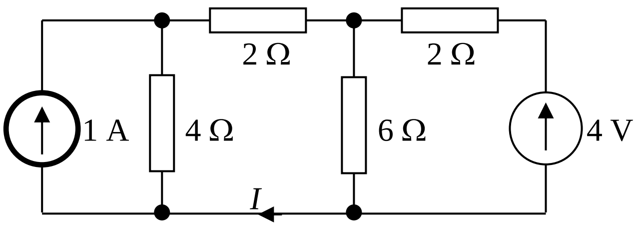

Fig. P.2.5.25

26. Find the value of *R* for which the two circuits shown are equivalent.

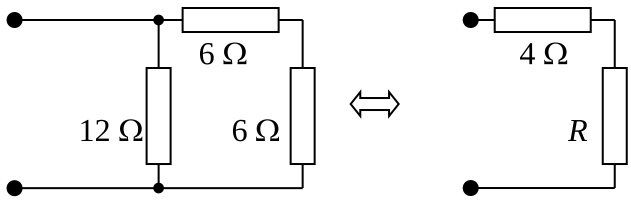

Fig. P.2.5.26

27. A source produces a terminal voltage of 10 V when supplying a current of 1 A. When the current increased to 2 A the voltage falls to 8 V. Find the Norton equivalent of the source.

28. A current source is made by connecting a voltage source of 10 V and negligible internal resistance in series with resistance of 100 Ω. Find the allowed range of load resistance if the current is to remain constant within 10% of its maximum value. For the calculated range of load resistance, find the range of its voltage.

2.6 MAXIMUM POWER TRANSFER THEOREM

Consider a source-load single loop circuit, as presented in Fig. 2.5.11a and the following problem:

For the given source, characterized by parameters E_o, R_t, find the load resistance R_l such that the maximum power available from the source is transferred to this load.

The power absorbed by the load is:

$$P_l = I^2 R_l = \left(\frac{E_o}{R_t + R_l} \right)^2 R_l = f(R_l)\,. \qquad (2.6.1)$$

To find the value of R_l that maximizes this power, the differential calculus to find where the derivative $\mathrm{d}P_l / \mathrm{d}R_l$ equals zero can be used:

$$dP_l / dR_l = 0 \Rightarrow R_l = R_t \qquad (2.6.2)$$

The maximum power delivered by a source represented by its Thevenin equivalent is attained when the load resistance R_l is equal to the internal resistance R_t.

The normalized plot $P_l / P_l^{\max} = f(R_l / R_t)$ is presented in Fig. 2.6.1, together with the system efficiency plot.

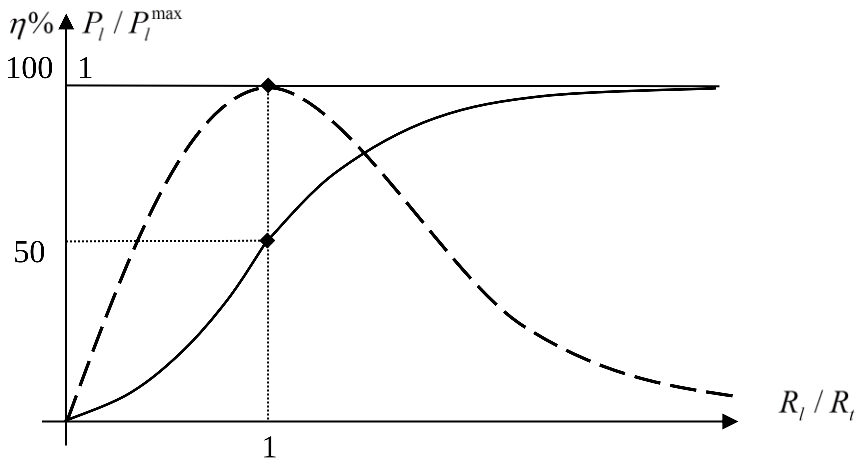

Fig. 2.6.1 Plots of the transferred power (dashed curve) and power transfer efficiency vs. load resistance

Efficiency of power transfer is defined as ratio of the power delivered to a load from a practical source to the power supplied by the emf.

$$\eta\% = \frac{P_l}{P_{E_o}}100\% = \frac{I^2 R_l}{I\,E}100\% = \frac{R_l}{R_l + R_t}100\% \qquad (2.6.3)$$

At the **maximum power transfer condition**:

$$P_l = P_l^{\max} = \frac{E_o^2}{4R_l}\,, \quad I = \frac{E_o}{2R_t} \qquad (2.6.4a)$$

$$\eta\% = 50\% \qquad (2.6.4b)$$

As it is clear, at the maximum power transfer condition only 50% of the power supplied is delivered to a load, the remaining 50% is lost on internal resistance. There is a tradeoff between power company and power consumer goals. Power company tries to keep its losses low by operating at high efficiency:

$$\frac{R_l}{R_t} >> 1 \qquad (2.6.5)$$

The goal of high efficiency is normally more important! Power consumer (electronic system) normaly wants to absorb the maximum power available from a source, i.e. wants to operate under the maximum power transfer condition (2.6.2). In communications circuits, a strong signal may be more important than a high percentage of efficiency.

The presented maximum power transfer theorem considers R_l as the independent variable, resistance of a single resistor. There is, however, an alternative approach. It is assumed that the load is a general two-terminal linear subcircuit, active or passive, and it contains one adjustable parameter, E_x or J_x or R_x or G_x, which varies the terminal voltage U. To find the value of U that maximizes the transferred power

$$P_l = UI = U\frac{E_o - U}{R_t} \qquad (2.6.6)$$

the differential calculus to find where the derivative dP_l / dU equals zero can be used:

$$dP_l / dU = \frac{E_o - 2U}{R_t} = 0 \Rightarrow U = E_o / 2 \qquad (2.6.7)$$

which is the condition on U for the maximum power transfer. At this voltage, the corresponding terminal current, the maximum power and the efficiency are given by equations (2.6.4).

For the given terminal voltage and current, the value of adjustable parameter can be calculated from subcircuit equations. Separation principle, discussed in Chapter 2.8, may be utilized.

Example 2.6.1

For the active subcircuit of Example 2.6.2 (Fig. 2.5.9), find a load resistance R_l such that maximum power is dissipated in R_l. Calculate the value of maximum power.

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$$R_l = R_t = 6.1\Omega,\ P_l^{\max} = \frac{7.7^2}{4 \cdot 6.1} = 2.43\,\mathrm{W}$$

♦

Example 2.6.2

In the circuit shown in Fig. 2.6.2, suppose E_x is adjustable. Find its value such that the maximum power is transferred from A to B. What is the value of $P_l^{\max}$.

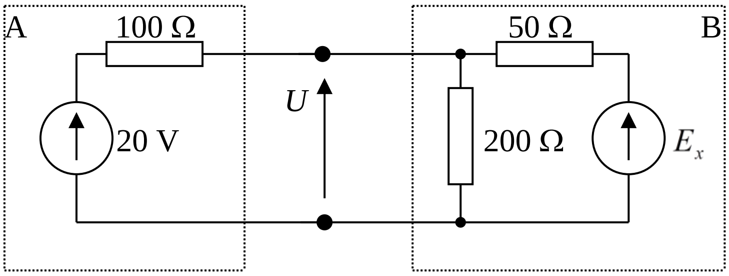

Fig. 2.6.2 Thevenin equivalent of A connected to loading subcircuit B with adjustable E_x

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At the maximum power transfer condition, from (2.6.7), $U = 20/2 = 10\,\mathrm{V}$.
Then, from (2.6.4), the terminal current is $I = 10/100 = 0.1\,\mathrm{A}$.
Applying KCL to the top terminal of B yields

$$0.1 = \frac{10}{200} + \frac{10 - E_x}{50} \Rightarrow E_x = 7.5\,\mathrm{V}.$$

Finally, from (2.6.4), $P_l^{\max} = 1\,\mathrm{W}$.

♦

Drill problems 2.6

1. What should be the load of a practical source $J_s = 10\,\mathrm{A}, G_t = 1\,\mathrm{S}$, such that the system efficiency is $\eta\% = 75\%$.
2. A practical linear source has been loaded, first by an ideal voltmeter, then, by ammeter of $R_{\mathrm{A}} = 100\,\Omega$. The indications are 15 V and 0.1 A, respectively. Draw $I - U$ relationship, find load resistance and the power transferred at the maximum power transfer condition.
3. A practical source $E_o = 10$ V, $R_t = 10\ \Omega$ has been loaded by a resistor of variable resistance $R \in < 2,8 > \Omega$. Find the minimum and the maximum power supplied.
4. A practical source $E_o = 4\,\mathrm{V}$, $R_t = 2\ \Omega$ has been loaded by a resistor. Find range of its resistance $< R_{\min}, R_{\max} >$ so that the power transferred is $0.5 \le P_R \le 1\mathrm{W}$.

5. A battery has open-circuit voltage of 9 V and short-circuit current of 3 A. Find the load resistance at 25% efficiency of the system.
6. Find the minimum and the maximum power transferred from a $J_s = 1.5$ A, $G_t = 0.25$ S source to a variable load ranging from 4 to 6 Ω.
7. Calculate the efficiency of the system presented in Fig. 2.5.14.
8. Find the maximum power absorbed by the load resistor.

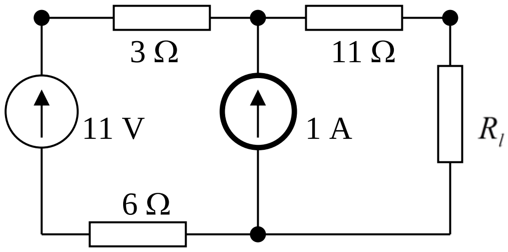

Fig. P.2.6.8

9. If a practical current source: $J_s = 2$ A, $G_t = 0.5$ S, and a voltage source: $E_o = ?$, $R_t = 4\Omega$ are connected back to back ("+" with "+"), is it a value of E_o for which there is no power transfer between them ? If so, find the voltage.
10. Find R such that maximum power is dissipated in R. Calculate its value.

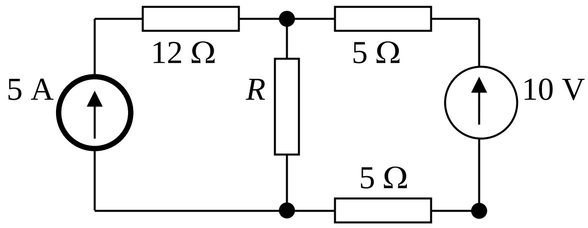

Fig. P.2.6.10

11. In a circuit of Problem 2.6.10, resistance R varies from 4 to 8 Ω. Find the minimum and the maximum power dissipated on this resistance.
12. Find an expression for the maximum power available from two identical sources, each characterized by emf E_o and internal resistance R_t, if they are connected: a) in series, b) in parallel.
13. Two active subcircuits characterized by Thevenin equivalents are as shown. Suppose $E_{o1} = 12$ V, $R_{t1} = 3\Omega$, $E_{o2} = 2$ V and R_{t2} adjustable. Find the value of R_{t2} such that the maximum power is transferred from subcircuit 1 to subcircuit 2. What is the value of this power ?

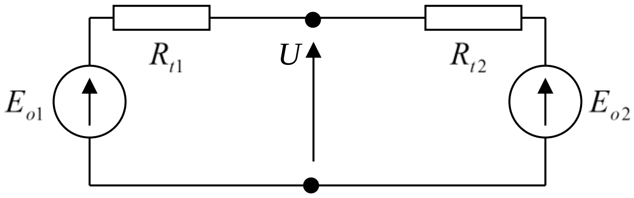

Fig. P.2.6.13

14. Two active subcircuits characterized by Thevenin equivalents are as shown in Fig. P.2.6.13. Suppose $R_{t1} = 3\Omega$, $E_{o2} = 2\,\mathrm{V}$, $R_{t2} = 2\Omega$ and E_{o1} adjustable. Find the value of E_{o1} such that the maximum power is transferred from subcircuit 2 to subcircuit 1. What is the value of this power ?

15. A subcircuit characterized by the Thevenin equivalent: $E_o = 10$ V, $R_t = 2.4\ \Omega$, is loaded by two resistors R_{l1}, R_{l2} connected in parallel. Suppose $R_{l1} = 6\Omega$ and R_{l2} adjustable. Find the value of R_{l2} such that the maximum power is transferred. What is the value of this power ?

2.7 TRANSFER FUNCTION, SUPERPOSITION THEOREM

TRANSFER FUNCTION

A circuit equations relate the circuit response(s) with its excitation(s),
for one-dimensional (SISO) case:

$$Y = f(X) \tag{2.7.1a}$$

for multi-dimensional (MIMO) case:

$$\mathbf{Y} = \mathbf{f}(\mathbf{X}) \tag{2.7.1b}$$

For a linear circuit and the voltage or current output (response), this relationship is linear and term of transfer function can be introduced.

One-dimensional case

Block diagram of a linear SISO circuit is presented in Fig. 1.2.1, Its input-output relationship

$$Y = K\,X \tag{2.7.2}$$

is presented in Fig. 2.7.1, where $X = E$ or J , $Y = U, V$ or I and K is the so called **transfer function**.

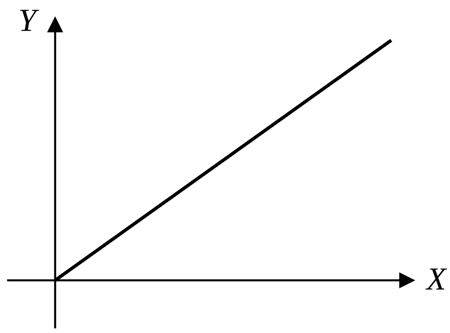

Fig. 2.7.1 Linear circuit output-input relationship

Example 2.7.1

Find the transfer function of a two-resistor voltage divider (Fig. 2.5.2a), with U_1 as the output.

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For the two-resistor voltage divider, relationship between the input voltage and the output voltage (2.5.6) can be expressed by means of transfer function, as presented in Fig. 2.7.2.

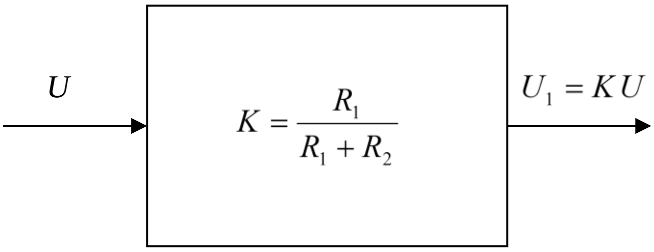

Fig. 2.7.2 Block diagram of two-resistor voltage divider

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<u>Multi-dimensional case</u>

Block diagram of MIMO circuit is presented in Fig. 2.7.3, where $X_i = E_i$ or J_i ; $i = 1,...,M$ and $Y_j = U_j, V_j$ or I_j ; $j = 1,...,N$.

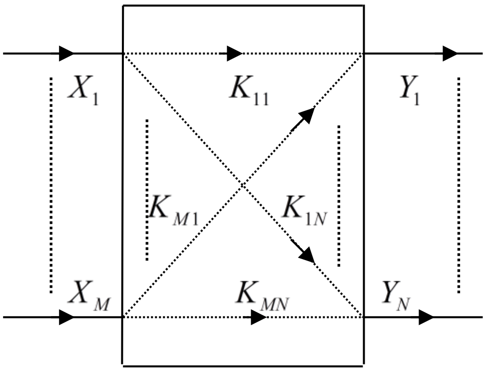

Fig. 2.7.3 Block diagram of MIMO circuit

A linear circuit of *M* inputs and *N* outputs is uniquely characterized by *M·N* transfer functions.

Transfer function

$$K_{X_iY_j} = K_{ij} = \frac{Y_j^{X_i}}{X_i} \qquad (2.7.3)$$

uniquely describes a linear circuit with respect to one input and one output signal, voltage or current.

Then, the output-input relationship is

$$Y_j = \sum_{i=1}^{M} K_{ij} X_i = \sum_{i=1}^{M} Y_j^{X_i} \; ; \; j = 1,...,N \qquad (2.7.4)$$

where the *i*-th component of the *j*-th output is:

$$Y_j^{X_i} = Y_j \Big|_{\substack{X_k=0 \\ k=1,\dots,M;\ k\neq i}} \tag{2.7.5}$$

Now, the superposition principle can be formulated.

SUPERPOSITION THEOREM

For a linear circuit excited from M independent sources, any voltage or current can be obtained by adding all individual voltages or currents, each caused by one source acting alone with all other sources set to zero. Zeroing of voltage source means shorting of its terminals, while zeroing of current source means opening of its terminals.

<u>Example 2.7.2</u>

A two-loop circuit is presented in Fig. 2.7.4. Find voltage U_3, by means of: a) nodal analysis, b) superposition principle.

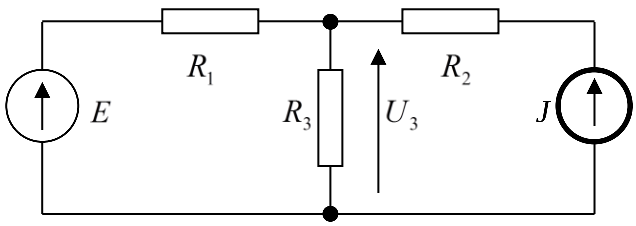

Fig. 2.7.4 Circuit for Example 2.7.2

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a)

Nodal equation:

$$U_3\left(\frac{1}{R_1}+\frac{1}{R_3}\right) = E\frac{1}{R_1} + J\ . \tag{2.7.6}$$

Then,

$$U_3 = E\frac{R_3}{R_1+R_3} + J\frac{R_1R_3}{R_1+R_3}\ . \tag{2.7.7}$$

Block diagram of the circuit is presented in Fig. 2.7.5.

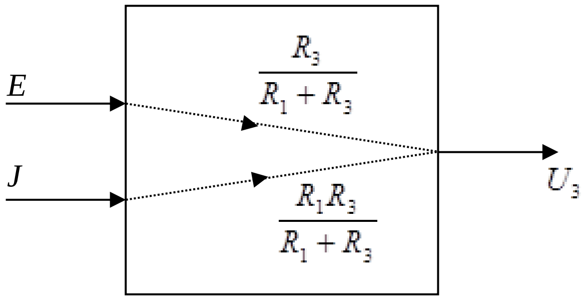

Fig. 2.7.5 Block diagram for Example 2.7.2

b)

Two auxiliary circuits obtained from the original one by zeroing one source are presented in Fig. 2.7.6.

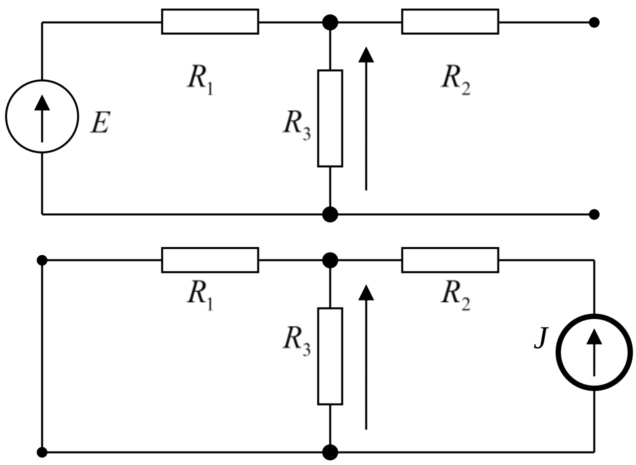

Fig. 2.7.6 Auxiliary circuits (Example 2.7.2) obtained by zeroing J or E

From analyses of Fig. 2.7.6 circuits:

$$U_3^E = \frac{R_3}{R_1 + R_3} E \tag{2.7.8a}$$

$$U_3^J = \frac{R_1 R_3}{R_1 + R_3} J \tag{2.7.8b}$$

When adding these two components, the total voltage U_3 is obtained.

$$U_3 = U_3^E + U_3^J \tag{2.7.9}$$

♦

Applications of superposition theorem:

1. Replacement of a complex (multiple-input) circuit analysis by series of analyses of single input circuits.
2. **Incremental analysis** – finding of increments of circuit responses resulting from an increment of single excitation (source).

Incremental analysis – Problem 1

For the given increment of the *i*-th excitation:

$$\Delta X_i = X_i^{\,2} - X_i^{\,1}, \tag{2.7.10}$$

find the corresponding increment of the *j*-th response:

$$\Delta Y_j = \Delta Y_j^{X_i}. \tag{2.7.11a}$$

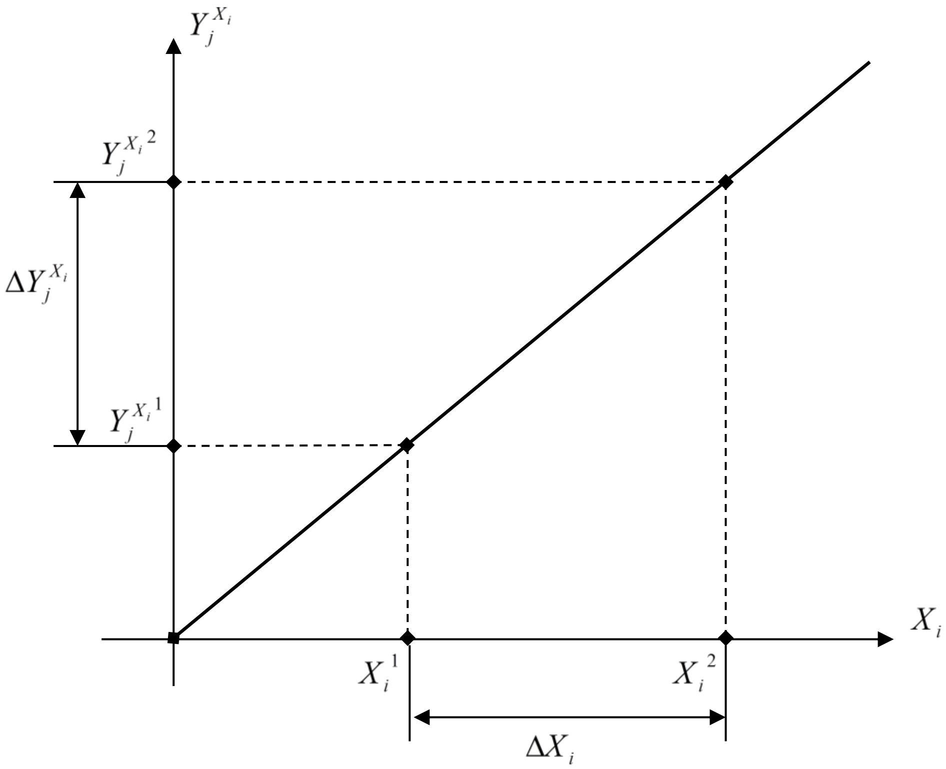

Fig. 2.7.7 Increments of circuit excitation and response

Increments of circuit excitation and response are depicted in Fig. 2.7.7. Then, increment of a circuit response is

$$\Delta Y_j = K_{ij} \Delta X_i \tag{2.7.11b}$$

Incremental analysis – Problem 2

For the given increment of the i-th excitation:

$$\Delta X_i = X_i^{\ 2} - X_i^{\ 1}, \tag{2.7.12a}$$

find increment of the k-th excitation:

$$\Delta X_k = X_k^{\ 2} - X_k^{\ 1}, \tag{2.7.12b}$$

such that

$$\Delta Y_j = \Delta Y_j^{X_i} + \Delta Y_j^{X_k} = 0\,. \tag{2.7.13a}$$

An increment of a circuit response resulting from increments of the i-th and the k-th excitation has to be zero, i.e. no change in a circuit response should be observed:

$$\Delta Y_j = K_{ij} \Delta X_i + K_{kj} \Delta X_k = 0 \tag{2.7.13b}$$

Then, the k-th excitation increment, necessary to compensate the effect of the i-th excitation increment, is

$$\Delta X_k = -\frac{K_{ij}\Delta X_i}{K_{kj}}, \tag{2.7.14}$$

Positive value of the increment means increase of the excitation, negative value means its decrease.

Example 2.7.2 – cont.

Find the increment of U_3 caused by the increment of *E*, if value of *E* increases three times. Then, find the increment of *J* necessary to compensate this increment of *E*.

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For the assumed increment of the voltage source:

$$\Delta E = 3E - E = 2E \tag{2.7.15}$$

the corresponding increment of the voltage U_3 (2.7.8a) is

$$\Delta U_3 = \frac{R_3}{R_1 + R_3} 2E \tag{2.7.16}$$

The increment of *J* necessary to compensate this change is:

$$\Delta J = -\frac{2E}{R_1}, \tag{2.7.17}$$

sign "–" means that decrease of the current source value is necessary.

♦

Drill problems 2.7

1. The emf has increased its value two times (up to 2*E*). Calculate the increment (value and sign) of the current *I*. Then, calculate the increment of the current source necessary to compensate this change.

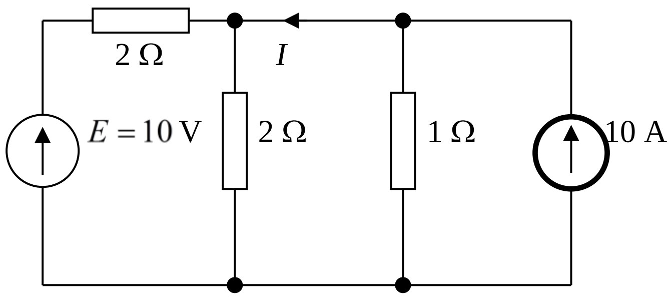

Fig. P.2.7.1

2. Find the gain K of the voltage adder, $U = K(U_1 + U_2)$.

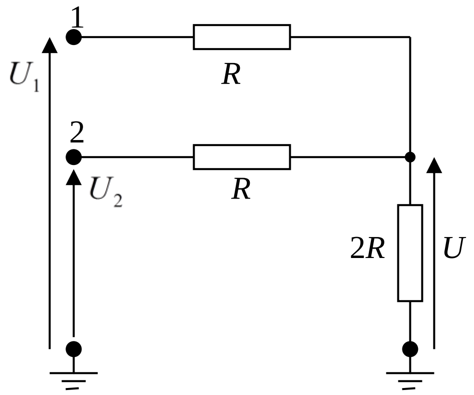

Fig. P.2.7.2

3. All independent sources (circuit inputs) and a shorted branch with the current I_x (circuit output) are extracted from a linear circuit, as shown. With sources J and E_1 on and $E_2 = 0$: $I_x = 20\,\mathrm{A}$, with J and E_2 on and $E_1 = 0$: $I_x = -5\,\mathrm{A}$, with all three sources on: $I_x = 12\,\mathrm{A}$. Find I_x if, a) J is doubled, b) E_2 is reversed.

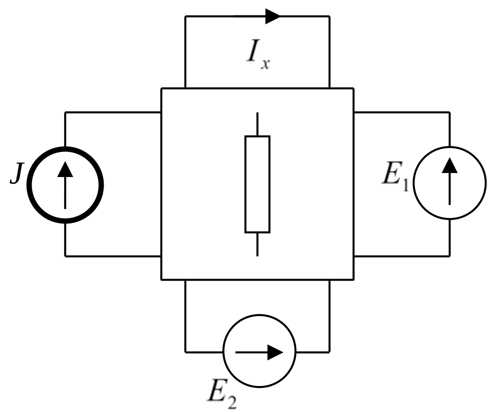

Fig. P.2.7.3

4. Use the superposition theorem to find the voltage in Problems 2.3.5 and 2.3.6.

5. All independent sources (circuit inputs) and a load branch with the current I_l (circuit output) are extracted from a linear circuit, as shown. This current is measured for two different values of E and J:

   1. $E = 7\,\mathrm{V}$, $J = 3\,\mathrm{A}$: $I_l = 1\,\mathrm{A}$,
   2. $E = 9\,\mathrm{V}$, $J = 1\,\mathrm{A}$: $I_l = 13\,\mathrm{A}$.

   Find, a) transfer functions K_{EI_l}, K_{JI_l}, b) I_l when $E = 15\,\mathrm{V}$, $J = 9\,\mathrm{A}$, c) increment of J necessary to compensate increase of E from 7 to 9 V.

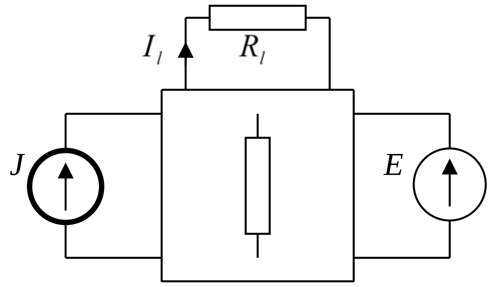

Fig. P.2.7.5

6. The circuit shown is driven by two independent sources. Find transfer functions in the linear relationship: $U = K_{EU}E + K_{JU}J$.

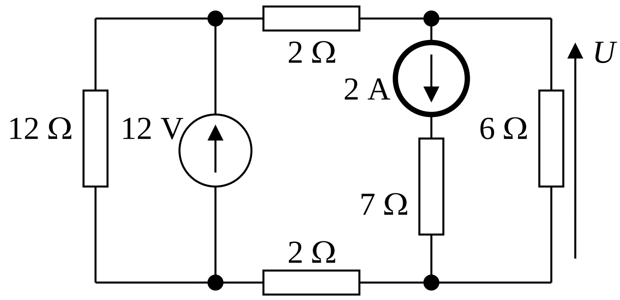

Fig. P.2.7.6

2.8 SUBSTITUTION THEOREM

Consider a circuit built of two subcircuits connected in *m* nodes, as presented in Fig. 2.8.1. Then, the following **substitution theorem**, can be formulated.

> Two subcircuits connected in *m* nodes can be isolated by means of $m-1$ pairs of voltage and/or current sources connected between the *m*-th node (selected arbitrarily reference node) and each of the other $m-1$ nodes. Value of such voltage source equals the voltage in original circuit, U_i. Value of such current source equals the current entering/leaving the node, I_i.

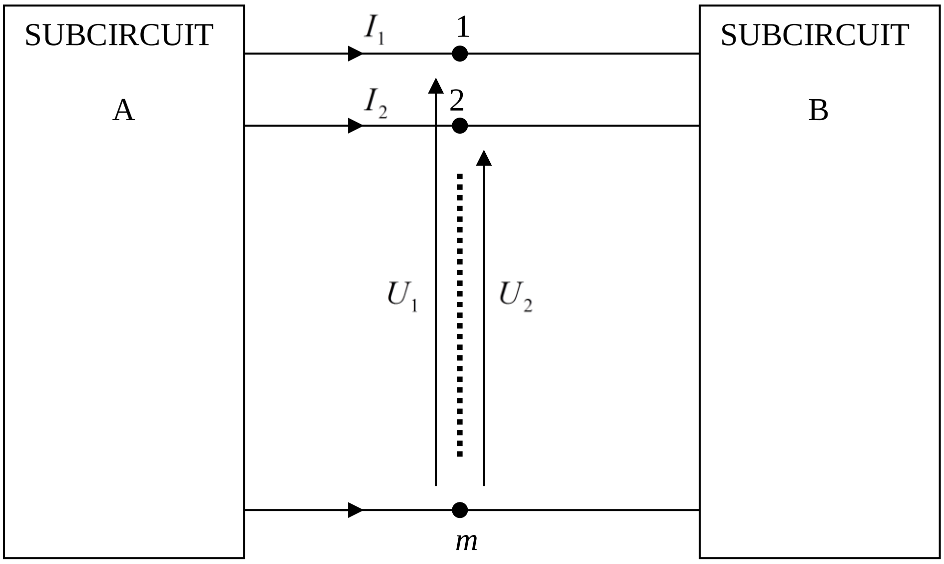

Fig. 2.8.1 A circuit built of two subcircuits connected in m nodes

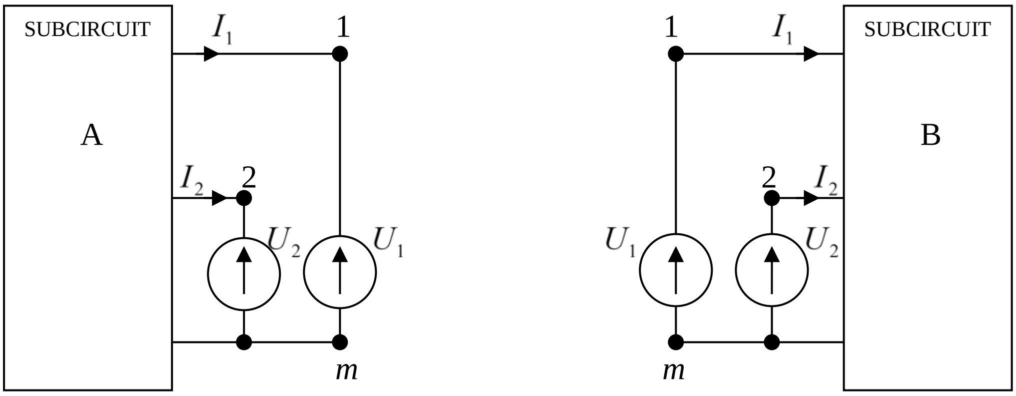

Fig. 2.8.2 Isolated subcircuits, *m*=3

For $m = 3$, subcircuits can be isolated by means of two pairs of sources, as presented in Fig. 2.8.2 for voltage isolating sources. Then, for calculation of voltages and currents inside subcircuit A, the subcircuit B may be replaced by two independent voltage sources and vice versa.

Applications of substitution theorem:

1. Independent analyses of subcircuits, if isolating voltages/currents are given, measured or pre-calculated or set by ideal sources.
2. Calculation or measurement of the power transferred from subcircuit A to subcircuit B.

For subcircuit A presented in Fig. 2.8.2, the power balance is:

$$P_{\mathrm{A}} = P_{U_1} + P_{U_2} = I_1 U_1 + I_2 U_2 \,. \tag{2.8.1}$$

If $P_{\mathrm{A}} > 0$, then subcircuit A supplies power, subcircuit B absorbs.

If $P_{\mathrm{A}} < 0$, then subcircuit A absorbs power, subcircuit B supplies.

In general,

If two subcircuits are connected in *m* nodes, then power transferred from one to the other can be measured by $m-1$ wattmeters or calculated, through calculation of $m-1$ pairs I_i, U_i.

<u>Example 2.8.1</u>

For subcircuits presented in Fig. 2.8.3 and the given measurements: $I_{\mathrm{A}} = 0.5\,\mathrm{A}, U_{\mathrm{V}} = 18\,\mathrm{V}$ (both meters are ideal), calculate the transferred power.

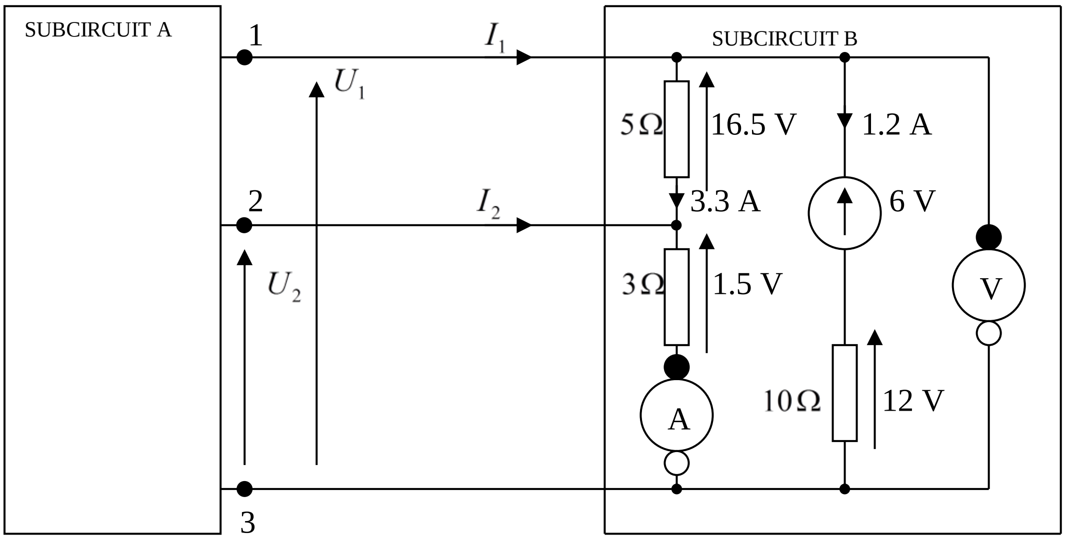

Fig. 2.8.3 Circuit for Example 2.8.1

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By alternate application of KCL, KVL and Ohm's law, subcircuit B currents and voltages can be calculated, as presented in Fig. 2.8.3. Then,

$$I_1 = -3.3 + 0.5 = -2.8\,\text{A},\; U_1 = 1.5\,\text{V}; \qquad (2.8.2)$$

$$I_2 = 1.2 + 3.3 = 4.5\,\text{A},\; U_2 = 18\,\text{V},$$

and the power supplied by subcircuit A is:

$$P_\text{A} = -4.2 + 81 = 76.8\,\text{W}. \qquad (2.8.3)$$

♦

Drill problems 2.8

1. Two subcircuits are modeled by Thevenin and Norton equivalents. Calculate the transferred power, its value and direction of transfer.

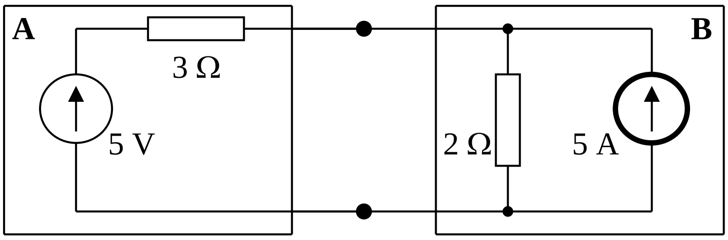

Fig. P.2.8.1

2. Find the power transferred from subcircuit A to subcircuit B.

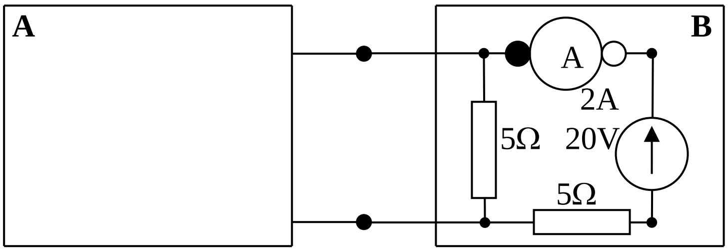

Fig. P.2.8.2

3. A subcircuit is separated from the rest of a circuit by entering/leaving currents, as shown. Find I and the power produced (or absorbed) by the 1 A current source.

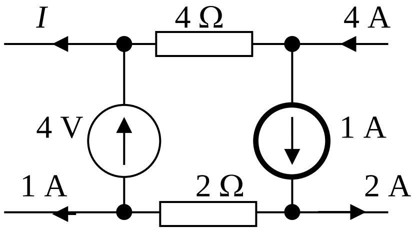

Fig. P.2.8.3

4. Find the current of 4 Ω resistor.

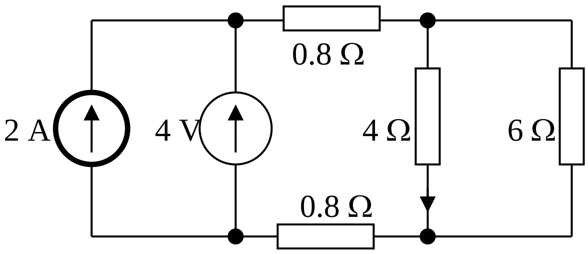

Fig. P.2.8.4

2.9 MULTI-TERMINAL ELEMENTS

A multi-terminal element is an element with *m* terminals available for external connections. After general description of multi-terminal elements, three-terminal element and four-terminal element are considered. Then, analysis of circuits containing multi-terminal elements is discussed.

ELEMENT DESCRIPTION – CONDUCTANCE MATRIX

<u>Passive multi-terminal element</u>

A general passive *m*-terminal element is presented in Fig. 2.9.1, node *m* is the reference one and the terminal currents and voltages satisfy the passive sign convention.

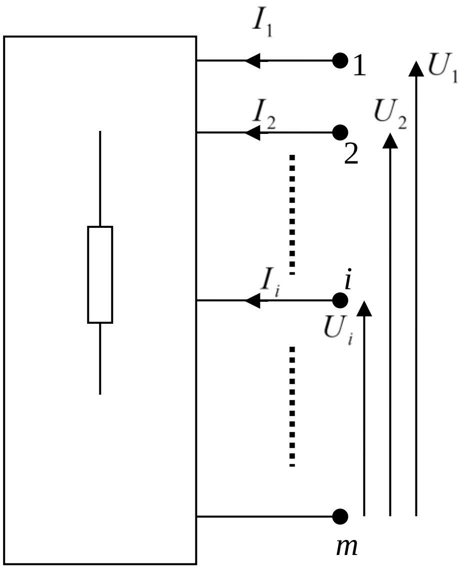

Fig. 2.9.1 Passive m-terminal element

The element can be uniquely described by $m-1$ equations relating external variables, $I_i, U_i;\ i = 1,\dots, n = m-1$. These equations can be equations expressing current by voltages:

$$\begin{bmatrix} I_1 \\ \vdots \\ I_n \end{bmatrix} = \begin{bmatrix} G_{11} & \cdots & G_{1n} \\ \vdots & \ddots & \vdots \\ G_{n1} & \cdots & G_{nn} \end{bmatrix} \cdot \begin{bmatrix} U_1 \\ \vdots \\ U_n \end{bmatrix} \tag{2.9.1}$$

$$\mathbf{I} = \mathbf{G} \cdot \mathbf{U} \tag{2.9.1a}$$

Then, an element is described by the **conductance matrix G**. Its diagonal element $G_{ii};\ i=1,...,n$, is a conductance between the *i*-th node and the reference one with all other nodes shorted to this *m*-th node:

$$G_{ii} = \frac{I_i}{U_i}\Big|_{U_k=0;\,k=1,\ldots,n;\,k\neq i} . \tag{2.9.2}$$

The off-diagonal element $G_{ij};\ i,j=1,...,n;\ i\neq j$, is the so called trans-conductance, ratio of the *i*-th terminal current to the *j*-th terminal voltage, with all nodes except the *j*-th shorted to the reference one:

$$G_{ij} = \frac{I_i}{U_j}\Big|_{U_k=0;\ k=1,..n;\ k\neq j} \tag{2.9.3}$$

For an *m*-terminal circuit of resistors, the conductance matrix is symmetrical, $G_{ij}=G_{ji}$. Then, the total number of its parameters (conductances) is:

$$M = [(m-1)^2-(m-1)]/2+m-1 = \frac{m(m-1)}{2} \tag{2.9.4}$$

These parameters can be

a) measured,
b) calculated, if a circuit structure is known.

Before discussing in details two, three and four-terminal element term of port will be introduced.

Port is a pair of terminals at which same current may enter and leave an element.

Two-terminal element (one-port)

A general two terminal element with voltage and current that satisfy the passive sign convention is presented in Fig. 2.1.2a. For this element: $m=2,\ M=1$ and the equation relating external variables (2.9.1) is simply the Ohm's law equation (2.1.5).

Three-terminal element

A general three terminal element is presented in Fig. 2.9.2. For this element: $m=3,\ M=3$ and equations relating external variables are:

$$\begin{aligned} I_1 &= G_{11}U_1+G_{12}U_2 \\ I_2 &= G_{21}U_1+G_{22}U_2 \end{aligned} \tag{2.9.5}$$

This element is characterized by three conductances: $G_{11}, G_{12}=G_{21}, G_{22}$.

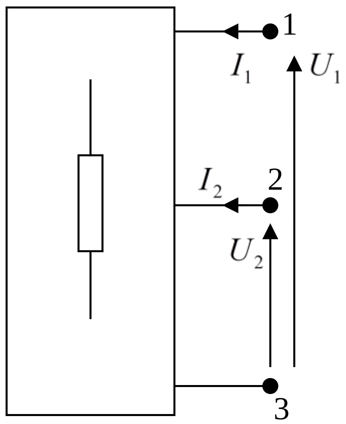

Fig. 2.9.2 Passive three-terminal element

Measurement circuits are presented in Fig. 2.9.3a and 2.9.3b, for G_{11} and G_{21}, respectively.

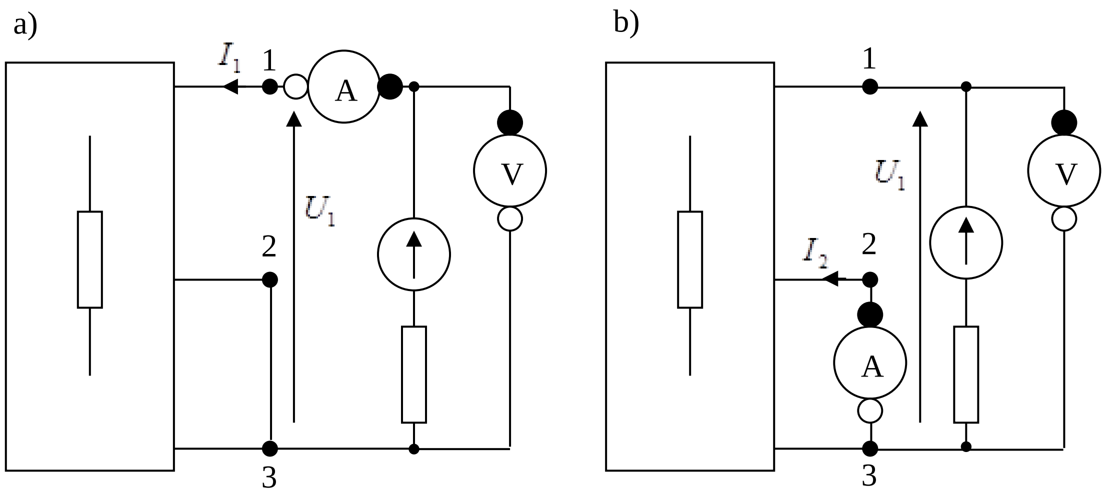

Fig. 2.9.3 Measurement circuits for measurement of: a) G_{11}, b) G_{21}

The conductance G_{11} is

$$G_{11} = \frac{I_1}{U_1};\ I_1 = I_\mathrm{A},\ U_1 = U_\mathrm{V} - I_\mathrm{A} R_\mathrm{A} \tag{2.9.6a}$$

The conductance G_{22} can be measured in the similar way, by shorting 1-3 and measuring I_2, U_2. Finally, the conductance G_{21} is

$$G_{21} = \frac{I_2 - G_{22} U_2}{U_1};\ I_2 = -I_\mathrm{A},\ U_2 = I_\mathrm{A} R_\mathrm{A},\ U_1 = U_\mathrm{V} \tag{2.9.6b}$$

For an ideal ammeter, $R_\mathrm{A} = 0$, and equations (2.9.6) can be simplified.

Example 2.9.1

Consider three resistors circuit of the T-shape structure, as presented in Fig. 2.9.4a.

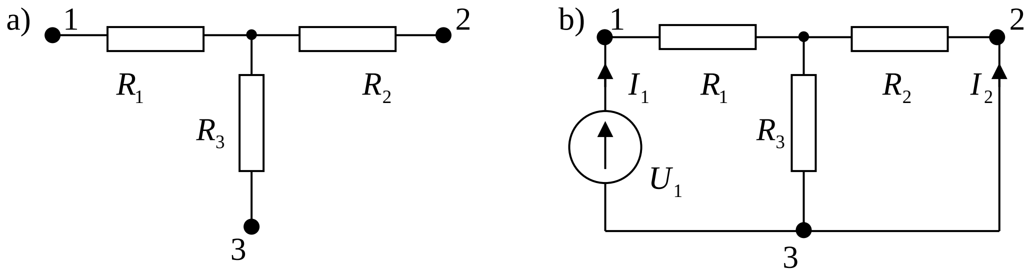

Fig. 2.9.4 T-shape circuit (Example 2.9.1) and circuit for calculation of G_{11} and G_{21}

The conductances G_{11} and G_{21} can be designated from Fig. 2.9.4b circuit. The terminal currents are

$$I_1 = U_1 /(R_1 + \frac{R_2 R_3}{R_2 + R_3}) , \tag{2.9.7a}$$

$$I_2 = -\frac{R_3}{R_2 + R_3} I_1 , \tag{2.9.7b}$$

then, the conductances are

$$G_{11} = \frac{I_1}{U_1} = \frac{R_2 + R_3}{R_1 R_2 + R_1 R_3 + R_2 R_3} \tag{2.9.8a}$$

$$G_{21} = \frac{I_2}{U_1} = -\frac{R_3}{R_1 R_2 + R_1 R_3 + R_2 R_3} \tag{2.9.8b}$$

The conductance G_{22}, conductance seen from terminals 2-3 when 1-3 are shorted, can be calculated from Fig. 2.9.4b subcircuit, with branches 1-3 and 2-3 swapped, and this conductance is

$$G_{22} = \frac{I_2}{U_2} = \frac{R_1 + R_3}{R_1 R_2 + R_1 R_3 + R_2 R_3} . \tag{2.9.8c}$$

♦

Two-port

A general two-port, presented in Fig. 2.9.5, is the special case of four-terminal element. It is described by the same set of equations as a three-terminal element (2.9.5). Port 1 between nodes 1 and 3 is the input port, port 2 between nodes 2 and 4 is the output port.

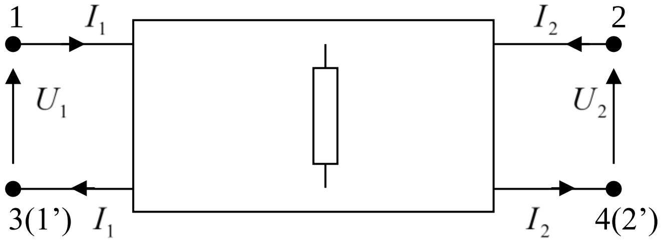

Fig. 2.9.5 Two-port

Active multi-terminal element

For an active multi-terminal element, terminal short-circuit currents should be added to equations (2.9.1).

$$\mathbf{I} = \mathbf{GU} + \mathbf{J} \tag{2.9.9}$$

Now, the conductances, short-circuit currents and the total number of parameters are

$$G_{ii} = \frac{I_i}{U_i}\Bigg|_{\substack{U_k=0;\,k=1,\ldots,n;\,k\neq i \\ \mathbf{J}=\mathbf{0}}} . \tag{2.9.10}$$

$$G_{ij} = \frac{I_i}{U_j}\Bigg|_{\substack{U_k=0;\,k=1,\ldots n;\,k\neq j \\ \mathbf{J}=\mathbf{0}}} \tag{2.9.11}$$

$$J_i = I_i\big|_{\mathbf{U}=\mathbf{0}} \tag{2.9.12}$$

$$M = \frac{m(m-1)}{2} + m - 1 \tag{2.9.13}$$

a)

To identify M parameters, no external source is necessary, M linearly independent measurements have to be performed, five measurements for a three-terminal element.

b)

<u>Example 2.9.1b</u>

Consider three resistors active circuit of the T-shape structure, as presented in Fig. 2.9.6a. After zeroing the voltage source, passive subcircuit of Fig. 2.9.4a is obtained and elements of conductance matrix can be found (2.9.8).
Short-circuit currents can be designated from Fig. 2.9.4b circuit, and they are

$$J_1 = -\frac{ER_2}{R_1R_2 + R_1R_3 + R_2R_3} \tag{2.9.14a}$$

$$J_2 = -\frac{ER_1}{R_1R_2 + R_1R_3 + R_2R_3} \tag{2.9.14b}$$

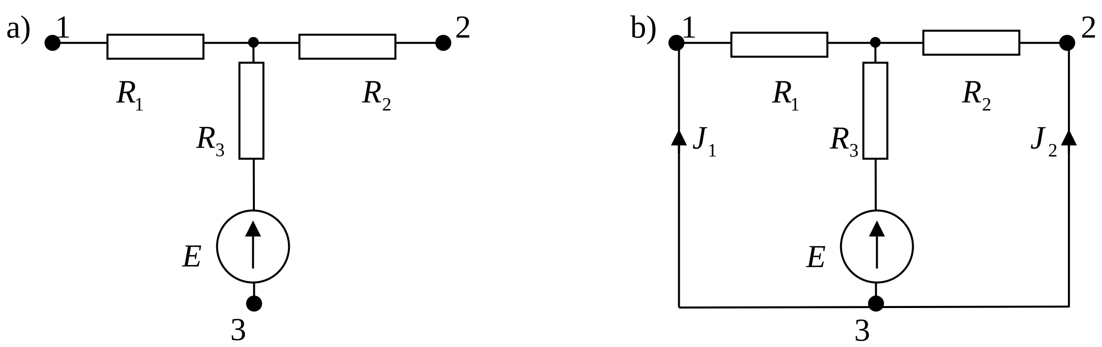

Fig. 2.9.6 T-shape subcircuit (Example 2.9.1) and circuit for calculation of **J**

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OTHER MATRICES OF MULTI-TERMINAL ELEMENT

A multi-terminal element of $m = n+1$ terminals or a multi-port of $n = m/2$ ports can be described by $n = m-1$ equations that express *n* external variables by other *n* external variables. Than, total of

$$N = \binom{2n}{n} \tag{2.9.15}$$

descriptions are possible.

The conductance matrix description, that expresses external currents by external voltages, has been discussed already.

The resistance matrix description, that expresses external voltages by external currents, is the other primary way of multi-terminal or multi-port description:

$$\mathbf{U} = \mathbf{RI} + \mathbf{E} \tag{2.9.16}$$

All other $N-2$ descriptions are the so called hybrid descriptions:

$$\mathbf{Y} = \mathbf{HX} + \mathbf{Z} \tag{2.9.17}$$

where,

$$\mathbf{Y} = \begin{bmatrix} I_1 \\ \vdots \\ I_k \\ U_{k+1} \\ \vdots \\ U_n \end{bmatrix}, \mathbf{H} = \begin{bmatrix} H_{11} & \cdots & H_{1n} \\ \vdots & \ddots & \vdots \\ H_{n1} & \cdots & H_{nn} \end{bmatrix}, \mathbf{X} = \begin{bmatrix} U_1 \\ \vdots \\ U_k \\ I_{k+1} \\ \vdots \\ I_n \end{bmatrix}, \mathbf{Z} = \begin{bmatrix} J_1 \\ \vdots \\ J_k \\ E_{k+1} \\ \vdots \\ E_n \end{bmatrix}. \tag{2.9.17a}$$

For three-terminal or two-port element, four hybrid descriptions can be formulated. The so called cascade matrix description (2.9.18), that expresses output variables by input variables, is frequently used.

$$\begin{bmatrix} U_2 \\ I_2 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \cdot \begin{bmatrix} U_1 \\ I_1 \end{bmatrix} + \begin{bmatrix} E_1 \\ J_2 \end{bmatrix} \tag{2.9.18}$$

For a resistive *m*-terminal circuit, both conductance matrix and resistance matrix are symmetrical. Hybrid matrix is non-symmetrical, however it also contains *M* (2.9.4) linearly independent parameters. In general, having one matrix description the other one can be found. The following relationships between the conductance matrix description and the resistance matrix description can be given:

$$\mathbf{R} = \mathbf{G}^{-1},\ \mathbf{E} = -\mathbf{G}^{-1}\mathbf{J} \tag{2.9.19a}$$

$$\mathbf{G} = \mathbf{R}^{-1},\ \mathbf{J} = -\mathbf{R}^{-1}\mathbf{E} \tag{2.9.19b}$$

Example 2.9.1a cont.

The circuit of Fig. 2.9.6a can be described by the resistance matrix:

$$U_1 = R_{11}I_1 + R_{12}I_2 + E_1 \tag{2.9.20}$$

$$U_2 = R_{21}I_1 + R_{22}I_2 + E_2$$

The parameters R_{11}, R_{21} can be designated from Fig. 2.9.7a circuit. They are

$$R_{11} = \frac{U_1}{I_1}\bigg|_{\substack{I_2=0\\ \mathbf{E}=\mathbf{0}}} = R_1 + R_3 \tag{2.9.21a}$$

$$R_{21} = \frac{U_2}{I_1}\bigg|_{\substack{I_2=0\\ \mathbf{E}=\mathbf{0}}} = R_3 = R_{12} \tag{2.9.21b}$$

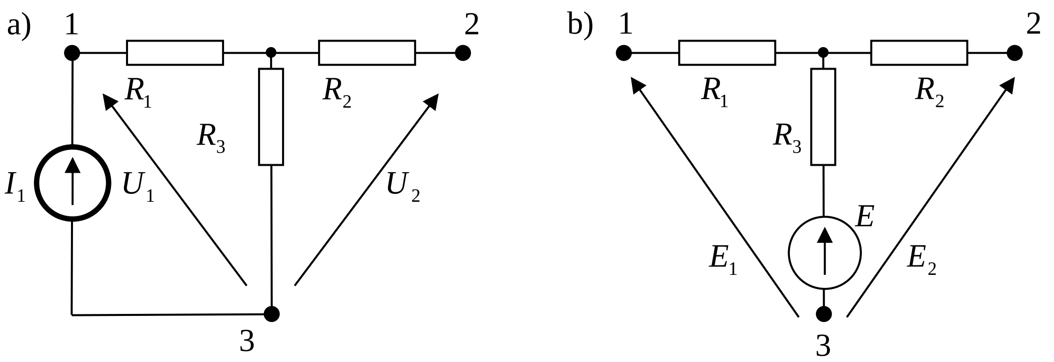

Fig. 2.9.7 (Example 2.5.1) Circuits for calculation of: a) R_{11}, R_{21}, b) $\mathbf{E}$

The resistance R_{22}, resistance seen from terminals 2-3 when 1-3 are opened, can be calculated in the similar way as R_{11}:

$$R_{22} = \frac{U_2}{I_2}\bigg|_{\substack{I_1=0\\ \mathbf{E}=\mathbf{0}}} = R_2 + R_3 \tag{2.9.21c}$$

The open-circuit voltages can be designated from Fig. 2.9.7b circuit.

$$E_1 = E_2 = E \tag{2.9.21d}$$

♦

ANALYSIS OF CIRCUITS WITH MULTI-TERMINAL ELEMENT(S)

Consider a circuit built of two-terminal elements, with one three-terminal element extracted, as presented in Fig. 2.9.8.

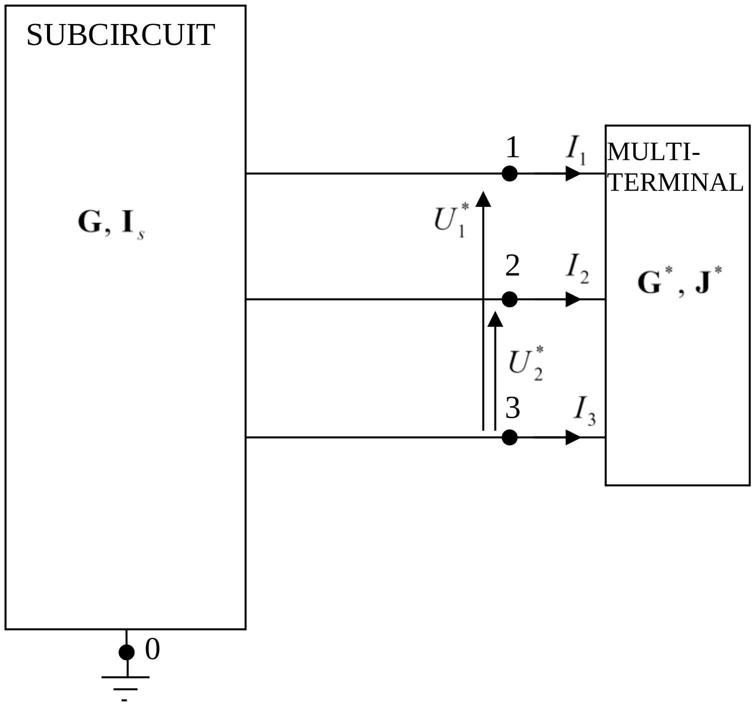

Fig. 2.9.8 Circuit with three terminal element extracted

The circuit nodal equations are:

$$\mathbf{GV} = \mathbf{I}_s - \mathbf{I} \tag{2.9.22}$$

where: $\mathbf{G}$ is the circuit conductance matrix,

$\mathbf{I}_s$ is vector of the circuit source currents at its nodes, internal nodes and nodes 1,2,3

$\mathbf{I}$ is vector of the *m*-terminal element currents, supplemented by $t-m$ zeroes:

$$\mathbf{I} = \begin{bmatrix} \mathbf{I}^* \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \mathbf{I}^* = \begin{bmatrix} I_1 \\ \vdots \\ I_m \end{bmatrix}. \tag{2.9.22a}$$

A general multi-terminal element equations (2.9.9) are

$$\mathbf{I}^* = \mathbf{G}^*\mathbf{V}^* + \mathbf{J}^* \quad . \tag{2.9.23}$$

For $m = 3$, taking into account that:

$$U_1^* = V_1 - V_3,\, U_2^* = V_2 - V_3,\, U_3^* = V_3,\, I_3 = -I_1 - I_2 \tag{2.9.24}$$

a three-terminal element equations are

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} G_{11}^* & G_{12}^* & -(G_{11}^* + G_{12}^*) \\ G_{21}^* & G_{22}^* & -(G_{21}^* + G_{22}^*) \\ -(G_{11}^* + G_{21}^*) & -(G_{12}^* + G_{22}^*) & \sum_{i=1}^{2}\sum_{j=1}^{2} G_{ij}^* \end{bmatrix} \cdot \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} + \begin{bmatrix} J_1^* \\ J_2^* \\ -(J_1^* + J_2^*) \end{bmatrix} \qquad (2.9.25)$$

Taking into account these equations in (2.9.22), the circuit nodal equations can be formulated. The strategy can be generalized into arbitrary number of multi-terminal elements case and the following algorithm can be formulated.

Algorithm 2.9.1 – Nodal analysis of circuits with *m*-terminal element(s)

1. Disconnect (extract) multi-terminal elements, find **G** and $\mathbf{I}_s$ of the obtained subcircuit.
2. Designate $\mathbf{G}^i$ and $\mathbf{J}^i$ of all multi-terminal elements; $i = 1,...,N$.
3. Overlap matrices $\mathbf{G}^i$ onto matrix **G** and vectors $-\mathbf{J}^i$ onto vector $\mathbf{I}_s$, for $i = 1,...,N$.

Example 2.9.2

All conductances and sources of the circuit presented in Fig. 2.9.9 are given, as well as the conductance matrix and the short-circuit currents of the three-terminal active element – its reference node is denoted by an asterisk. Find the circuit nodal equations.

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Circuit nodes are numbered: 0,1,2,3,4 – node 0 is the reference one. Three-terminal element nodes are numbered: (1), (2), (3). Then, nodal equations (2.9.26) can be formulated.

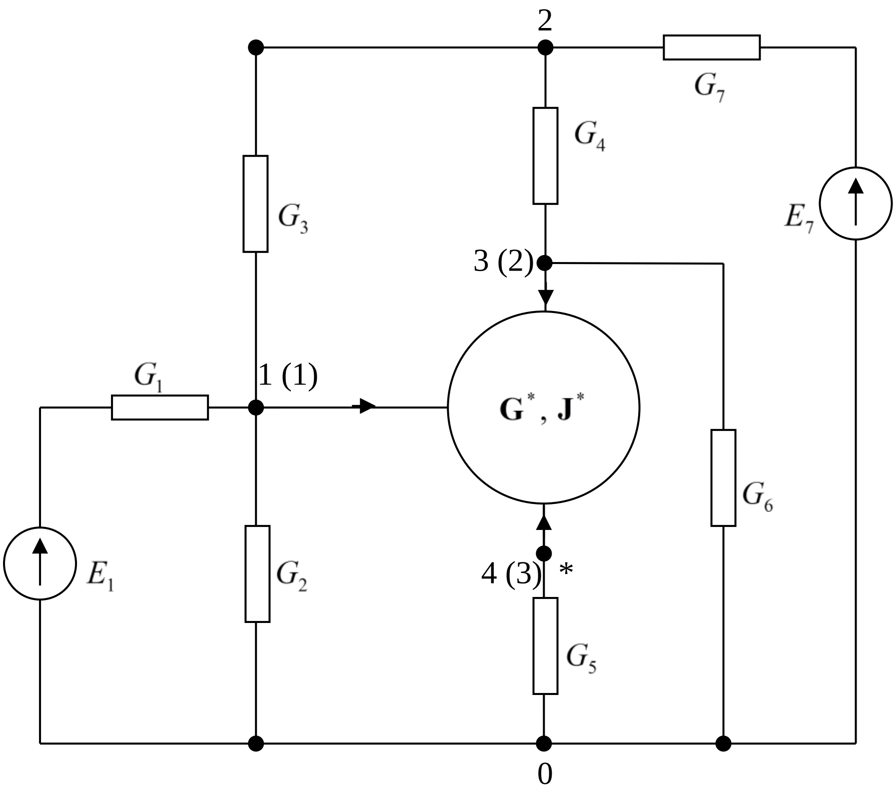

Fig. 2.9.9 Circuit for Example 2.9.2

$$
\begin{array}{cccc}
1\ (1) & 2 & 3\ (2) & 4\ (3)
\end{array}
\qquad (2.9.26)
$$

$$
\begin{bmatrix}
G_1+G_2+G_3+G_{11}^* & -G_3 & +G_{12}^* & -(G_{11}^*+G_{12}^*) \\
-G_3 & G_7+G_3+G_4 & -G_4 & 0 \\
G_{21}^* & -G_4 & G_4+G_6+G_{22}^* & -(G_{21}^*+G_{22}^*) \\
-(G_{11}^*+G_{21}^*) & 0 & -(G_{12}^*+G_{22}^*) & G_5+\sum_{i=1}^{2}\sum_{j=1}^{2}G_{ij}^*
\end{bmatrix}
\cdot
\begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix}
=
\begin{bmatrix} G_1E_1-J_1^* \\ G_7E_7 \\ -J_2^* \\ J_1^*+J_2^* \end{bmatrix}
$$

♦

Drill problems 2.9

1. What is the total number of parameters that characterize an active four-terminal element.
2. Find matrix **R** and vector **E**.

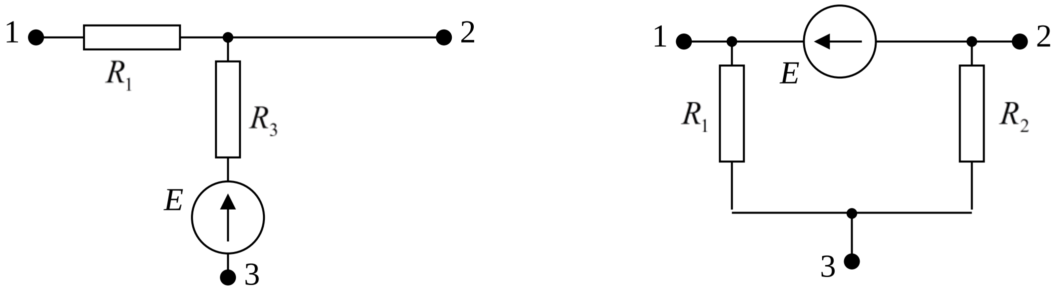

Fig. P.2.9.2

3. The three-terminal passive element is characterized by the following resistances: $R_{11}=R_{22}=1\Omega$, $R_{12}=R_{21}=0.5\Omega$. Find an ideal voltmeter indication.

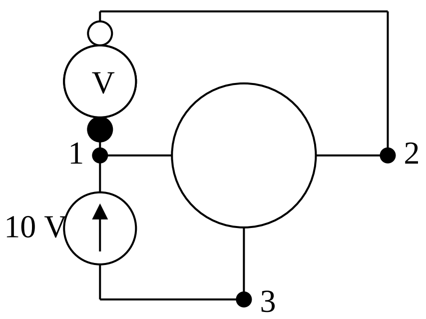

Fig. P.2.9.3

4. A three-terminal element is characterized by the resistances $R_{11} = R_{12} = R_{21} = 1\Omega$, $R_{22} = 2\Omega$ and the open-circuit voltages: $E_1 = -1\text{V}$, $E_2 = 2\text{V}$. An ideal ammeter is connected between terminals 1 and 3, an ideal voltmeter is connected between terminals 2 and 3. Find their indications.
5. Find matrix **G** and vector **J** for the subcircuits shown in Fig. P.2.9.1.
6. Find matrices **R**, **G** and **C** of the two-port shown. Find the expressions and values for $R_1 = 5\Omega,\ R_2 = 20\Omega,\ R_3 = 10\Omega$

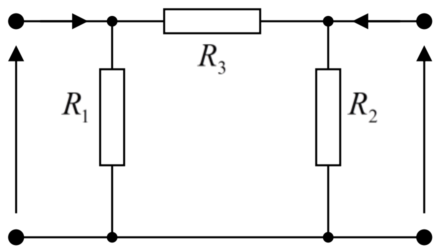

Fig. P.2.9.6

7. Find matrix **C** of the passive two-port for which $\mathbf{R} = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$ in Ω.
8. If the passive two-port shown has the conductance matrix $\mathbf{G} = \begin{bmatrix} 10 & -5 \\ -5 & 20 \end{bmatrix}$ in mS, what are the indications of ideal meters ?

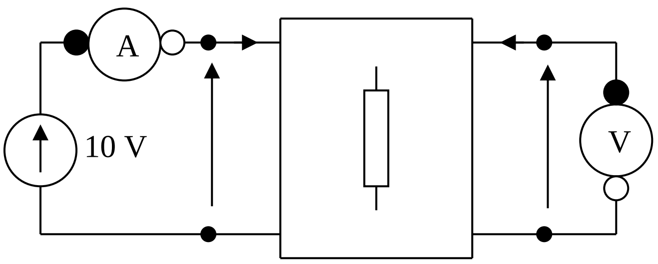

Fig. P.2.9.8

2.10 DEPENDENT (CONTROLLED) ELEMENTS

Arbitrary dependent element - description

Two-terminal elements are normally characterized by an analytic function, linear or nonlinear, of one argument (2.1.4). Another category of elements can be distinguished, namely dependent or controlled elements. A controlled element is described by the following relationship:

$$I = f(U, X) \text{ or } U = g(I, X) \tag{2.10.1}$$

Then, such element is described by a family of $I - U$ characteristics, with X as the second parameter, so called control variable, which can be: temperature *t*, lightning flux Φ, other voltage U_c or other current I_c. The two latter elements are called the controlled sources and they will be discussed in details.

Controlled sources - description

Controlled source is a source that provides a current or voltage that is dependent on other current or voltage elsewhere in the circuit.

Four types of controlled sources can be distinguished:

a) **V**oltage **C**ontrolled **V**oltage **S**ource (VCVS),
b) **C**urrent **C**ontrolled **V**oltage **S**ource (CCVS),
c) **V**oltage **C**ontrolled **C**urrent **S**ource (VCCS),
d) **C**urrent **C**ontrolled **C**urrent **S**ource (CCCS).

Then, two branches are assigned to each controlled source: source branch and control variable branch, as depicted in Fig. 2.7.1a, b, c, d, for ideal (resistiveless) sources. These elements equations are:

VCVS: $U = E = k_{UE} U_c$ (2.10.2a)

CCVS: $U = E = k_{IE} I_c$ (2.10.2b)

VCCS: $I = J = k_{UJ} U_c$ (2.10.2c)

CCCS: $I = J = k_{IJ} I_c$ (2.10.2d)

where, k_{UE} [V/V], k_{IE} [V/A], k_{UJ} [A/V], k_{IJ} [A/A] are control coefficients, constants characterizing corresponding sources.

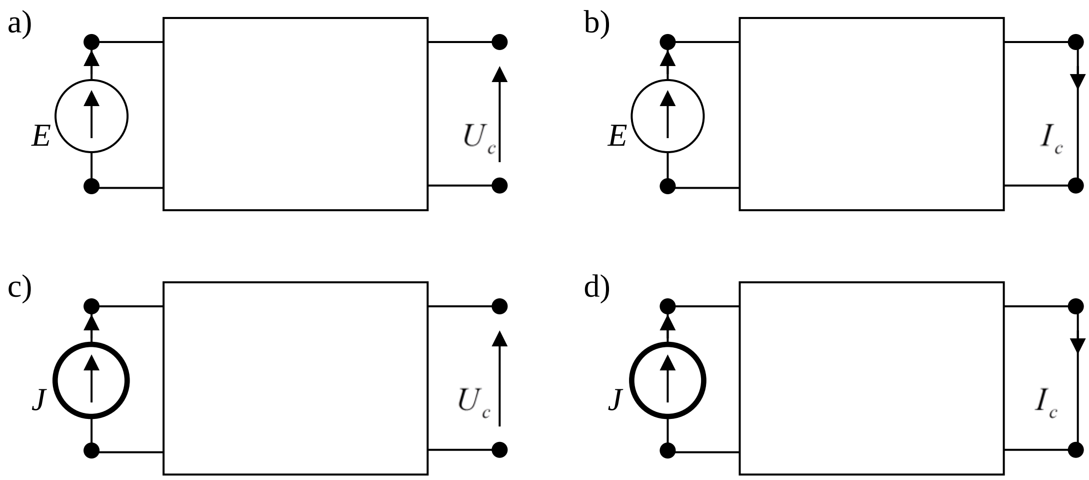

Fig. 2.10.1 a) VCVS, b) CCVS, c) VCCS, d) CCCS

Families of $I-U$ relationships characterizing VCVS and CCCS are presented in Fig. 2.10.2a and d.

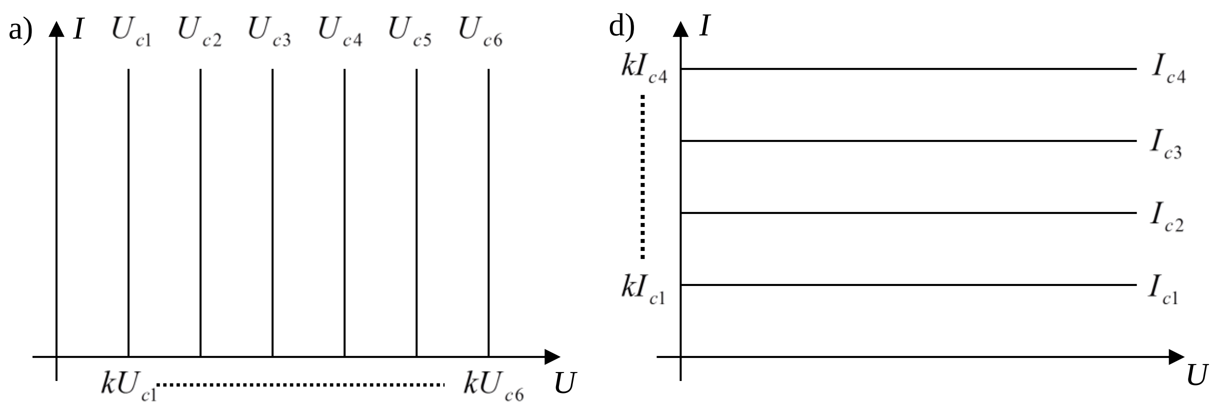

Fig. 2.10.2 Family of $I-U$ relationships characterizing a) VCVS, d) CCCS

Use of controlled sources to element modeling

Controlled sources are used in modeling of circuit elements, such as transistor, operational amplifier or any other multi-terminal element.

Transistor

A transistor circuit symbol and the simplified model, for the common emitter mode of operation, are presented in Fig. 2.10.3. As can be seen, the CCCS is used. Then, after linearization of a diode characteristic (2.12.1c), i.e. its replacement by the voltage source U_f, the transistor equations are:

$$I_{\mathrm{B}} = (1/R_{\mathrm{B}})U_{\mathrm{BE}} - U_f / R_{\mathrm{B}} \tag{2.10.3}$$

$$I_{\mathrm{C}} = (\beta / R_{\mathrm{B}})U_{\mathrm{BE}} - U_f \beta / R_{\mathrm{B}}$$

$$\begin{bmatrix} I_{\mathrm{B}} \\ I_{\mathrm{C}} \end{bmatrix} = \mathbf{G} \begin{bmatrix} U_{\mathrm{BE}} \\ U_{\mathrm{CE}} \end{bmatrix} + \mathbf{J} \tag{2.10.3a}$$

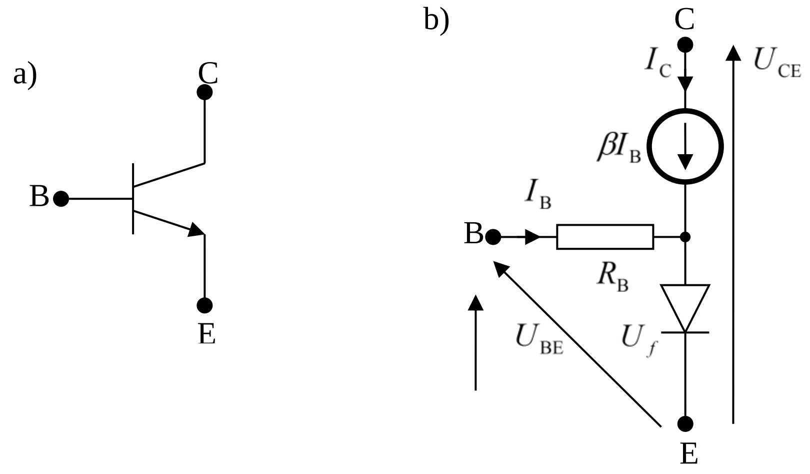

Fig. 2.10.3 Transistor a) circuit symbol, b) simplified model

For the assumed model, the transistor is characterized by the following conductance matrix and short-circuit current vector:

$$\mathbf{G} = \begin{bmatrix} 1/R_B & 0 \\ \beta/R_B & 0 \end{bmatrix}, \ \mathbf{J} = \begin{bmatrix} -U_f / R_B \\ -U_f \beta / R_B \end{bmatrix} \tag{2.10.3a}$$

and then, practically by two parameters (for a silicon transistor $U_f = 0.7$ V):

- current gain β,
- Base resistance R_B.

It should be observed, that resistance matrix **R** does not exist.

Operational amplifier

An operational amplifier (op-amp) circuit symbol and the model are presented in Fig. 2.10.4. As can be seen, the VCVS is used.

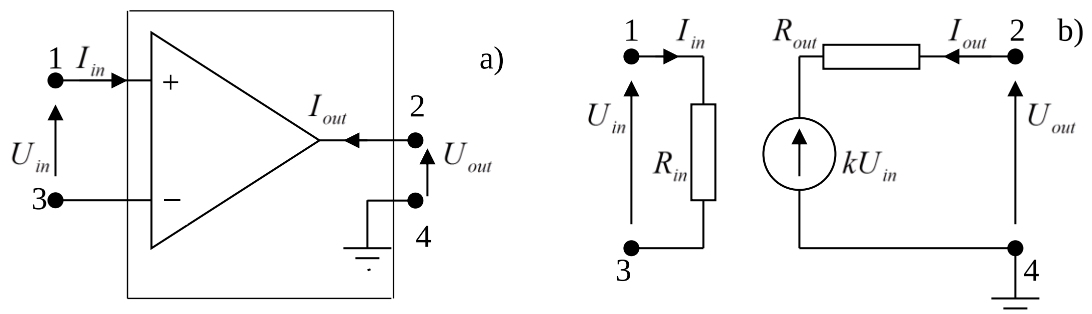

Fig. 2.10.4 Op-amp a) circuit symbol, b) model

For the assumed model, an op-amp is characterized by equations (2.10.4), i.e. by two resistances and one control coefficient.

$$I_{in} = U_{in} / R_{in} \tag{2.10.4}$$

$$I_{out} = -kU_{in} / R_{out} + U_{out} / R_{out}$$

$$\mathbf{G} = \begin{bmatrix} 1/R_{in} & 0 \\ -k/R_{out} & 1/R_{out} \end{bmatrix} \tag{2.10.4a}$$

For an ideal op-amp: $R_{in}=\infty \rightarrow I_{in}=0$; $R_{out}=0 \rightarrow U_{out}=kU_{in}$; $k=\infty$.

Arbitrary three-terminal or two-port element

An arbitrary linear active three-terminal element (Fig. 2.9.2) or two-port (Fig. 2.9.5) can be described by equations (2.9.20). These equations are KVL equations and equivalent circuit built of two-terminal elements can be constructed, as presented in Fig. 2.10.5. For a two-port element, connection denoted by the bold line should be removed.

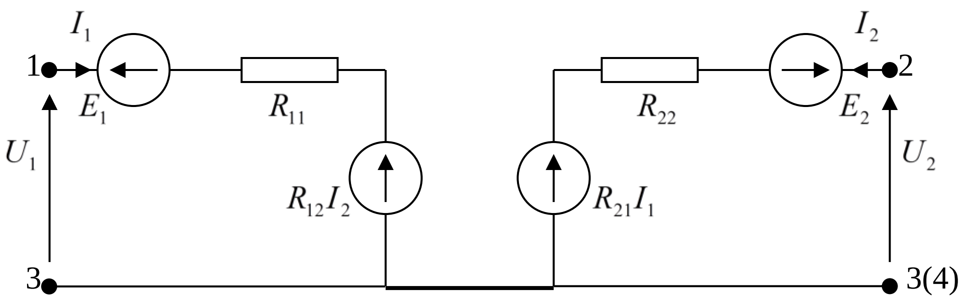

Fig. 2.10.5 Model of three-terminal or two-port element

<u>Analysis of circuits containing controlled sources</u>

If a circuit contains controlled source(s), then such circuit nodal equations should be supplemented by equation(s) of controlled source(s), with controlling variable(s) expressed by nodal voltages.

<u>Example 2.10.1</u>

Find nodal equations of the circuit presented in Fig. 2.10.6.

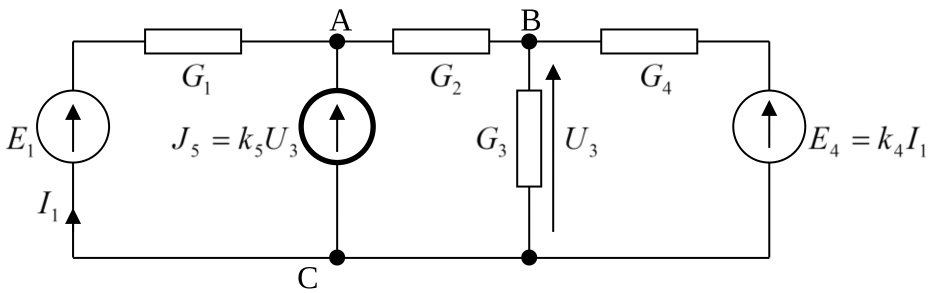

Fig. 2.10.6 Circuit for Example 2.10.1

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For the assumed $V_C = 0$, the circuit nodal equations are:

$$\begin{aligned} &V_A(G_1+G_2) - V_B G_2 = E_1 G_1 + J_5 \\ &-V_A G_2 + V_B(G_2+G_3+G_4) = E_4 G_4 \end{aligned} \tag{2.10.5}$$

Equations of the controlled sources, with controlling variables expressed by nodal voltages are:

$$J_5 = k_5 U_3 = k_5 V_{\mathrm{B}} \tag{2.10.6}$$

$$E_4 = k_4 I_1 = k_4 (E_1 - V_{\mathrm{A}}) G_1$$

After setting equations (2.10.6) into (2.10.5) and reordering, the following system is obtained:

$$V_{\mathrm{A}}(G_1 + G_2) - V_{\mathrm{B}}(G_2 + k_5) = E_1 G_1 \tag{2.10.7}$$

$$-V_{\mathrm{A}}(G_2 - G_1 G_4 k_4) + V_{\mathrm{B}}(G_2 + G_3 + G_4) = E_1 G_1 G_4 k_4$$

It should be observed, that the conductance matrix is not symmetrical, $G_{12} \neq G_{21}$. This is due to the presence of the controlled sources.

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The next example illustrates a strategy of determination of the Thevenin equivalent when two-terminal circuit contains dependent sources.

<u>Example 2.10.2</u>

Find the circuit $U - I$ relationship and then the Thevenin equivalent.

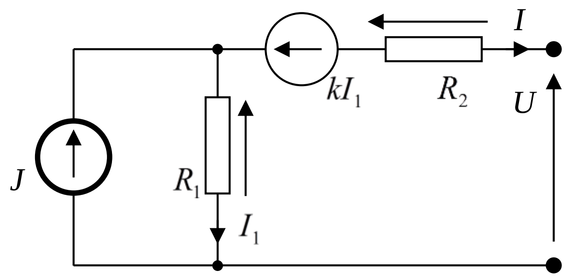

Fig. 2.10.7 Circuit for Example 2.10.2

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By attaching a fictitious external current *I*, KVL can be formulated

$$U = -IR_2 - kI_1 + R_1 I_1 \tag{2.10.8a}$$

From KCL

$$I_1 = J - I \tag{2.10.8b}$$

and the circuit $U - I$ relationship is

$$U = -IR_2 + (R_1 - k)J - I(R_1 - k) = (R_1 - k)J - (R_1 + R_2 - k)I \tag{2.10.9}$$

Then, the Thevenin equivalent parameters are:

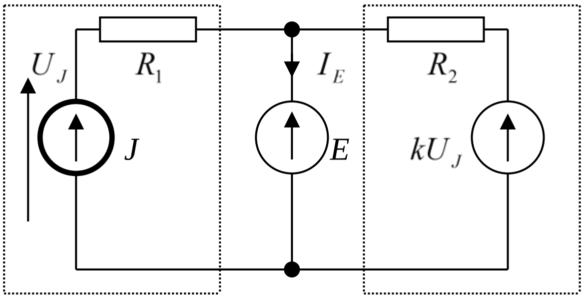
$$E_o = (R_1 - k)J, \; R_t = R_1 + R_2 - k \tag{2.10.10}$$

♦

The next example illustrates the proper use of superposition when there are dependent sources present in a circuit.

Example 2.10.3

Two subcircuits are separated by an ideal voltage source as shown. Find I_E. The circuit parameters are: $J = 5\,\text{A}$, $E = 20\,\text{V}$, $k = 2\,\text{V/V}$, $R_1 = 3\Omega$, $R_2 = 10\Omega$.



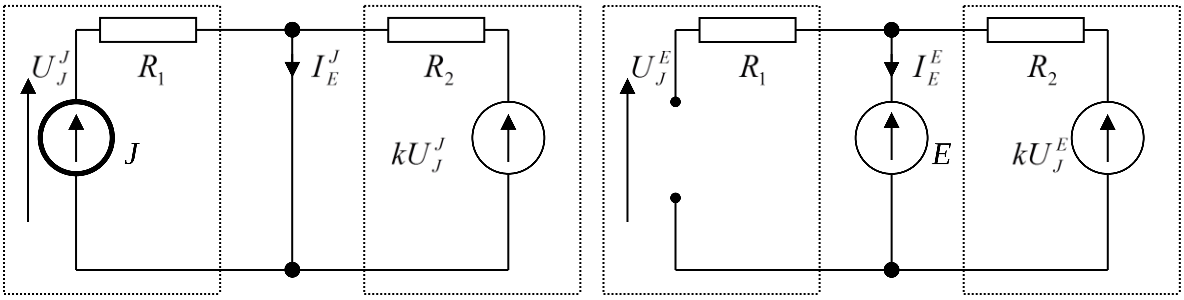


Fig. 2.10.8 Circuit for Example 2.10.3 and its superposition components

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An ideal voltage source isolates two subcircuits, however the substitution theorem can not be applied as the controlling variable and the dependent source are located in different subcircuits.

When superposition is applied, then only independent sources give the superposition components. Thus, the example circuit can be divided into two subcircuits, as shown. The computed components of the current are:

$$I_E^J = J + \frac{kU_J^J}{R_2} = J + \frac{kR_1 J}{R_2} = 8\,\text{A}, \quad I_E^E = \frac{kU_J^E - E}{R_2} = \frac{kE - E}{R_2} = 1\,\text{A}. \tag{2.10.11}$$

Then, the total current is: $I_E = I_E^J + I_E^E = 9\,\text{A}$

♦

Drill problems 2.10

1. Find model of a three-terminal/two-port element, if the element is described by the conductance matrix and short-current vector.
2. Draw $I - U$ characteristic or a practical controlled source: a) VCCS, b) VCVS, c) CCCS, d) CCVS, characterized by the following parameters: $R_t = 10\Omega$, $k = 2$ [X/X], for the controlling variable of 5 [X], where X means V or A.

3. Model a passive two-port characterized by the following resistances: $R_{11} = 0.5\Omega, R_{12} = R_{21} = 2\Omega,\ R_{22} = 4\Omega$. Use controlled sources and resistors.

4. Find the power absorbed by the load resistance $R_l = 1\Omega$.

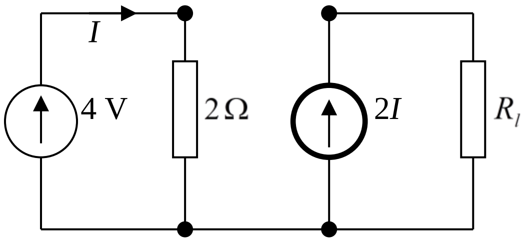

Fig. P.2.10.4

5. Find the mesh current.

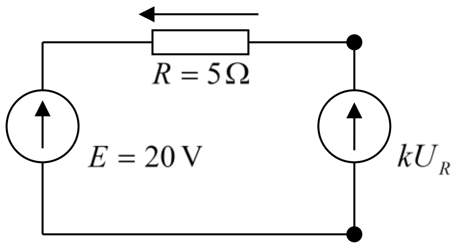

Fig. P.2.10.5

6. Find the collector resistance R_C that gives $U_{CE} = 5\,V$. Assume a diode voltage drop of $0.7\,V$. The supply voltages are $E_B = 1.7\,V$, $E_C = 10\,V$ and the Base resistance is $R_B = 100\,k\Omega$.

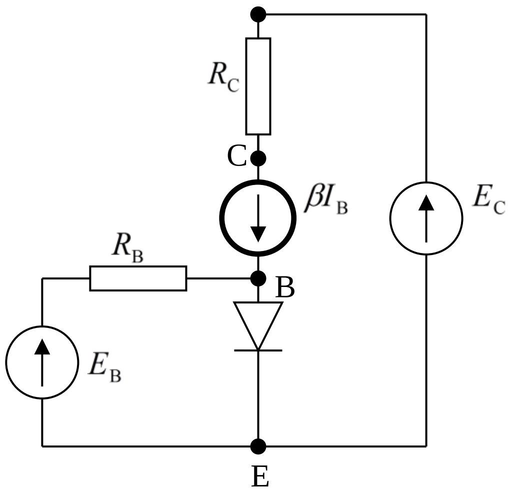

Fig. P.2.10.6

7. A practical source of $E_o = 1\text{mV}$, $R_t = 5\Omega$ is connected across the input terminals of an op-amp and the load resistance of $R_l = 1\text{k}\Omega$ is connected between the output and ground.

Determine the load voltage U_l. Use the idealized op-amp model, $R_{in} = \infty$, $R_{out} = 0$, with open-loop gain of $k = 10^5$ V/V .

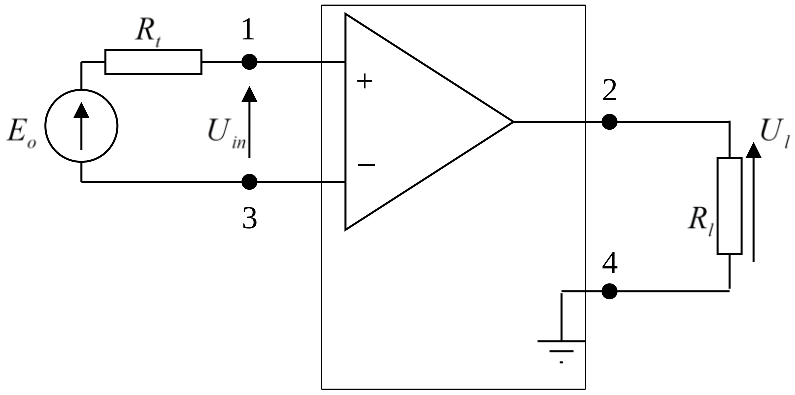

Fig. P.2.10.7

2.11 DESIGN TOLERANCES, SENSITIVITY ANALYSIS

A MIMO circuit of Fig. 2.7.3 is uniquely characterized by its L constants (primary parameters) $P_1,...,P_L$. Any transfer function K_{ij} (secondary parameter) is a function of these constants, $K_{ij} = K_{ij}(P_1,...,P_L)$, any output signal is a function of these constants and input signals, $Y_j = Y_j(X_1,...,X_M, P_1,...,P_L)$. Both transfer functions and output signals can be considered as circuit variables designated by circuit constants and/or circuit inputs

$$F_k = F_k(X_1,...,X_P);\ X_{M+l} = P_l;\ l = 1,...,L \tag{2.11.1}$$

<u>Example 2.11.1</u>

A voltage divider presented in Fig. 2.11.1 is characterized by two parameters, resistances R_1, R_2 . Then, two circuit variables, transfer functions are selected:

- input resistance: $F_1 = R_{in} = U_{in} / I = R_1 + R_2$ (2.11.2a)
- gain: $F_2 = K = U_{out} / U_{in} = \dfrac{R_2}{R_1 + R_2}$ (2.11.2b)

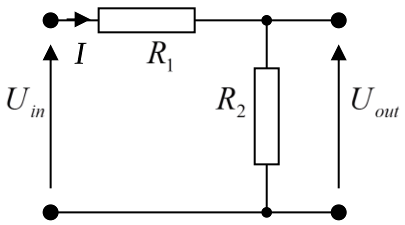

Fig. 2.11.1 Voltage divider

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Then, ideal and practical circuits are distinguished.

Ideal circuit: All circuit parameters have **nominal values**:

$$X_i = X_i^n;\ i = 1,...,P \tag{2.11.3}$$

Practical circuit: All circuit parameters are characterized by nominal values and design tolerances, i.e. their values lay within **tolerance margins**:

$$X_i \in < X_i^-, X_i^+ >;\ i = 1,...,P \tag{2.11.4}$$

Based on tests, made during the manufacturing process, the probability distribution for each parameter can be found. Presented in Fig. 2.11.2 **normal** or Gauss **distribution** is the most commonly used. This distribution is described by the following equation (index *i* has been omitted for simplicity of description):
where σ is the so called **standard deviation**.

$$p(X) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(X - X^n)^2}{2\sigma^2} \right) \tag{2.11.5}$$

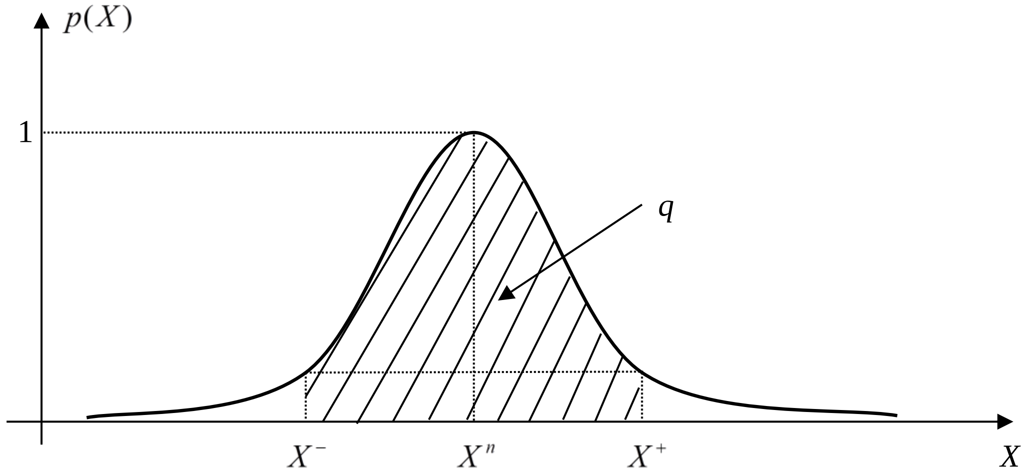

Fig. 2.11.2 Normal distribution of circuit (element) parameter

For the given standard deviation, tolerance margins are related with the probability distribution by the following equation:

$$q = \int_{X^n - \Delta X}^{X^n + \Delta X} p(X)dX \tag{2.11.6}$$

where q is the production yield, e.g. for $q = 0.95$, 95% of the production is classified as “healthy”.

For the assumed parameter deviation ΔX $\Delta X = X^+ - X^n = X^n - X^- > 0$, production yield q can be calculated from (2.11.6). If q is assumed, then the acceptable deviation can be calculated, practically deviation of

$$\Delta X = (2 \div 3)\sigma \tag{2.11.7}$$

is accepted.

An element is normally characterized by its parameter deviation to nominal value ratio, so called **parameter tolerance**:

$$tol_X = \frac{\Delta X}{X^n} \tag{2.11.8}$$

For P parameters characterizing a circuit, its tolerance region can be defined.

Tolerance region (tolerance box) is a parallelepiped in the parameter space $\Re^P$ with planes parallel with coordinate axes, and designated by tolerance margins of circuit parameters $X_1,\ldots,X_P$.

<u>Example 2.11.1 – cont.</u>

The nominal values of resistances and their tolerances are: $R_1^n = R_2^n = 1\Omega$, $tol_1 = 0.1,\ tol_2 = 0.05$. Graph the tolerance region.

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The tolerance region is presented in Fig. 2.11.3, nominal point is denoted.

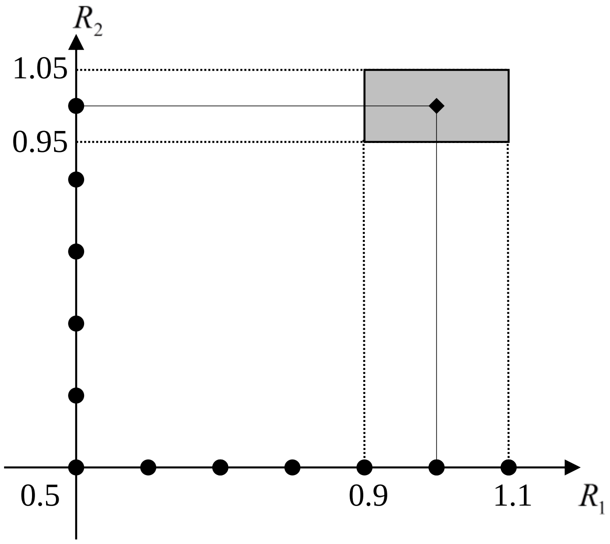

Fig. 2.11.3 Tolerance region for Example 2.11.1

♦

Presence of design tolerances has to be taken into account at a circuit design stage. Two approaches are possible:

1. Parameter design tolerances are given by the design-engineer. Finding of maximum deviations of circuit variables, caused by these tolerances, is the task.
2. Design specifications, acceptable deviations of circuit variables, are given by the design-engineer. Finding of parameter design tolerances is the task.

<u>Designation of the maximum design deviation of circuit variable</u>

For each circuit variable F (index has been omitted for simplicity of description), its maximum deviations, due to design tolerances of circuit parameters, can be found. Two different techniques are possible:

- *worst case* analysis,
- sensitivity analysis.

***Worst case* analysis**

It is assumed that within the tolerance region, first derivatives of a circuit variable function (2.11.1) do not change sign:

$$\mathrm{sgn}(\partial F / \partial X_i) = \text{const} \;\text{ for }\; X_i \in < X_i^-, X_i^+ >;\; i = 1,..., P \tag{2.11.9}$$

Then, the boundary values of a circuit variable, due to a presence of parameter design tolerances, are calculated by setting in function (2.11.1) the boundary values of parameters:

$$F^+ = F(X_1^*,..., X_P^*) \tag{2.11.10a}$$

where, $X_i^* = \begin{cases} X_i^+ & \text{if } (\partial F / \partial X_i)^n > 0 \\ X_i^- & \text{if } (\partial F / \partial X_i)^n < 0 \end{cases}$

$$F^- = F(X_1^*,..., X_P^*) \tag{2.11.10b}$$

where, $X_i^* = \begin{cases} X_i^+ & \text{if } (\partial F / \partial X_i)^n < 0 \\ X_i^- & \text{if } (\partial F / \partial X_i)^n > 0 \end{cases}$

and

$$(\partial F / \partial X_i)^n = S_{X_i}^F \tag{2.11.11}$$

is the 1st derivative calculated at the nominal point $\mathbf{X}^n$, the so called **sensitivity** of a circuit variable F with respect to small changes of parameter X_i in a close neighborhood of the nominal point, the 1st order sensitivity. For M circuit variables and P circuit parameters, the $M \times P$ sensitivity matrix can be created

$$\mathbf{S} = \begin{bmatrix} S_{X_1}^{F_1} & \cdots & S_{X_P}^{F_1} \\ \vdots & \ddots & \vdots \\ S_{X_1}^{F_M} & \cdots & S_{X_P}^{F_M} \end{bmatrix} \tag{2.11.11a}$$

Finally, the maximum deviation caused by parameter tolerances is

$$\Delta F_{\max} = \left| F^+ - F^n \right| \cong \left| F^- - F^n \right| \tag{2.11.12}$$

<u>Example 2.11.1 – cont.</u>

The boundary values of circuit variables are calculated from the following equations:

$$R_{in}^+ = R_{in}(R_1^+, R_2^+) = R_1^+ + R_2^+ = 2.15\Omega \tag{2.11.13a}$$

$$R_{in}^- = R_{in}(R_1^-, R_2^-) = R_1^- + R_2^- = 1.85\Omega$$

$$K^+ = K(R_1^-, R_2^+) = \frac{R_2^+}{R_1^- + R_2^+} = 0.5366\,\text{V/V} \tag{2.11.13b}$$

$$K^- = K(R_1^+, R_2^-) = \frac{R_2^-}{R_1^+ + R_2^-} = 0.4634\,\text{V/V}$$

and the maximum deviations, caused by the design deviations of parameters are:

$$\Delta R_{in,\max} = 0.15\Omega,\ \Delta K_{\max} = 0.0366\,\text{V/V} \tag{2.11.14}$$

♦

Sensitivity analysis

Consider the 1st order approximation of the circuit variable function (2.11.1), its Taylor's series expansion around the nominal point:

$$F(\mathbf{X}^n + \mathbf{\Delta X}) \cong F(\mathbf{X}^n) + \sum_{i=1}^{P} (\partial F / \partial X_i)^n \Delta X_i \tag{2.11.15}$$

Then, the deviation of a circuit variable can be expressed by the 1st order sensitivities and parameter deviations:

$$\Delta F = F(\mathbf{X}^n + \mathbf{\Delta X}) - F(\mathbf{X}^n) \cong \sum_{i=1}^{P} S_{X_i}^F \Delta X_i \tag{2.11.16}$$

The **relative sensitivity** can be introduced:

$$Sr_{X_i}^F = (\partial F / \partial X_i)^n /(F^n / X_i^n) = S_{X_i}^F \frac{X_i^n}{F^n} \tag{2.11.17}$$

and then, the relative deviation of a circuit variable is

$$\Delta F / F^n \cong \sum_{i=1}^{P} Sr_{X_i}^F \, tol_{X_i} \tag{2.11.18}$$

To find the maximum deviation, signs of sensitivities should be disregarded:

$$\Delta F_{\max} \cong \sum_{i=1}^{P} \left| S_{X_i}^F \right| \Delta X_i \tag{2.11.19}$$

$$(\Delta F / F^n)_{\max} = tol_F \cong \sum_{i=1}^{P} \left| Sr_{X_i}^F \right| tol_{X_i} \tag{2.11.20}$$

Example 2.11.1 – cont.

Sensitivities of the selected circuit variables are:

$$S_{R_1}^{R_{in}} = S_{R_2}^{R_{in}} = 1\Omega/\Omega \tag{2.11.21a}$$

$$S_{R_1}^K = \left(-\frac{R_2}{(R_1 + R_2)^2} \right)^n = -\frac{1}{4}\, 1/\Omega,\ S_{R_2}^K = \left(\frac{1}{R_1 + R_2} - \frac{R_2}{(R_1 + R_2)^2} \right)^n = \frac{1}{4}\, 1/\Omega \tag{2.11.21b}$$

Then, the maximum deviations are:

$$\Delta R_{in,\max} \cong 0.15\Omega,\ \Delta K_{\max} \cong 0.15/4 = 0.0375\,\text{V/V}\ .$$

As can be seen, these values are very close the exact values obtained from the worst case analysis (2.11.14).

♦

Normally, an analytic form of a circuit variable function (2.11.1) is not known. Then, two different methods of sensitivity calculations, other than the explicit one, are used.

- Adjoint Circuit method, based on Tellegen's Theorem.
- Direct method.

In the Tellegen's theorem based **adjoint circuit method**, an adjoint circuit is created. This circuit is obtained from the original (nominal) one by zeroing all sources (shorting voltage sources, opening current sources). Its excitation is designated by the considered circuit variable. Next, based on analyses of two circuits, the adjoint and the original one, all sensitivities of this variable, one row of the sensitivity matrix, are calculated, Tellegen's Theorem is applied. Details of this method are not discussed.
As M circuit variables are considered, then, to find all $M \cdot P$ sensitivities, $M+1$ analyses have to be performed: original circuit analysis + M analyses of adjoint circuits.

In the **direct method**, two analyses are performed, the original (nominal) circuit analysis and analysis of the nominal circuit with an increment added to one parameter:

$$X_i = X_i^n + \Delta X_i, \quad X_j = X_j^n; \; j = 1,...,P; \; j \neq i \tag{2.11.22}$$

From the results of analyses, increments of all circuit variables are designated and sensitivities with respect to small increment of X_i, one column of the sensitivity matrix, are calculated:

$$S_{X_i}^{F_k} = \frac{\Delta F_{k,X_i}}{\Delta X_i} \tag{2.11.23}$$

As P circuit variables are considered, then, to find all $M \cdot P$ sensitivities, $P+1$ analyses have to be performed: original circuit analysis + P analyses of circuits with one parameter incremented.

<u>Designation of parameter design tolerances</u>

It is assumed that for the selected M circuit variables, design specifications are set by the design-engineer:

$$F_j \in < F_j^{\min}, F_j^{\max} >; \; j = 1,...,M \tag{2.11.24}$$

That way, the acceptability region in the circuit variable space $\Re^M$ is defined. Next, this region is mapped into the parameter space. Finally, design centering and tolerancing is performed. The greatest parallelepiped (tolerance box) that can be included in the obtained region, for the assumed 100% production yield, or the greatest parallelepiped that overlaps this region and secures the yield less than 100% but greater than the assumed value, is designated. Various methods of mapping and then design centering and tolerancing are used and they are not discussed here.

<u>Example 2.11.2</u>

Voltage divider presented in Fig. 2.11.1 is considered and the following specifications are assumed:

$$R_{in} = 2\Omega \pm 10\%, \; K = 0.5\,\mathrm{V/V} \pm 10\% \tag{2.11.24}$$

Then, the acceptability region in the parameter space is defined by the following inequalities:

$$1.8 \le R_1 + R_2 \le 2.2 \tag{2.11.25a}$$

$$0.45 \le \frac{R_2}{R_1 + R_2} \le 0.55 \tag{2.11.25b}$$

From (2.11.25a) the following two boundary lines are designated:

$$R_{in}^{\max} : R_2 = -R_1 + 2.2 \tag{2.11.26a}$$

$$R_{in}^{\min} : R_2 = -R_1 + 1.8$$

From (2.11.25b) the other two boundary lines are designated:

$$K^{\max} : R_2 = 1.22 R_1 \tag{2.11.26b}$$

$$K^{\min} : R_2 = 0.82 R_1$$

This boundary lines and the obtained acceptability region are presented in Fig. 2.11.4.

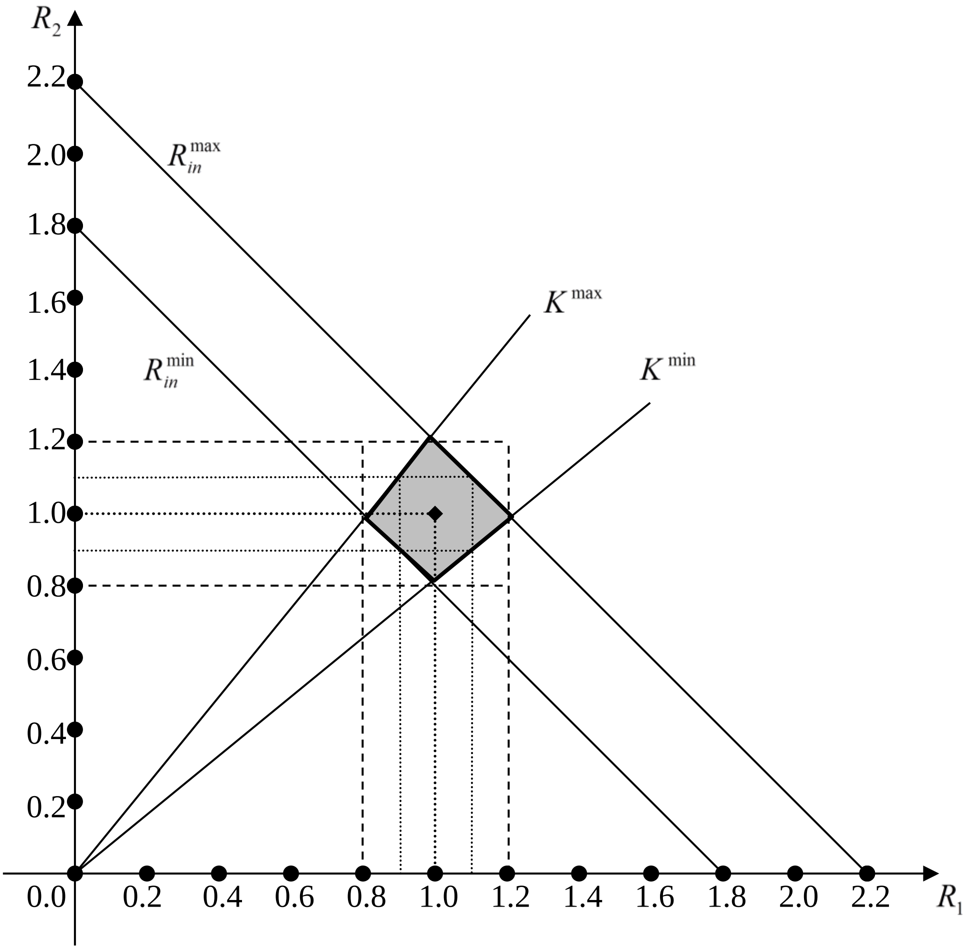

Fig. 2.11.4 Example 2.11.2 acceptability region with marked tolerance regions

Design centering and tolerancing is the next step. For this simple example, central location of the nominal point can be easily deduced: $R_1^n = R_2^n = 1\Omega$, as marked by +++ lines. Then, for the assumed 100% yield, the greatest tolerance box included in the acceptability region can be found: $\Delta R_1 = \Delta R_2 = 0.1\Omega$, as marked by the dotted lines. Another tolerance box: $\Delta R_1 = \Delta R_2 = 0.2\Omega$, marked by the dashed lines, overlaps the acceptability region. For this box, the production yield is less than 100%.

♦

Drill problems 2.11

1. Given the voltage divider: $R_1 = 10\Omega,\ R_2 = 20\Omega$, find all four voltage sensitivities
2. In the voltage divider of Problem 2.11.1, $U_{in} = 30\,\mathrm{V}$ and resistors have 10% tolerance. Find the design deviations of both voltages. Use both the *worst case* and sensitivity approach.
3. Given the current divider: $R_1 = 10\Omega,\ R_2 = 20\Omega$, find all four current sensitivities.
4. In the current divider of Problem 2.11.3, $I_{in} = 30\,\mathrm{A}$ and resistors have 10% tolerance. Find the design deviations of currents. Use both the *worst case* and sensitivity approach.

2.12 ANALYSIS OF NONLINEAR CIRCUITS

Before discussing different approaches to analysis of nonlinear circuits, i.e. circuits that contain at least one nonlinear element, two nonlinear elements commonly used in electronic circuits are presented.

- SEMICONDUCTOR DIODE

A circuit symbol and $I-U$ relationship of a semiconductor diode are presented in Fig. 2.12.1, for both ideal and practical diode.

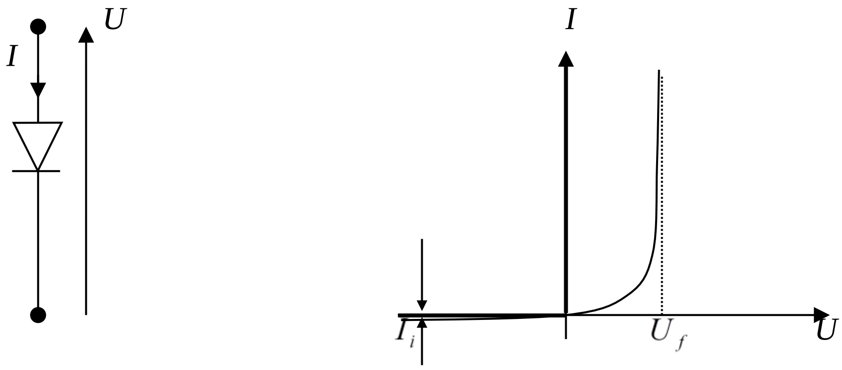

Fig. 2.12.1 Semiconductor diode circuit symbol and *I*–*U* relationship

Ideal diode (bold line) $I-U$ relationship:

$I=0$ for $U<0$ (inverse polarization), (2.12.1a)

$U=0$ for $I\geq 0$ (forward polarization)

Then,

forward polarized ideal diode is a short-circuit,

inverse polarized diode is an open-circuit.

Practical diode (thin line) $I-U$ relationship:

$$I=I_i(\exp^{U/U_0}-1) \tag{2.12.1b}$$

where,

$I_i=10^{-15}\div 10^{-12}$ A (inverse current),

$U_0\cong 25$ mV .

The practical diode $I-U$ relationship can be linearized, and then,

forward polarized diode is practically an ideal voltage source U_f ,

inverse polarized diode is practically an open-circuit:

$I=0$ for $U<U_f$ (inverse polarization), (2.12.1c)

$U=U_f$ for $I\geq 0$ (forward polarization).

For a silicon diode, its forward voltage is equal to $U_f=0.7$ V.

- SEMICONDUCTOR ZENER'S DIODE

A circuit symbol and $I-U$ relationship of a semiconductor Zener's diode are presented in Fig. 2.12.2, for both ideal and practical diode.

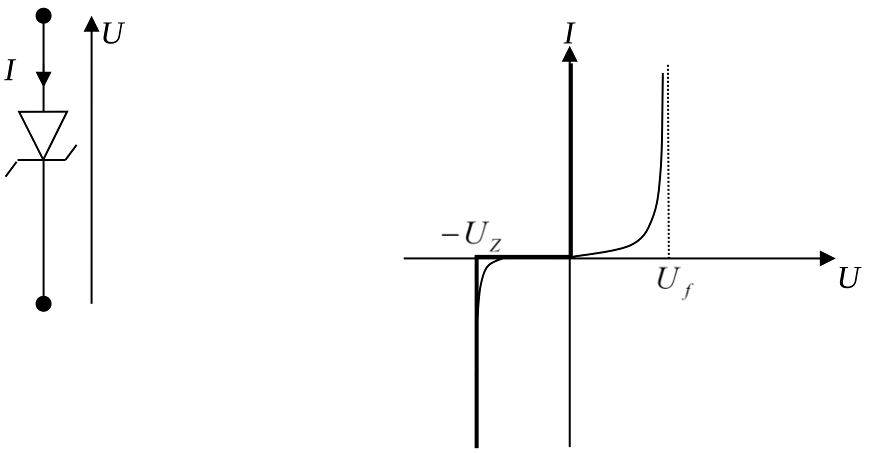

Fig. 2.12.2 Semiconductor Zener's diode circuit symbol and *I*−*U* relationship

Three different approaches to nonlinear circuit may be distinguished.

1. Graphical analysis.
2. Analysis based on **P**iece-**W**ise-**L**inear (PWL) approximation of nonlinearities.
3. Analysis based on the Newton-Raphson iteration technique.

GRAPHICAL ANALYSIS

A series and a parallel connection of elements is considered at first. Then, graphical analysis of a single-loop circuit is discussed, and finally, analysis of a complex circuit with only one nonlinear element is considered.

Series connection of elements

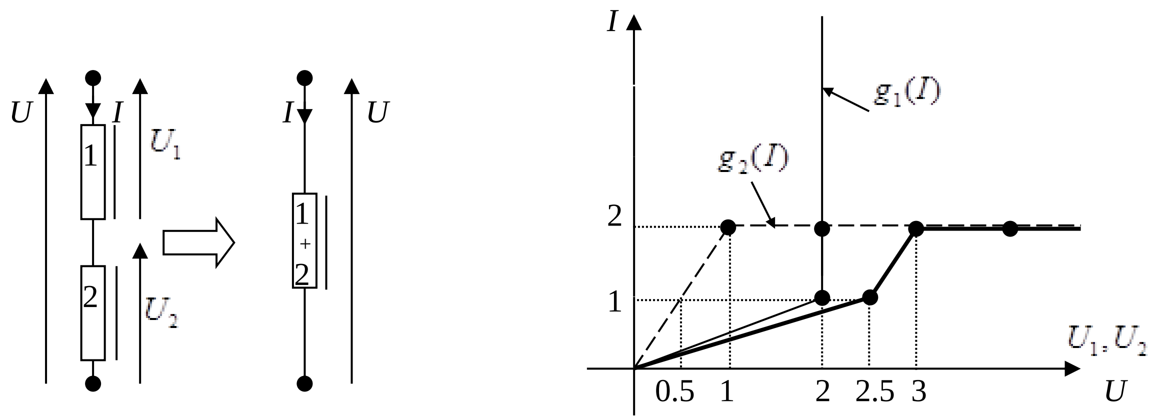

Fig. 2.12.3 Two nonlinear elements connected in series, their *I*−*U* relationships (thin solid and dashed line) and total *I*−*U* relationship (bold line)

Consider two bilateral nonlinear elements characterized by PWL $I-U$ relationships and connected in series, as presented in Fig. 2.12.3. The element $I-U$ relationships are

$$U_1 = g_1(I),\ U_2 = g_2(I) \tag{2.12.2}$$

Then, taking into account KVL, characteristic of the equivalent element is

$$`U = U_1 + U_2 = g_1(I) + g_2(I) = g(I) \tag{2.12.2a}$$

The total $I-U$ relationship for the series connection of nonlinear elements is obtained by graphical adding the voltages of elements at various values of current.

For PWL relationships, these values are designated by the $I-U$ tables of elements. For the exemplary elements, given in Table 2.12.1 and in Fig. 2.12.3, the total $I-U$ relationship is presented in the same Fig. 2.12.3.

Table 2.12.1

Exemplary $I-U$ relationships

I_1	0	1	2
U_1	0	2	2

I_2	0	2	2
U_2	0	1	2

Parallel connection of elements

Consider the same two bilateral nonlinear elements characterized by the PWL $I-U$ relationships, connected this time in parallel, as presented in Fig. 2.12.4.

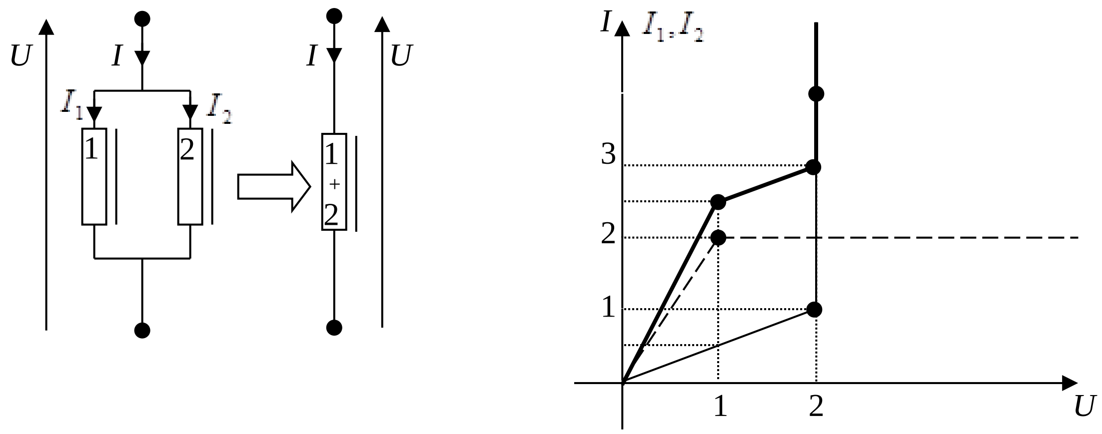

Fig. 2.12.4 Two nonlinear elements connected in parallel, their I–U relationships (thin solid and dashed line) and total I–U relationship (bold line)

The element $I-U$ relationships are

$$I_1 = f_1(U),\ I_2 = f_2(U) \tag{2.12.2}$$

Then, taking into account KCL, characteristic of the equivalent element is

$$I = I_1 + I_2 = f_1(U) + f_2(U) = f(U) \tag{2.12.2a}$$

The total $I-U$ relationship for the parallel connection of nonlinear elements is obtained by graphical adding the currents of elements at various values of voltage.

For PWL relationships, these values are designated by the $I-U$ tables of elements. For the exemplary elements, given in Table 2.12.1 and in Fig. 2.12.4, the total $I-U$ relationship is presented in the same Fig. 2.12.4.

Single-loop circuit

A single-loop nonlinear circuit is presented in Fig. 2.12.5a. The nonlinear element $I-U$ relationship (2.12.3a) is presented in Fig. 2.12.5b.

$$I = f(U) \tag{2.12.3a}$$

From the mesh KLV equation and the resistor Ohm's law, its current can be expressed by the nonlinear element voltage:

$$I = \frac{E-U}{R} \tag{2.12.3b}$$

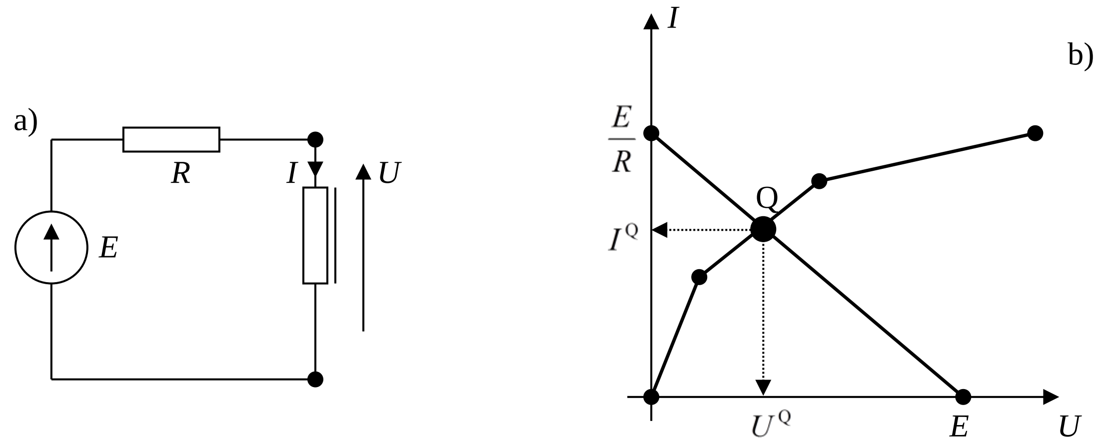

Fig. 2.12.5 a) Single-loop circuit, b) Nonlinear element I–U relationship and load line

The nonlinear element $I-U$ relationship (2.12.3a) and the linear element equation (2.12.3b), the so called **load line** equation, form a set of two equations describing the circuit. This set can be solved graphically, coordinates of a crossing point designate the circuit operating point, the so called **Q** (quiescent)**-point**.

Circuit with one nonlinear element

Consider a circuit built of two parts: a linear part and a nonlinear part, as presented in Fig. 2.12.6. It is assumed that the nonlinear part is built of few nonlinear elements. If so, its total $I-U$ relationship can be found by graphically adding of the component characteristics, as discussed previously. The Thevenin equivalent of the linear part can be found and then, the nonlinear circuit of Fig. 2.12.6 can be transformed into the single-loop circuit of Fig. 2.12.5a, and next, the graphical method can be utilized to find the Q-point.

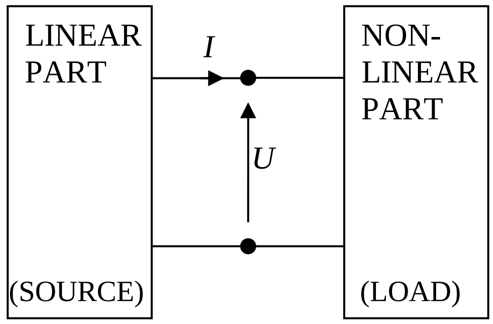

Fig. 2.12.6 Nonlinear circuit separated intolinear and nonlinear part

Algorithm 2.12.1 – Graphical analysis of nonlinear circuit

Step 1.	If nonlinear part consists of more than one element, find graphically the total $I-U$ relationship.
Step 2.	Find the Thevenin equivalent of the linear part.
Step 3.	Find, graphically the Q-point voltage U^{Q} of the obtained single-loop circuit.
Step 4.	To find voltages and/or currents inside the linear part, separate this part by means of the voltage source U^{Q}, and perform analysis of the obtained linear circuit.

Example 2.12.1

A nonlinear circuit is shown in Fig. 2.12.7. For the given parameters of linear elements: $R_1 = 10\,\Omega, R_2 = 5\,\Omega, E_2 = 12\,\mathrm{V}$, and nonlinear element $I-U$ relationship presented in Table 2.12.2, find the power supplied by the voltage source.

Table 2.12.1

Example 2.12.2 $I-U$ relationship

I [A]	0.0	0.1	0.5	0.5
U [V]	0.0	6	10	12

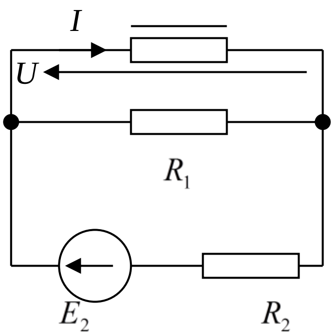

Fig. 2.12.7 Example 2.12.1 nonlinear circuit

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The linear part open-circuit voltage and total resistance are

$$E_o = \frac{R_1}{R_1 + R_2} E_2 = 8\,\mathrm{V}\,, \tag{2.12.4a}$$

$$R_t = \frac{R_1 R_2}{R_1 + R_2} = \frac{10}{3}\ \Omega. \tag{2.12.4b}$$

To draw the load line, voltage increment of 1 V has been assumed, as denoted in Fig. 2.12.8. For this increment and the calculated resistance of 10/3 Ω, the current increment is

$$\Delta I = \Delta U / R = 0.3\ \mathrm{A}. \tag{2.12.5}$$

From the graphical construction, as presented in Fig. 2.12.8, the Q-point coordinates are: $I^{\mathrm{Q}} = 0.225\,\mathrm{A}, U^{\mathrm{Q}} = 7.25\,\mathrm{V}$.

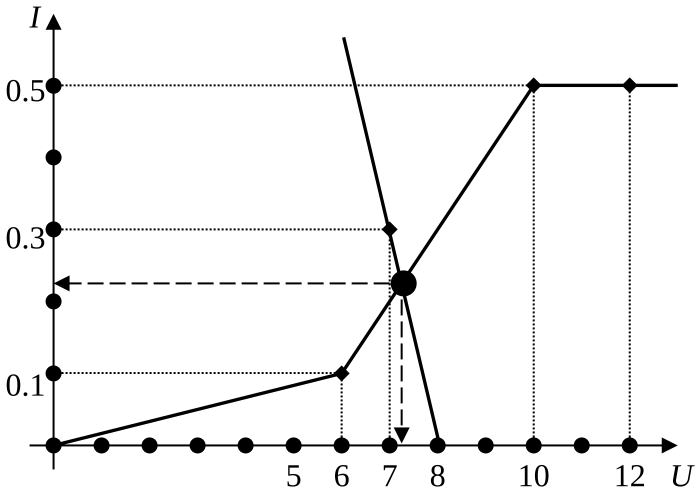

Fig. 2.12.8 Graphical designation of Example 2.12.1 Q-point.

To find the power supplied by the source, a circuit shown in Fig. 2.12.9 is analyzed.

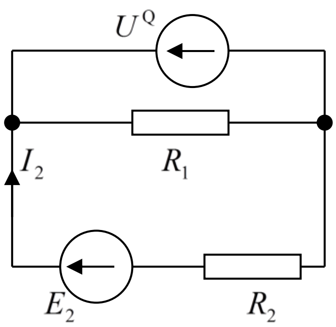

Fig. 2.12.9 Example 2.12.1 linear circuit after separation of nonlinear element

The source current, calculated by means of the superposition principle, is:

$$I_2 = \frac{E_2}{R_2} - \frac{U^{\mathrm{Q}}}{R_2} = 0.95 \text{ A}, \tag{2.12.6}$$

and then, the power supplied by E_2 is

$$P_{E_2} = I_2 E_2 = 11.4 \text{ W}. \tag{2.12.7}$$

♦

ANALYSIS BASED ON PWL APPROXIMATION

A nonlinear element can be characterized by an analytic function (2.1.4) or by its tabularized PWL approximation. Each linear segment is located on a straight line described by equation (2.12.8a) or (2.12.8b).

$$U = E + RI \tag{2.12.8a}$$

$$I = J + GU \tag{2.12.8b}$$

This means that a nonlinear element operating at the given linear segment can be replaced by its Thevenin or Norton equivalent circuit depicted in Fig. 2.5.6 (for simplicity of description indices in (2.12.8) have been omitted). Then, the PWL approximation based algorithm of nonlinear circuit analysis can be formulated.

Algorithm 2.12.2 – PWL approximation based analysis of nonlinear circuit

Step 1. Perform PWL approximation of characteristics of all nonlinear elements.

Step 2. For each nonlinear element, assume location of the Q point, i.e. choose one segment of each linearized characteristic and then, replace nonlinear element by its Thevenin or Norton equivalent.

Step 3. Perform a linear circuit analysis, designate its Q-point.

Step 4. Compare the obtained location of the Q-point with the assumed one. If locations are the same, save the obtained solution.

Step 5. Repeat Steps 2, 3 and 4, for all combinations of segments.

Remarks.

- Steps 2, 3 and 4 have to be repeated if a circuit has multiple solutions. If a circuit has only one solution, then, the algorithm is terminated after finding it. The question of whether a circuit has one solution or multiple solutions is not discussed.
- For circuits with multiple solutions, there are effective algorithms that allow significant reduction of combinations of segments that have to be analyzed. This subject is not discussed.
- The PWL approximation based analysis is allows to find all the solutions.

Example 2.12.1 – cont.

The PWL approximation based analysis is applied. The tabularized $I-U$ relationship (Table 2.12.1) has three segments. These segments are described by the following equations:

I. $U=60I$ (2.12.9-I)

II. $U=5+10I$ (2.12.9-II)

III. $I=0.5$ (2.12.9-III)

Thevenin or Norton equivalents are presented in Fig. 2.12.10.

At first, location of the Q-point on the segment I is assumed,

$$0 \le U \le 6\,\mathrm{V} \tag{2.12.10-I}$$

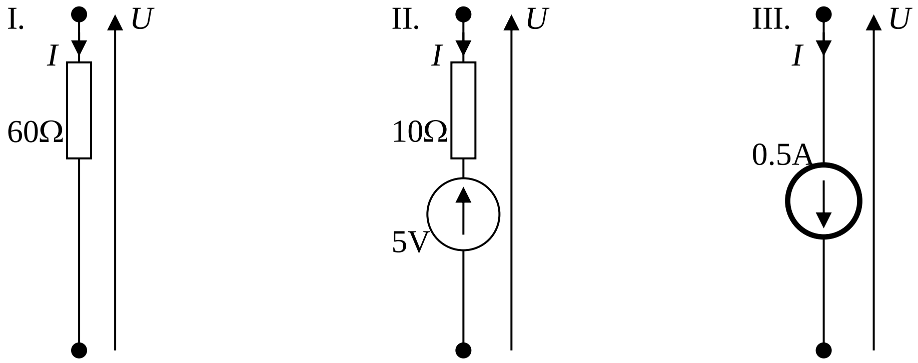

Fig. 2.12.10 Thevenin and Norton equivalents of Table 2.12.1 I−U relationship

Then, the circuit of Fig. 2.12.11-I is analyzed. The obtained voltage is

$$U=E_2\frac{\dfrac{RR_1}{R+R_1}}{R_2+\dfrac{RR_1}{R+R_1}}=7.58\,\mathrm{V} \tag{2.12.11-I}$$

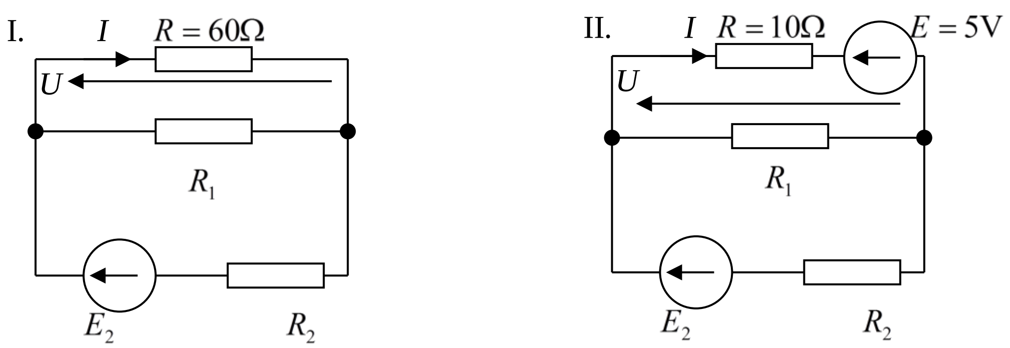

Fig. 2.12.11 Example 2.12.1 linear circuits for the first two segments of nonlinear element

The voltage is located outside the assumed range (2.12.10-I) and the next segment has to be assumed,

$$6 \le U \le 10\,\mathrm{V} \tag{2.12.10-II}$$

Then, circuit presented in Fig. 2.12.11-II is analyzed. The obtained voltage is

$$U = E_2 \frac{\dfrac{RR_1}{R+R_1}}{R_2 + \dfrac{RR_1}{R+R_1}} + E \frac{\dfrac{R_1R_2}{R_1+R_2}}{R + \dfrac{R_1R_2}{R_1+R_2}} = 6 + 1.25 = 7.25\,\mathrm{V} \tag{2.12.11-II}$$

That way, the solution has been found. This solution is consistent with the solution obtained by means of the graphical method.

♦

ANALYSIS BASED ON NEWTON-RAPHSON ITERATION SCHEME

It is assumed that all nonlinear elements are characterized by analytic functions (2.1.4). In a close neighborhood of the Q-point nonlinear characteristic can be linearized, by means of Taylor's expansion (2.12.12), as presented in Fig. 2.12.12.

$$I = I^{\mathrm{Q}} + (dI/dU)^{\mathrm{Q}}(U - U^{\mathrm{Q}}) = J^{\mathrm{Q}} + G^{\mathrm{Q}}U \tag{2.12.12}$$

where

$$(dI/dU)^{\mathrm{Q}} = G^{\mathrm{Q}} \tag{2.12.12a}$$

is called the **dynamic conductance** at the Q-point and

$$J^{\mathrm{Q}} = I^{\mathrm{Q}} - G^{\mathrm{Q}}U^{\mathrm{Q}} \tag{2.12.12b}$$

is the short-circuit current at the Q-point. Then, for the given Q-point, Norton (or Thevenin) equivalent of each element can be found, as depicted in Fig. 6.12.13.

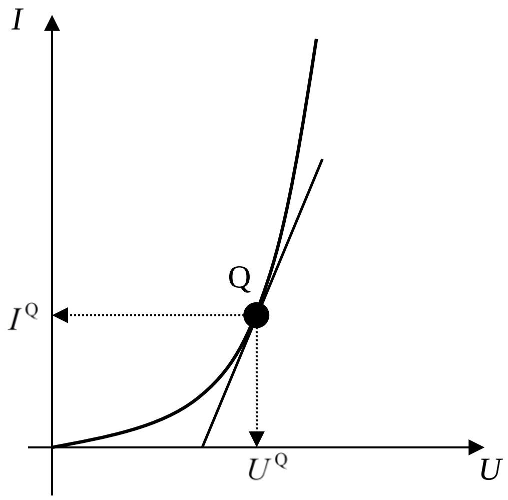

Fig. 2.12.12 Nonlinear characteristic linearized at the Q-point

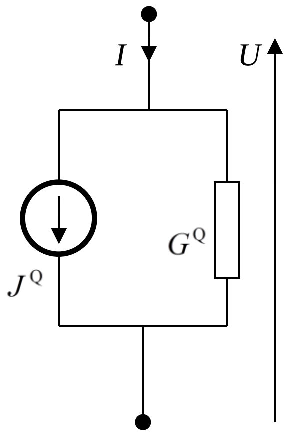

Fig. 2.12.13 Norton equivalent at the Q-point

Newton-Raphson iteration scheme will be formulated, first, for one-dimensional case, then for multi-dimensional case.

Algorithm 2.12.3a – Newton-Raphson iteration scheme, one-dimensional case

Step 1. Set $i = 0$. Assume a trial solution U^0 .

Step 2. Linearize $I = f(U)$ at U^i , find the Norton equivalent.

Step 3. Find solution of the obtained linear circuit, U^{i*} .

Step 4. Check a distance between the assumed U^i and the obtained U^{i*} :

$$\left| U^{i*} - U^i \right| \tag{2.12.13}$$

If this distance is greater than the assumed ε , then set $i = i + 1$,

$$U^i = U^{(i-1)*} \tag{2.12.14}$$

and GO TO Step 2, end the algorithm otherwise.

Example 2.12.2

Find the Q-point of the single-loop circuit presented in Fig. 2.12.14 - diode is characterized by equation (2.12.1b).

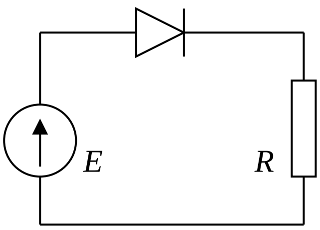

Fig. 2.12.14 Single-loop circuit of Example 2.12.2

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At each iteration ($i = 0,1,2,...,n$), for the given coordinates of the i-th iteration starting-point (I^i, U^i), parameters of the diode Norton equivalent (J^i, G^i) are designated and system of the following linear equations is solved:

$$I = J^i + G^i U \tag{2.12.15}$$

$$I = \frac{E - U}{R}\ .$$

The obtained solution: U^{i*} designates location of the next iteration starting-point (2.12.14). The graphical construction of the first two iterations is presented in Fig. 2.12.15.

For the assumed trial solution denoted by 0, solution denoted by 0^* is obtained. This solution designates starring point of the 1st iteration, denoted by 1. Then, next solution, denoted by 1^* is obtained and the process repeats. As can be observed, iterations converge to the circuit Q-point, denoted by n. Practically, this point is reached after the 3rd iteration.

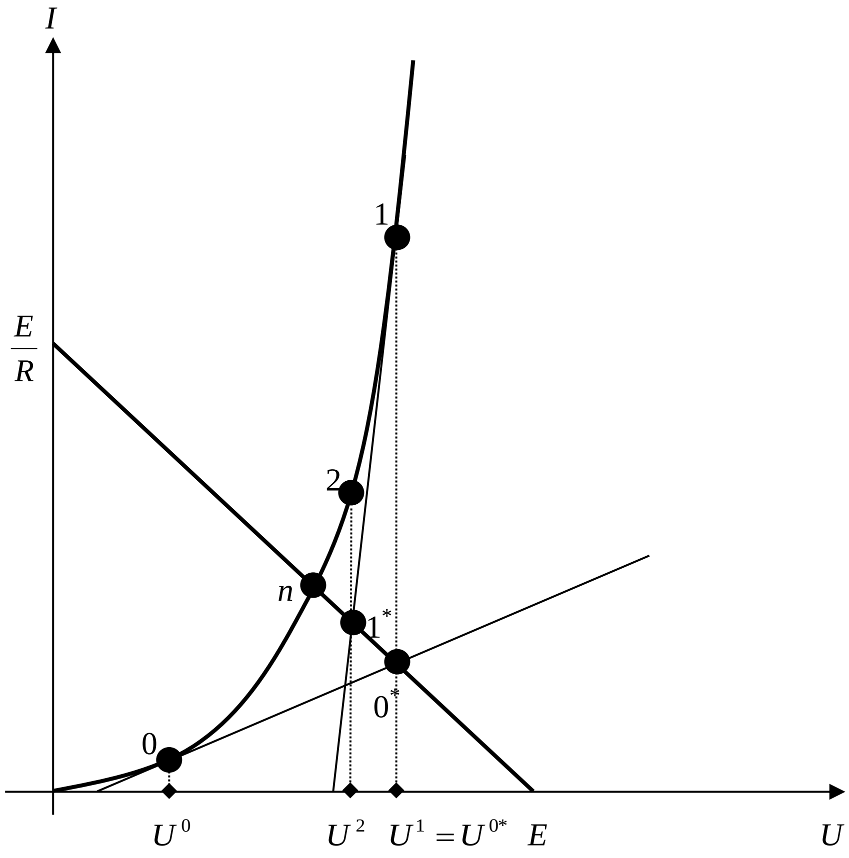

Fig. 2.12.15 Example 2.12.2 – Graphical construction of the first two iterations

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In multi-dimensional case, after linearization of nonlinearities by means of Taylor's expansion, i.e. after replacement of nonlinear elements by their Norton equivalents, nodal equations are formulated and solved, to find the new solution.
Algorithm 2.12.3a – Newton-Raphson iteration scheme, multi-dimensional case

Step 1. Set $i=0$. Assume a trial solution $\mathbf{V}^0$.

Step 2. Linearize nonlinear characteristics at $\mathbf{V}^i$, find Norton equivalents.

Step 3. Find solution of the obtained linear circuit, $\mathbf{V}^{i*}$.

Step 4. Check the distance between the assumed $\mathbf{V}^i$ and the obtained $\mathbf{V}^{i*}$:

$$\left\|\mathbf{V}^i - \mathbf{V}^{i*}\right\| \tag{2.12.13a}$$

If this distance is greater than the assumed ε, then, set $i = i+1$,

$$\mathbf{V}^i = \mathbf{V}^{(i-1)*} \tag{2.12.14a}$$

and GO TO Step 2, end the algorithm otherwise.

Remarks

- Simple iteration scheme (2.12.14) can be replaced by the more complex one, where the new starting point $\mathbf{V}^i$ is calculated from the previous one $\mathbf{V}^{i-1}$ and the last obtained solution $\mathbf{V}^{(i-1)*}$.
- Iterations may diverge and assumption of the maximum number of iterations is necessary.

These problems are beyond the scope of this book.

Drill problems 2.12

1. Calculate the power supplied by the ideal current source $J = 2.7$ mA, and powers absorbed by the resistor $R = 1\text{k}\Omega$ and diode given by the $I-U$ characteristic.

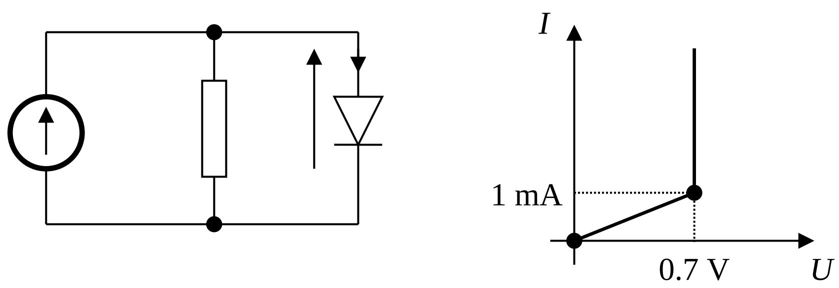


Fig. P.2.12.1

2. Calculate mesh currents - diode $I-U$ characteristic is the same as in Problem 2.12.1.

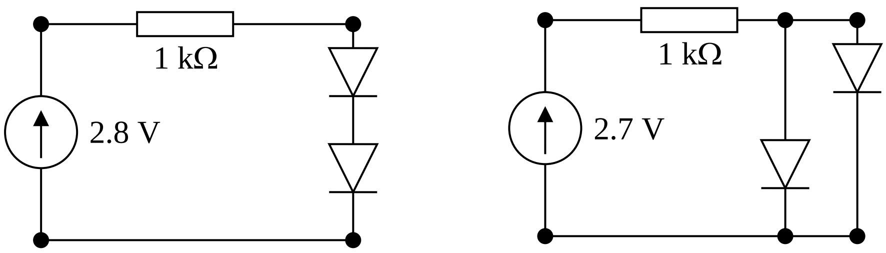


Fig. P.2.12.2

3. Find Thevenin and Norton equivalents for both segments of practical sources given by the presented $I-U$ relationships.

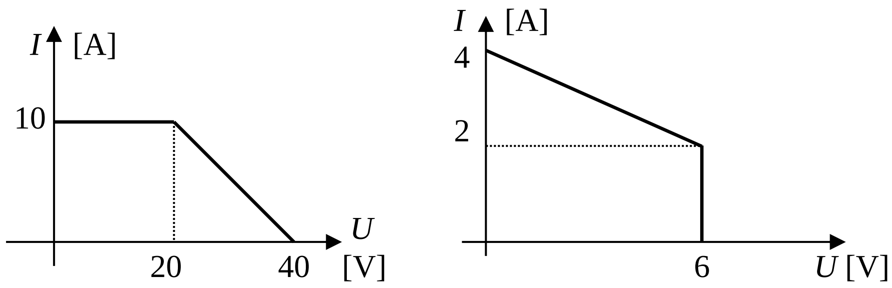


Fig. P.2.12.3

4. Practical sources of Problem 2.12.3 are loaded by a $R = 2\Omega$ resistor. Find the power absorbed.
5. Draw the total $I-U$ relationships for: a) ideal (2.12.1a), b) practical (2.12.1c) diodes.

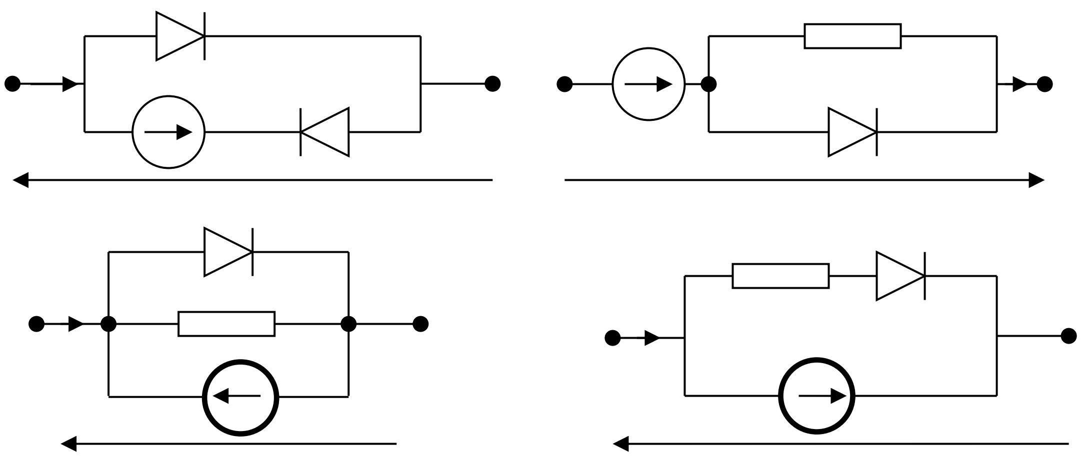


Fig. P.2.12.5

6. Find the series resistance *R*, so that 10V Zener's diode operates at 10 mA current. Supply voltage is 12.5 V, load resistance is 1000 Ω.

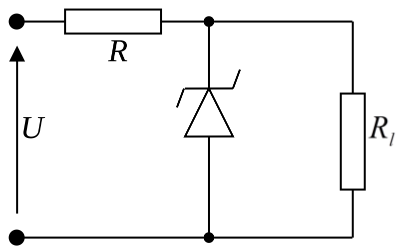


Fig. P.2.12.6

7. In Problem 2.12.6 circuit, find the acceptable range of load resistance so that the diode current ranges from 5 to 15 mA.

8. Find the coordinates of the nonlinear element Q point.

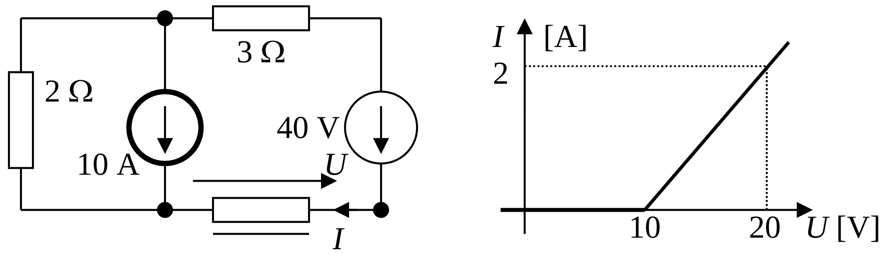


Fig. P.2.12.8

9. Find the acceptable range of the load resistance, if the acceptable range of its voltage is $5 \pm 0.5\,\mathrm{V}$ and the supply voltage may deviate from the nominal value of 12 V by ± 1 V.

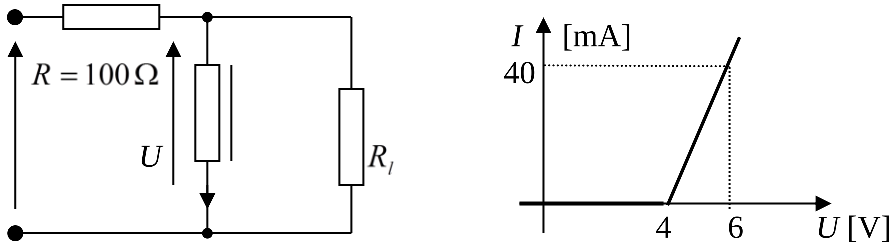


Fig. P.2.12.9

10. Resistor $R = 2\Omega$ and the nonlinear element, characterized by the given Table, are connected in series. Find the range of current that flows through the combination if the supply voltage ranges from 7 to 13 V.

Table P.2.12.10

$I - U$ relationship of P.2.12.10

I [A]	0	0.25	1	3
U [V]	0	5	8	10

2.13 NETWORK ANALOGIES – MAGNETIC CIRCUITS

The presented laws, principles and theorems of dc circuits can be applied to circuits and networks other than electric (electronic), such as pneumatic network, hydraulic network or magnetic circuit. Table 2.13.1 presents analogies between electric circuit variables and non-electric circuit/network variables. Such circuit/network is described by nonlinear equations and can be analyzed by means of methods presented in Chapter 2.12.

Table 2.13.1 Analogous electrical and pneumatic/hydraulic or magnetic circuit quantities

Electric circuit	Pneumatic or hydraulic network	Magnetic circuit
voltage drop U	pressure difference ΔP	magnetic voltage U_m
potential V	pressure P	------
electromotive force E	pump capacity ΔP_p	magnetomotive force F
current I	flow Φ	magnetic flux Φ
resistance R	------	magnetic resistance R_m

Magnetic circuits will be discussed in details.

Magnetic field is a region in space wherein a magnetic body (pole) M [Wb] experiences a magnetic force F_m [N].

Magnetic field is uniquely defined in its every point by the **magnetic field intensity** or magnetizing force.

Magnetic field intensity is defined as the magnetic force per unit magnetic body at a particular point of space.

$$H = F_m / M \tag{2.13.1}$$

Its unit is newton per weber or ampere-turn per meter, $[\text{N/Wb}] = [\text{At/m}]$

Magnetic line of force is the path along which an isolated magnetic pole moves within a magnetic field. It is considered that magnetic lines of force are passing perpendicularly through the given magnetic element, as presented in Fig. 2.13.1 – **S**outh and **N**orth poles are denoted. Magnetic flux and magnetic flux density can be defined.

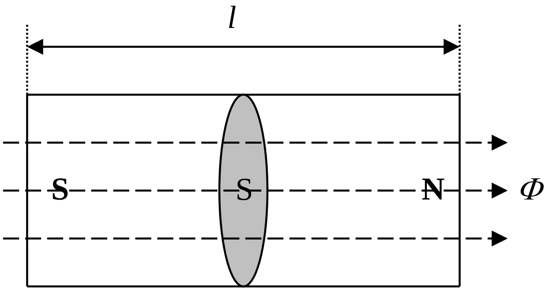


Fig. 2.13.1 Magnetic element with denoted poles and magnetic lines of force

Magnetic flux Φ is total number of lines passing through an area *S*. Its unit is **weber** [Wb]=[V·s].

Magnetic flux density *B* is defined as number of lines passing perpendicularly through an area of 1 m^2:

$$B = \Phi / S \tag{2.13.2}$$

Its unit is **tesla**, [T]=[V·s/m^2].

Magnetic element material is characterized by its **magnetization curve** or $B-H$ curve:

$$B = f(H) \tag{2.13.3}$$

In case of **diamagnetics**, $B-H$ relationship is linear:

$$B = \mu_r \mu_0 H = \mu H \tag{2.13.3a}$$

where, μ_r - dimensionless relative magnetic permeability of the material,

$\mu_0 = 4\pi 10^{-7}$ [V·s/A·m] - magnetic permeability of the free space,

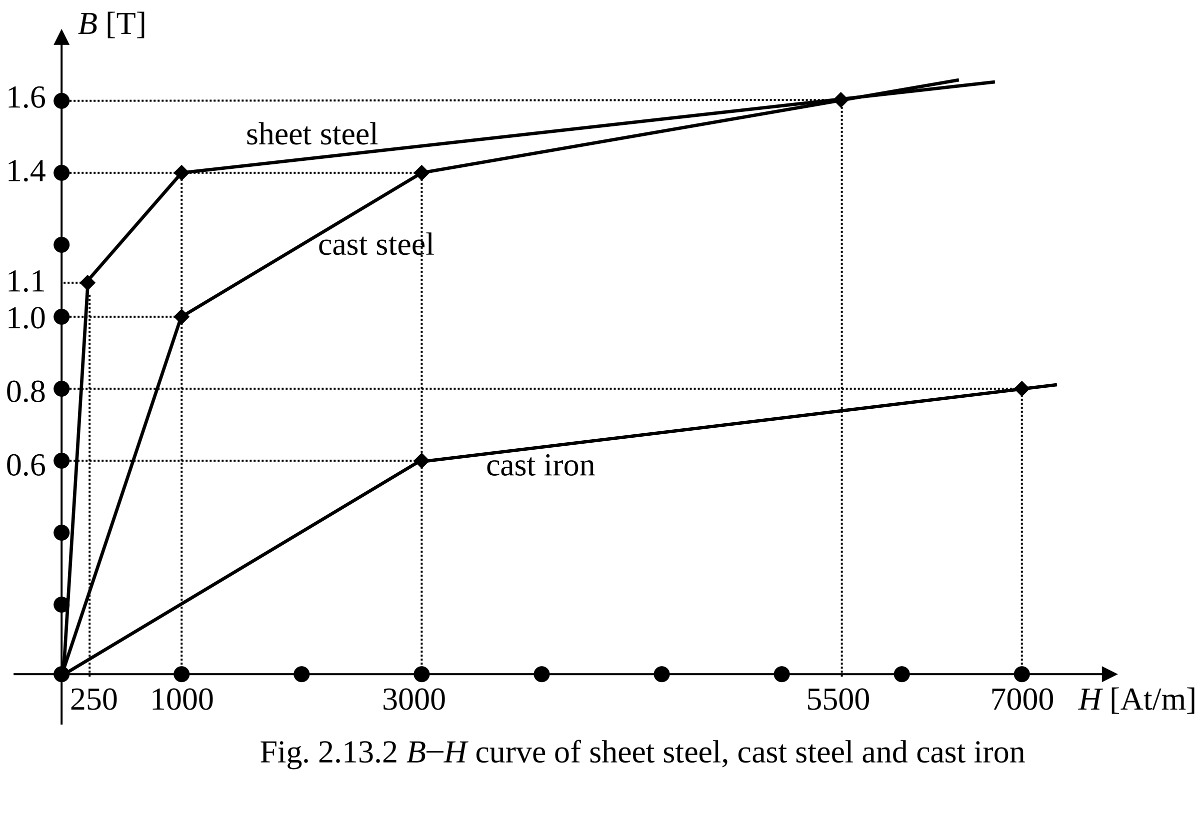


Fig. 2.13.2 *B*–*H* curve of sheet steel, cast steel and cast iron

In case of **ferromagnetics**, $B-H$ relationship is nonlinear. Magnetization curves of three most common ferromagnetics are presented in Fig. 2.13.2. Operating point should be located on the steepest segment, not on the saturation part of the characteristic. The higher is material permeability, designated by the steepest segment, the better are its magnetic properties. Then, sheet steel reveals the best magnetic properties. Its relative permeability is 3500 and it is seven times greater than the cast iron permeability, which equals 480. For comparison, an air core relative permeability is 1.

For the given magnetic flux flowing through an element its **magnetic voltage** can be defined:

$$U_m = Hl \tag{2.13.4}$$

Its unit is **ampere-turn** [At].

Then, a magnetic element of Fig 2.13.1 can be modeled by a nonlinear resistor, as presented in Fig. 2.13.3. Its $\Phi - U_m$ relationship can be found by rescaling the material $B - H$ curve, taking into account the element dimensions, cross section area S and mean magnetic length l, as shown in Fig. 2.13.1 and Fig. 2.13.6 for different cores.

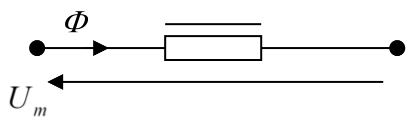


Fig. 2.13.3 Electrical model of magnetic element

For a linear (diamagnetic) element, from (2.13.3a):

$$BS = \Phi = \frac{\mu Hl}{l} S = \frac{\mu S}{l} U_m . \tag{2.13.5}$$

Then, magnetic element Ohm's law can be formulated:

$$\Phi = U_m / R_m \tag{2.13.6}$$

where,

$$R_m = \frac{l}{\mu_r \mu_0 S} \tag{2.13.7}$$

is the so called **magnetic resistance** or **reluctance**, in [At/Wb].

Magnetic field induced around a current carrying conductor is further considered. Fig. 2.13.4 presents such conductor perpendicularly crossing the plane, with conventional current direction, a) from the plane, b) to the plane. Direction of the magnetic field can be specified by the **right hand rule**.

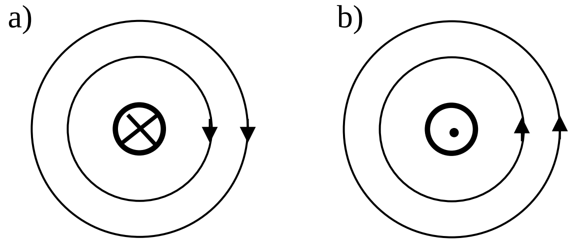


Fig. 2.13.4 Current carrying conductor perpendicularly crossing the plane

If a current carrying conductor is grasped in the right hand, with the thumb pointing in the direction of the conventional current, the fingers will then point in the direction of the magnetic lines of force.

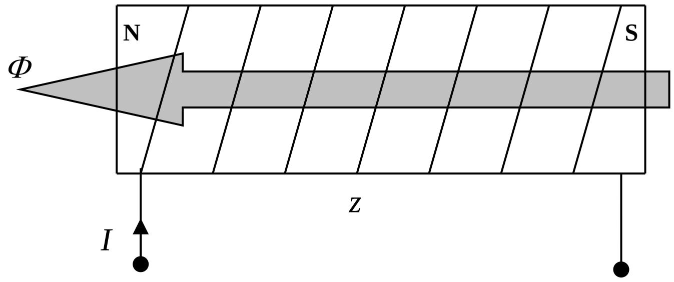


Fig. 2.13.5 Solenoid coil

Fig. 2.13.5 presents **electromagnet** or solenoid coil or simply **coil**, a wire wound around a core, with the total number of *z* turns. Such coil exhibits the magnetic field when energized. The value of flux that develops in a coil depends on the current *I* and the number of turns z.

The product of *I* and *z* is called the **magnetomotive force** (mmf):

$$F = Iz \tag{2.13.8}$$

Its unit is ampere-turn, [At] and it is an analog to emf.

A coil as an element of electric circuit is considered in next chapters. In this chapter analysis of magnetic circuit is considered. Electric model of such circuit is built of nonlinear resistors, mmf and eventually a linear resistor, if an air gap is present. A nonlinear resistor is described by an element $\Phi - U_m$ relationship, a linear resistor is described by Ohm's law (2.13.6), fluxes are related by KCL (2.13.9) and magnetic voltages are related by KVL (2.13.10).

The algebraic sum of magnetic fluxes entering or leaving arbitrary node *i* equals zero.

$$\sum_{\bullet i} \Phi = 0 \tag{2.13.9}$$

where, $\sum_{\bullet i}$ denotes algebraic sum of fluxes entering or leaving the node *i*.

The algebraic sum of magnetic voltages around arbitrary loop *i* equals zero,

$$\sum_{\text{O} i} U_m = 0\,, \tag{2.13.10}$$

where, $\sum_{\text{O} i}$ denotes algebraic sum of magnetic voltages around the loop *i*.

Then, methods of nonlinear circuit analysis, presented in Chapter 2.12, can be applied. Graphical method is preferred, due to simplicity of magnetic circuit, which is practically always a single-loop circuit or eventually two-loop circuit. Toroidal-core single-loop circuit and exemplary rectangular-core single-loop circuit and two-loop circuit are presented in Fig. 2.13.6a, b and c, respectively, together with their equivalent diagrams.

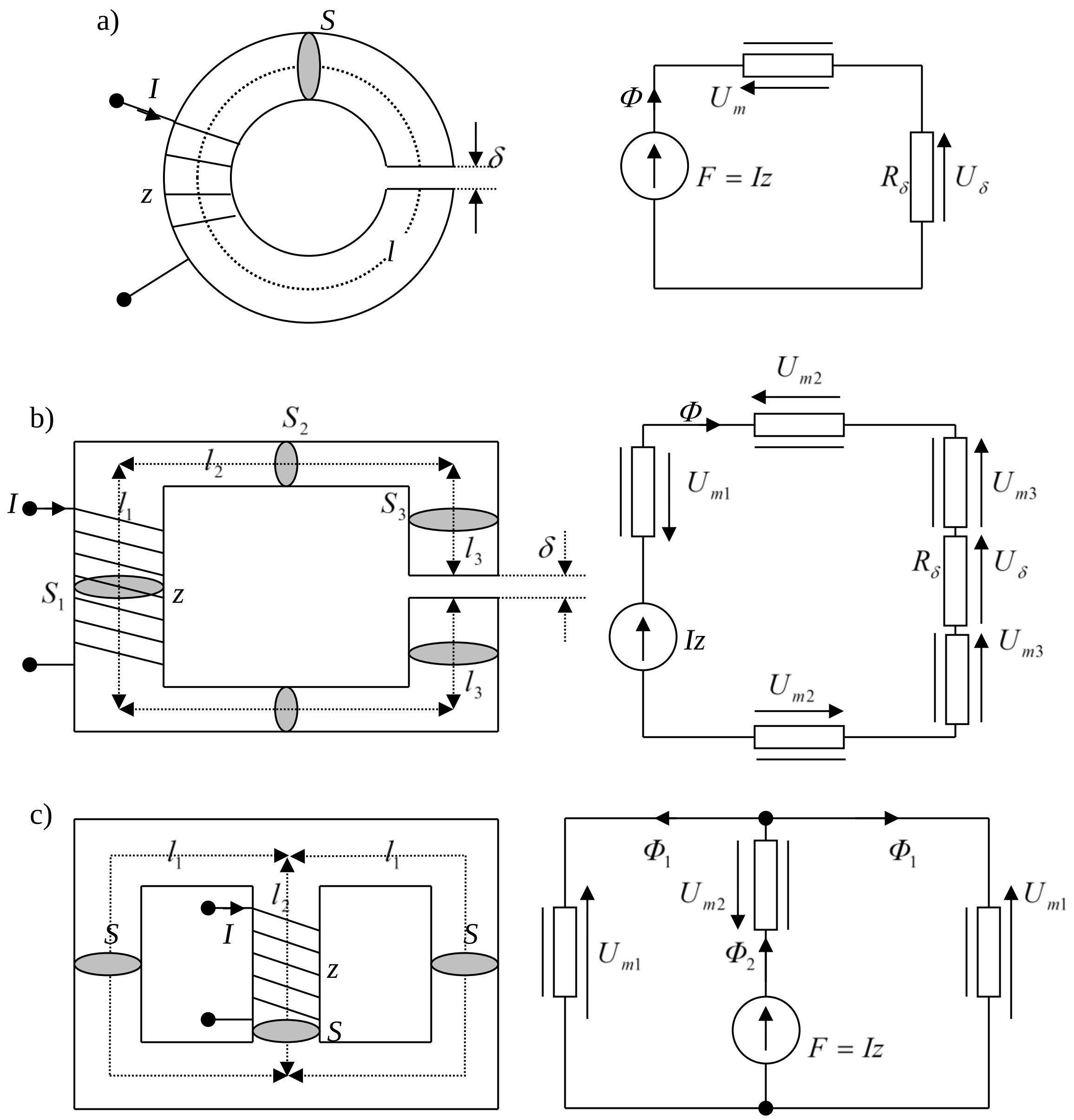


Fig. 2.13.6 Exemplary magnetic circuits and their electric models, a) toroidal core circuit, b) single-loop circuit, c) two-loop circuit

<u>Drill problems 2.13</u>

1. A coil of 200 turns is wrapped on a homogenous sheet-steel core having a cross section of $2\,\mathrm{cm}^2$ and mean length of 20 cm. If a flux of $2.5\cdot 10^{-4}$ Wb is developed in the core, what current must flow in a coil of $z = 100$ turns.

2. A cast iron core has a cross section of $0.5\,\mathrm{cm}^2$ and mean length of 10 cm. If a coil placed on the core develops 100 At, determine the flux produced in the core.

3. TRANSIENT ANALYSIS

Transient analysis is unsettled or temporary state of a circuit after throwing a switch or change in the applied voltage or current excitation.

Transient analysis is considered in the time period starting from the **initial time**, taken as $t_0 = 0_+ = 0$, and ending at the **steady state time** t_∞. Then, any transient response (circuit variable in a transient state) is characterized by time-domain function $y = y(t)$. Its boundary values are:

- **initial value**, $y(0) = Y_0$,
- **steady state value**, $y(\infty) = Y_\infty$.

A circuit transient analysis equations are integro-differential equations. These equations can be solved in the original time-domain or in operator-domain, after Laplace transformation. Definition and properties of the Laplace transform, together with transforms of the selected singularity functions and ordinary functions that describe circuit excitations and responses are presented in Appendix A.

At first, analysis of a transient state caused by changing topology of a circuit with time-invariant (dc) excitation, by opening or closing a switch or simply moving it from one position to the other, will be considered.

Then, methods of transient analysis in circuits with arbitrary aperiodic excitation will be discussed.

Before presenting methods of transient analysis, *i-u* relationships of circuit elements and Kirchhoff's laws in time-domain and operator-domain are presented.

3.1 KIRCHHOFF'S LAWS and PASSIVE ELEMENT LAWS

KIRCHHOFF'S LAWS

Kirchhoff's laws, discussed in Chapter 2.2 for constant values of currents and voltages, can be generalized into time-varying values.

Kirchhoff's Current Law

At any instant of time, the algebraic sum of currents entering or leaving arbitrary node or cutset equals zero:

$$\sum_{\bullet j} i = 0 . \qquad (3.1.1)$$

where,

$\sum_{\bullet j}$ denotes algebraic sum of instantaneous currents entering or leaving the j-th node (cutset), by the convention:

+ , if current arrowhead is directed to the node (cutset),
– , if current arrowhead is directed from the node (cutset).

Using the linearity rule (A3), in the s-domain KCL law becomes

$$\sum_{\bullet j} I(s) = 0 \,. \tag{3.1.1a}$$

Kirchhoff's Voltage Law

At any instant of time, the algebraic sum of voltages around arbitrary loop or closed path equals zero,

$$\sum_{\circ j} u = 0 \,, \tag{3.1.2}$$

where,

$\sum_{\circ j}$ denotes algebraic sum of voltages around the j-th loop (closed path), by the convention:

+ , if, voltage arrowhead has clockwise direction,
– , if voltage arrowhead has anticlockwise direction.

Using the linearity rule (A3), in the s-domain KCL law becomes

$$\sum_{\circ j} U(s) = 0 \,, \tag{3.1.2a}$$

PASSIVE ELEMENT LAWS

Ideal passive circuit elements are considered. These elements can be divided into two classes:

- Energy dissipating elements: resistors.
- Energy storage elements: capacitors and coils.

Resistor

A linear resistor presented in Fig. 3.1.1 is characterized by Ohm's law (3.1.3):

$$u = Ri \tag{3.1.3a}$$

$$i = Gu \tag{3.1.3b}$$

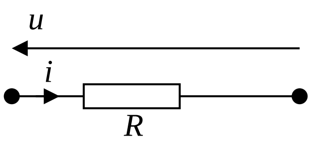


Fig. 3.1.1 Circuit symbol for a linear resistor

A resistor is uniquely characterized by its resistance *R*, in ohms [Ω]=[V/A], or conductance *G*, in siemens [S]=[A/V]. Resistance (conductance) is a circuit constant, constant of proportionality relating the current and the voltage.

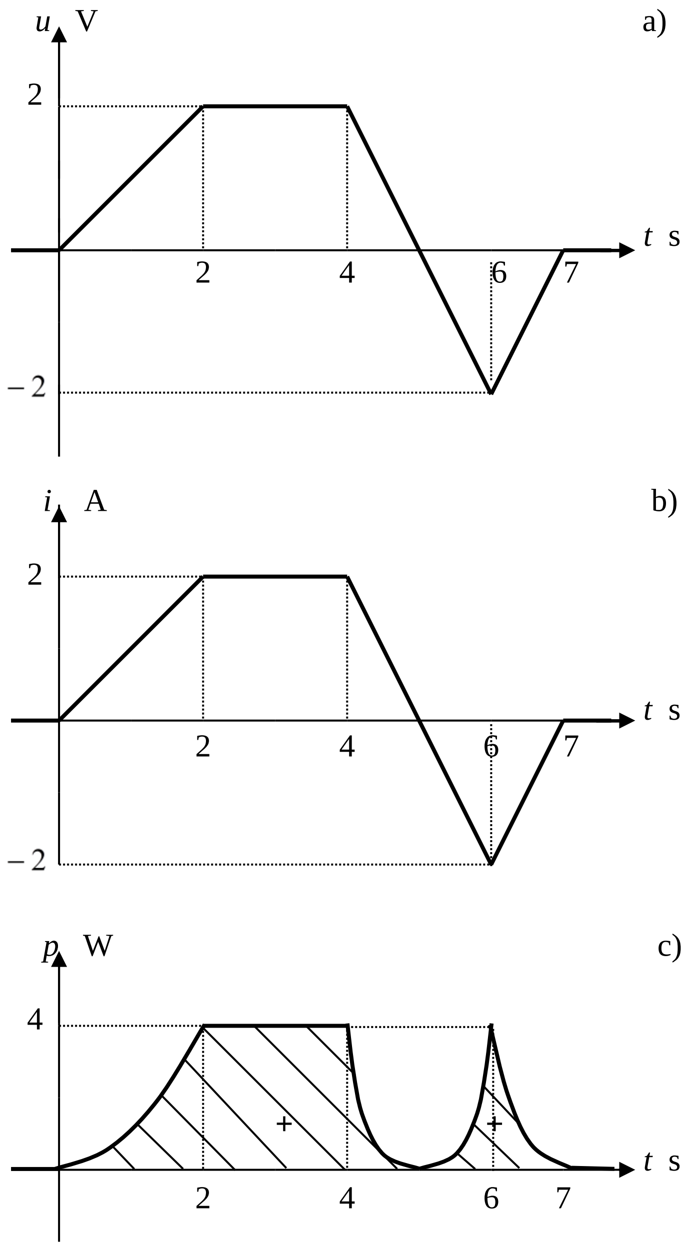


Fig. 3.1.2 Voltage, current and instantaneous power waveforms in a 1Ω resistor.

The resistor instantaneous power is:

$$p = i^2 R = u^2 G \geq 0 \tag{3.1.4}$$

Then, energy dissipated between the initial time $t_0 = 0$ and arbitrary time $t < \infty$ is always nonnegative

$$w(t) = w = R\int_0^t i^2 dt = G\int_0^t u^2 dt \tag{3.1.5}$$

The total energy dissipated is

$$W_\infty = R\int_0^\infty i^2 dt = G\int_0^\infty u^2 dt \tag{3.1.5a}$$

Fig. 3.1.2 presents exemplary plots of u, i and p in 1Ω resistor - the total energy dissipated is equal to the hatched area, $W_\infty = 38/3$ J.

Transforming both sides of Ohm's law (3.1.3) (linearity rule (A3) is utilized), yields

$$U(s) = RI(s) \tag{3.1.6a}$$

$$I(s) = GU(s) \tag{3.1.6b}$$

The s-domain equivalent of a resistor is presented in Fig. 3.1.3

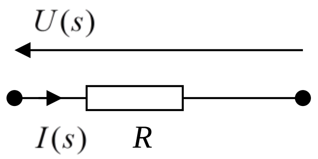


Fig. 3.1.3 s-domain equivalent of a resistor

Capacitor

A capacitor is an element that consists of two conducting bodies (plates) that are separated by a dielectric. A linear capacitor presented in Fig. 3.1.4 is characterized by $q-u$ relationship (3.1.7):

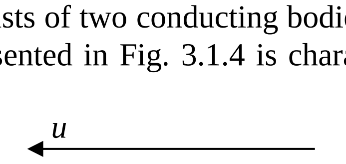


Fig. 3.1.4 Circuit symbol for a capacitor

$$q = Cu \tag{3.1.7}$$

A capacitor is uniquely characterized by its **capacitance** C, in **farads** [F]=[C/V]=[A·sec/V]. Capacitance is a circuit constant, constant of proportionality relating the charge and the voltage.

Differentiating (3.1.7), a capacitor $i-u$ relationship can be found:

$$i = C\frac{du}{dt} \tag{3.1.8a}$$

The voltage in terms of the current can be found by integrating both sides of (3.1.8a) between times $t_0 = 0$ and *t*:

$$u = \frac{1}{C}\int_0^t i dt + U_0 \tag{3.1.8b}$$

where

$$U_0 = u(0) = q(0)/C \tag{3.1.8c}$$

is the voltage on *C* at time $t_0 = 0$, the capacitor initial condition. The integral term in (3.1.8b) represents the voltage that accumulates on the capacitor in the interval from $t_0 = 0$ to *t*, whereas U_0 is that, which accumulates from $t = -\infty$ to t_0. The voltage $u(-\infty)$ is taken to be zero.

The **principle of conservation of charge** implies that the **voltage on a capacitor is always continuous**, may not change abruptly, even though the current may be discontinuous.

In particular, the voltage on a capacitor may not change abruptly at the inception of transient state, at $t = t_0 = 0$. If $t = 0_-$ is an instant of time just before $t = 0$, then

$$U_0 = u(0_-) \tag{3.1.8d}$$

what means that a capacitor initial condition is designated by its voltage just before inception of the transient state.

Capacitor is the energy storage element. The energy stored in the electric field between $t = -\infty$ and arbitrary time $t < \infty$ is

$$w = \int_{-\infty}^{t} iu\, dt = C\int_{-\infty}^{t} u\, du = \frac{Cu^2}{2}\bigg|_{-\infty}^{t} \tag{3.1.9}$$

As $u(-\infty) = 0$, then,

$$w = \frac{Cu^2}{2} \tag{3.1.9a}$$

It should be observed, that energy stored at time *t* is always nonnegative and it is designated by the capacitor constant and instantaneous value of voltage, the way of reaching this value is meaningless. Energy stored at the steady state condition is:

$$W = \frac{CU_\infty^2}{2} = const \tag{3.1.9b}$$

This energy can be fully recovered. Assume that voltage across a 1 F capacitor changes as presented in Fig. 3.1.2a. Then, the current and power waveforms are as presented in Fig. 3.1.5b and c.

The capacitor stores energy ($p > 0$), then gives it back ($p < 0$), stores again and gives back. In Fig. 3.1.5, energy stored is denoted by "+", energy given back is denoted by "–".

- $t = 0$: no energy is stored, $W_0 = 0$ J,
- $t \in (0,2)$: capacitor is charged, energy is absorbed,
- $t = 2$: $W_2 = 2$ J,
- $t \in (2,4)$: no flow of energy, $w = const = 2$ J,
- $t = 4$: $W_4 = 2$ J,
- $t \in (4,5)$: discharging of capacitor, energy is given back,
- $t = 5$: no energy is stored, $W_5 = 0$ J,
- $t \in (5,6)$: capacitor is charged, energy is absorbed,
- $t = 6$: $W_6 = 2$ J,
- $t \in (6,7)$: discharging of capacitor, energy is given back,
- $t \geq 7$: no energy is stored, $w = 0$ J.

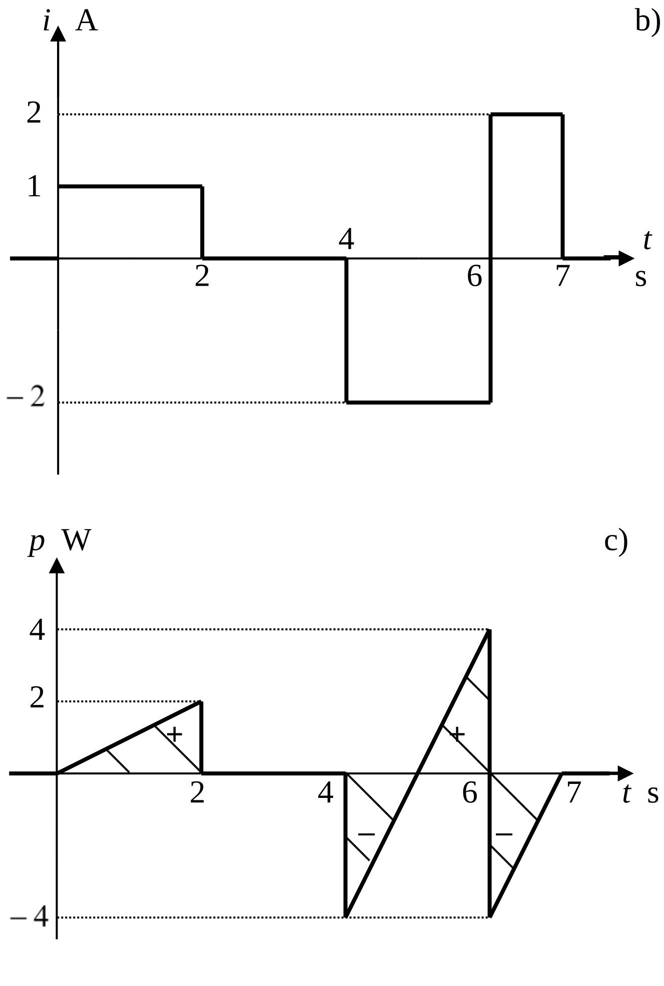


Fig. 3.1.5 Current and power waveforms in a 1 F capacitor for Fig. 3.1.2a voltage waveform

Transforming both sides of capacitor law (3.1.8a) (linearity rule (A3) and differentiation rule (A5) are utilized), yields

$$I(s) = sCU(s) - CU_0 \tag{3.1.10a}$$

Solving this equation for $U(s)$ or applying integration rule (A4) to (3.1.8b), yields

$$U(s) = \frac{1}{sC} I(s) + \frac{U_0}{s} \tag{3.1.10b}$$

Then, based on Kirchhoff's equations, capacitor *s*-domain equivalents can be found. They are presented in Fig. 3.1.6.

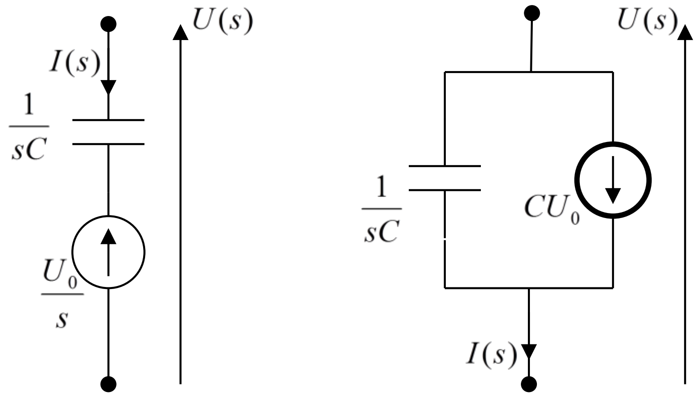


Fig. 3.1.6 *s*-domain equivalents of a capacitor

Coil (Inductor)

A coil or inductor is an element that consists of a coiled conducting wire around a core. A coil with toroidal core is presented in Fig. 3.1.7, together with its electric analog.

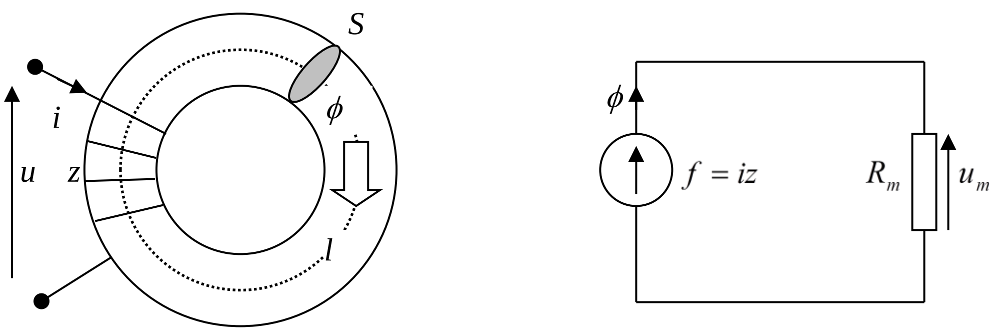


Fig. 3.1.7 Toroidal core coil and its electric analog

A current flowing through the coil produces a magnetic flux ϕ. The **total flux** linked by the z turns of the coil, denoted by ϕ_t, is

$$\phi_t = z\phi \tag{3.1.11}$$

This total flux is commonly referred to as the **flux linkage**.

A linear core characterized by the linear $B-H$ relationship (2.13.3a) is considered. Then, for the given dimensions, the core magnetic resistance (2.13.7) can be designated. Finally, taking into account electric analogies, Ohm's law and KVL, the total flux can be expressed by the coil current. For the core presented in Fig. 3.1.7, this flux is

$$\phi_t = \frac{iz^2}{R_m} = i\frac{z^2\mu_r\mu_o S}{l} \qquad (3.1.12)$$

Then, constant of proportionality relating the total flux and the current, a circuit constant characterizing uniquely the coil, can be introduced. This constant *L* is called the **inductance**. Its unit is **henry** [H]=[Wb/A]=[V·s/A].

$$\phi_t = Li \qquad (3.1.13)$$

In general, inductance is directly proportional to the square of the number of turns and core permeability, then it is proportional to the core dimensions. For the core presented in Fig. 3.1.7, the inductance is

$$L = \frac{z^2\mu_r\mu_o S}{l} \qquad (3.1.14)$$

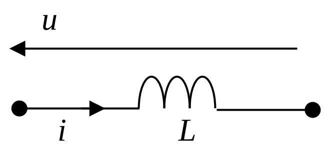


Fig. 3.1.8 Circuit symbol for a coil

Circuit symbol for a coil is presented in Fig. 3.1.8. To find a coil $i-u$ relationship, Faraday's law should be recalled.

When the magnetic flux linking a coil changes, a voltage directly proportional to the rate of flux change is induced in a coil:

$$u = z\frac{d\phi}{dt} = \frac{d\phi_t}{dt} \qquad (3.1.15)$$

Faraday's law (3.1.15) with (3.1.13) yields a coil $i-u$ relationship:

$$u = L\frac{di}{dt} \qquad (3.1.16a)$$

The current in terms of the voltage can be found by integrating both sides of (3.1.16a) between times $t_0 = 0$ and *t*:

$$i = \frac{1}{L}\int_0^t u dt + I_0 \qquad (3.1.16b)$$

where

$$I_0 = i(0) = \phi_t(0)/L \qquad (3.1.16c)$$

is the current at time $t_0 = 0$, the coil initial condition.

The **principle of conservation of flux** implies that the **current through a coil is always continuous**, may not change abruptly, even though the voltage may be discontinuous.

In particular, the current may not change abruptly at the inception of transient state, at $t = t_0 = 0$. If $t = 0_-$ is an instant of time just before $t = 0$, then

$$I_0 = i(0_-) \tag{3.1.16d}$$

what means that a coil initial condition is designated by its current just before inception of the transient state.

Coil is the energy storage element. The energy stored in the magnetic field between $t = -\infty$ and arbitrary time $t < \infty$ is

$$w = \int_{-\infty}^{t} iu\, dt = L\int_{-\infty}^{t} i\, di = \frac{Li^2}{2}\Bigg|_{-\infty}^{t} \tag{3.1.17}$$

As $i(-\infty) = 0$, then,

$$w = \frac{Li^2}{2} \tag{3.1.17a}$$

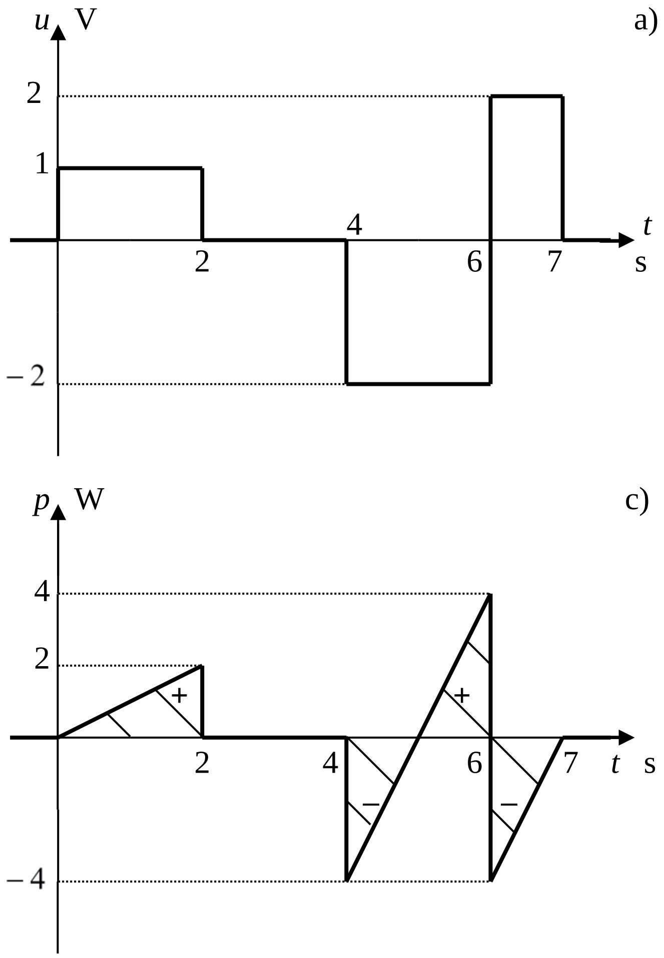


Fig. 3.1.9 Voltage and power waveforms in a 1 H coil and Fig. 3.1.2b current waveform

It should be observed, that energy stored at time t is always nonnegative and it is designated by the coil constant and instantaneous value of current, the way of reaching this value is meaningless. Energy stored at the steady state condition is:

$$W_\infty = \frac{LI_\infty^2}{2} = const \tag{3.1.17b}$$

This energy can be fully recovered. Assume that the current through a 1 H coil changes as presented in Fig. 3.1.2b. Then, the voltage and power waveforms are as presented in Fig. 3.1.9a and c.

The coil stores energy ($p > 0$), then gives it back ($p < 0$), stores again and gives back. In Fig. 3.1.9, energy stored is denoted by "+", energy given back is denoted by "–".

- $t = 0$: no energy is stored, $W_0 = 0$ J,
- $t \in (0,2)$: energy is absorbed,
- $t = 2$: $W_2 = 2$ J,
- $t \in (2,4)$: no flow of energy, $w = const = 2$ J,
- $t = 4$: $W_4 = 2$ J,
- $t \in (4,5)$: energy is given back,
- $t = 5$: no energy is stored, $W_5 = 0$ J,
- $t \in (5,6)$: energy is absorbed,
- $t = 6$: $W_6 = 2$ J,
- $t \in (6,7)$: energy is given back,
- $t \geq 7$: no energy is stored, $w = 0$ J.

Transforming both sides of coil law (3.1.16a) (linearity rule (A3) and differentiation rule (A5) are utilized), yields

$$U(s) = sLI(s) - LI_0 \tag{3.1.18a}$$

Solving this equation for $I(s)$ or applying integration rule (A4) to (3.1.16b), yields

$$I(s) = \frac{1}{sL}U(s) + \frac{I_0}{s} \tag{3.1.18b}$$

Then, based on Kirchhoff's equations, capacitor s-domain equivalents can be found. They are presented in Fig. 3.1.10.

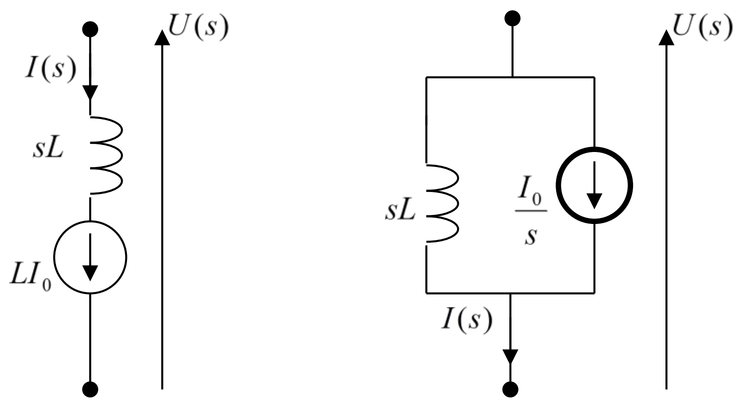


Fig. 3.1.10 s-domain equivalents of a coil

Passive elements – Summary

The three passive elements: resistor, capacitor and coil, are characterized by three circuit constants: resistance *R*, capacitance *C* and inductance *L*, and described by four circuit variables: voltage, current, charge and total flux. Graph depicted in Fig. 3.1.11 presents relationships between these variables. As should be observed, only total flux and charge are not related.

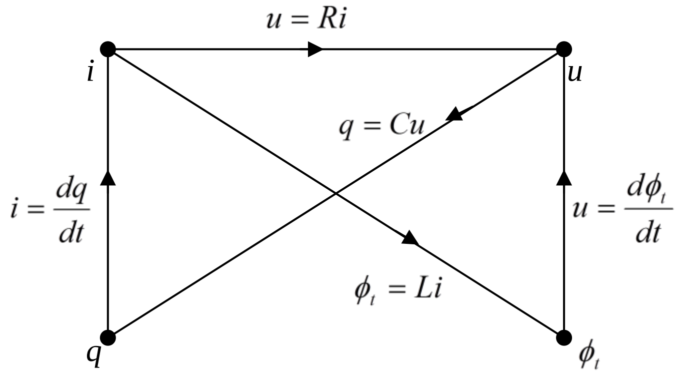


Fig. 3.1.11 Graph representation of relationships between four circuit variables

It has been assumed that all elements are ideal. Practical capacitor and coil are discussed in Chapter 4, their circuit models built of ideal elements are presented.

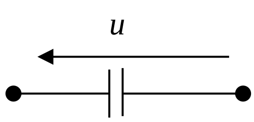
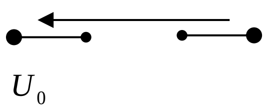
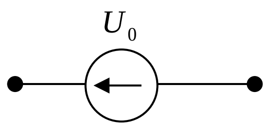
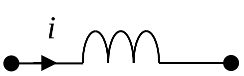
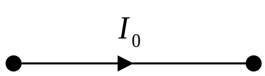
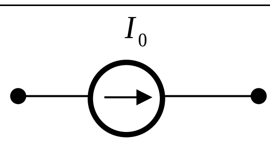
Coil and Capacitor boundary behavior

Taking into account $i-u$ relationships of energy storage elements and flux or charge conservation principle, their boundary behavior can be analyzed.

- At the initial time, $t = t_0 = 0$, the capacitor voltage is equal to the initial condition (3.1.8c), the coil current is equal to the initial condition (3.1.16c), what results from the charge or flux preservation principle. Then, at this instant of time, capacitor may be replaced by the dc voltage source U_0, coil may be replaced by the dc current source I_0 and the dc analysis can be performed, to find all other circuit variables.
- At the steady state, at $t = 0_-$ or $t = \infty$, all circuit variables are constant, including coil currents and capacitor voltages. Then, the coil voltage is zero and it can be replaced by the short circuit, the capacitor current is zero and it can be replaced by the open circuit.

Table 3.1.1

Capacitor and coil models at boundary conditions

Element	$t = 0_-$	$t = 0$	$t = \infty$
			
			

Coil and capacitor models at boundary conditions are presented in Table 3.1.1.
Knowledge of the circuit order and the response boundary values allow to predict the general form of the response. In the 1st order circuit this knowledge, together with knowledge of the time constant, allows to give the exact solution. This approach is discussed in the next section of this Chapter.

All three *s*-domain element equations, (3.1.6), (3.1.10) and (3.1.18), contain a term that relates the voltage $U(s)$ and the current $I(s)$. Energy storage element equations also contain a term designated by the element initial condition. The factor of proportionality between the voltage and the current in the first term is the element impedance $Z(s)$. **Impedance** is defined as the ratio of $U(s)$ to $I(s)$ when initial condition $X_0 = U_0$ or I_0 is zero.

$$Z(s) = U(s)/I(s)\big|_{X_0=0} \tag{3.1.19a}$$

The reciprocal of the impedance is called the **admittance** $Y(s)$

$$Y(s) = 1/Z(s) = I(s)/U(s)\big|_{X_0=0} \tag{3.1.19b}$$

Impedances and admittances of the three elements are presented in Table 3.1.2

Table 3.1.2 Impedances and admittances of passive elements

Element	Impedance $Z(s)$	Admittance $Y(s)$
Resistor	R	$G = 1/R$
Capacitor	$1/sC$	sC
Coil	sL	$1/sL$

A resistor equivalent consists only of the impedance *R*. The energy storage element equivalent consists of a pure impedance connected in series with an initial condition designated voltage source or a pure admittance connected in parallel with an initial condition designated current source.

For zero initial conditions and the introduced term of impedance/admittance, an element equations in the *s*-domain can be presented in the generalized form

$$U(s) = Z(s)I(s) \tag{3.1.20a}$$

$$I(s) = Y(s)U(s) \tag{3.1.20b}$$

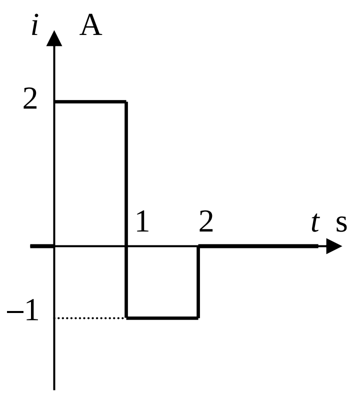
also called Ohm's low in *s*-domain.

Term of impedance/admittance can be generalized on arbitrary two-terminal passive circuit. Its equivalent impedance or admittance can be found, in the same way as equivalent resistance or conductance in the dc circuit. Such impedance or admittance is a complex function of *s*. Homogenous circuits are exceptions – any combination of resistors can be replaced by a single resistor, any combination of capacitors can be replaced by a single capacitor and any combination of inductors can be replaced by a single inductor.

Drill problems 3.1

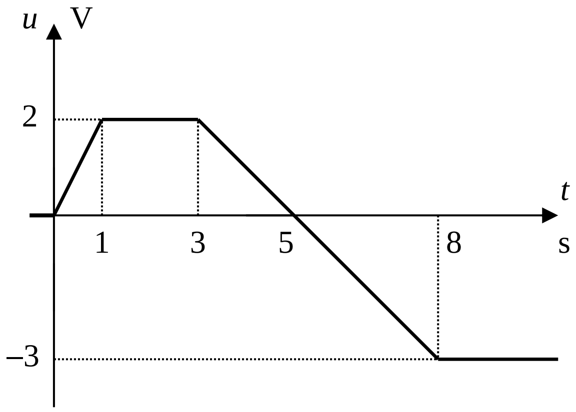
1. Find equivalent capacitance of two capacitors connected in series (parallel).
2. Write down (with reasoning) the dc voltage divider equations for two capacitors connected in series.
3. Find equivalent inductance of two coils connected in series (parallel).
4. Write down (with reasoning) the dc current divider equations for two inductors connected in parallel.
5. A coil of 200 turns of wire is wound on a steel core having a mean length of 0.1 m and a cross section of 4 10^{-4} m^2. The relative permeability at the rated current of the coil is 1000. Determine the inductance of the coil.
6. Constant current of 5 mA produces flux of $2.5 \cdot 10^{-4}$ Wb in a coil of 200 turns. What energy is stored in this coil.
7. Constant current of 5 mA charges $1\mu\text{F}$ capacitor for 10 seconds. What energy is stored after this period of time (energy initially stored is zero).
8. A coil of 200 turns is wrapped on a sheet-steel core ($B-H$ curve - Fig. 2.9.2) having a cross section of 2 cm^2 and a mean length of 20 cm. If a flux of $2.5 \cdot 10^{-4}$ Wb is developed in a core, what current must flow in the coil ?
9. A cast iron-core ($B-H$ curve - Fig. 2.9.2) has a cross section of 0.5 cm^2 and a mean length of 10 cm. If a coil placed on the core develops 100 A_t, determine the flux produced in the core. Find cast iron permeabilities and inductances for z=10^4 turns (I=10 mA).
10. What constant current is required to charge a $2\mu\text{F}$ from 0 to 5 V in 2 ms.
11. A constant current of 10 mA is charging a $2\mu\text{F}$ capacitor. If the capacitor initial voltage is zero, find the charge, voltage and energy stored after 10 ms.
12. Two capacitors, 10 and $40\mu\text{F}$, are connected in series to a 100 V source. What energy is stored in each ? What charge is stored in each ?
13. The given current flows through 1F capacitor. Calculate the maximum energy stored and the total energy stored, if the initially stored energy is zero.

Fig. P.3.1.13



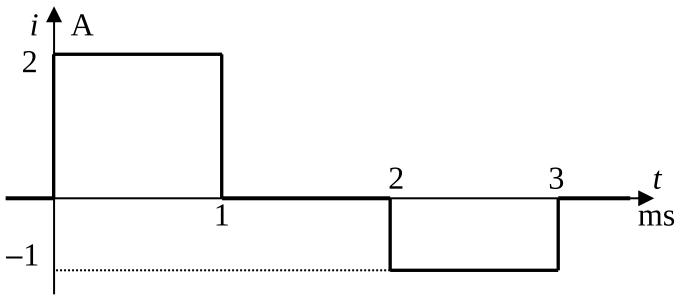
14. If the voltage across an $1\mu\text{F}$ capacitor changes as shown, plot its current, designate the maximum energy stored.

Fig. P.3.1.14



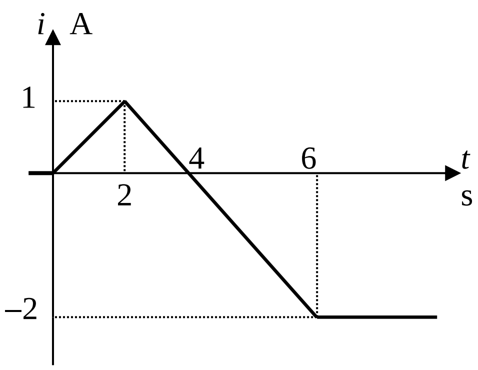
15. For the given current that flows through an $1\mu\text{F}$ capacitor, plot the corresponding voltage (assume $U_0 = 0$).

Fig. P.3.1.15

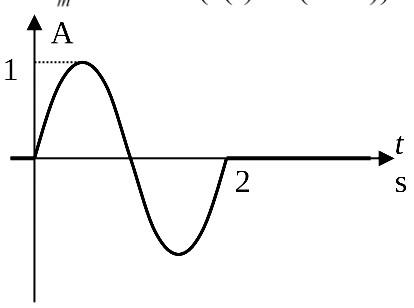


16. If the current in an 0.1 H coil changes as shown, plot voltage across the coil and designate the maximum energy stored.

Fig. P.3.1.16

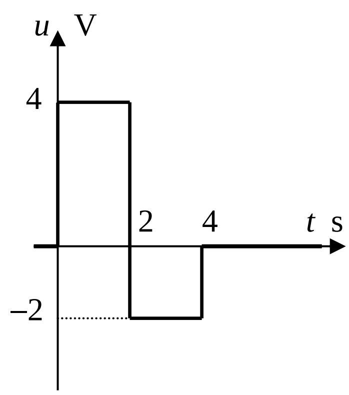


$i = I_m \sin 2\pi / T(\mathbf{1}(t) - \mathbf{1}(t - T))$



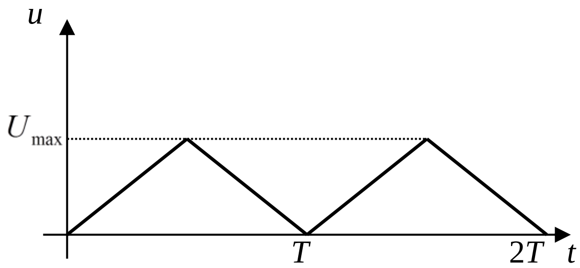
17. Find the current *i* in a 0.5 H inductor if $i(0) = 0$ and the voltage is as shown. What is the maximum energy stored and at what time ?

Fig. P.3.1.17



18. The voltage across a $1\mu\text{F}$ capacitor is the triangular waveform. Draw the current waveform. What is the maximum energy stored and at what times ? Assume: $T = 2\,\text{ms}$, $U_{max} = 10\,\text{V}$.

Fig. P.3.1.18



20. Given a coil of 10 turns and 0.5 mH. Find the inductance after adding/subtracting of 5 turns.

3.2 TRANSIENT ANALYSIS in CIRCUITS with STEP EXCITATION

The transient response caused by changing topology of a circuit with time-invariant (dc) excitation, by moving a switch at $t = 0$, is considered at first. It is generally assumed that at $t = 0_-$, all circuit variables (currents and voltages) are zero or/and constant.

In general, such response may be sum of two components:

- natural response or zero-input response,
- forced response or zero-state response.

Natural response or **zero-input response**

$$y_n(t) = y_n \tag{3.2.1a}$$

is the result of initial capacitive and/or inductive energy stored within a circuit.

Forced response or **zero-state response**

$$y_f(t) = y_f \tag{3.2.1b}$$

is the result of excitation, independent sources acting within a circuit.

Then, the total response, so called **complete response**

$$y = y_n + y_f \tag{3.2.1}$$

is a superposition of the initial condition response with all independent sources zeroed and the response to independent sources with the initial conditions zeroed. Block diagram interpretation of this strategy is presented in Fig. 3.2.1, where x is the excitation, x_0 is the initial condition.

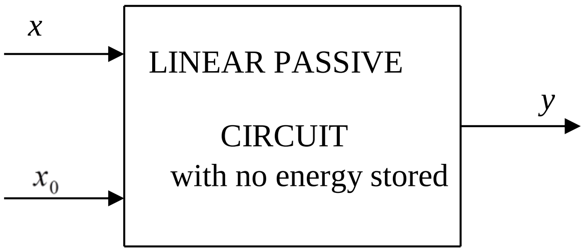


Fig. 3.2.1 Block diagram of single output (transient response) double input (excitation + initial condition) circuit

A forced response with zero initial conditions will be considered at first, $x_0 = 0$. Analysis of the 1st order circuits, and then, analysis of the 2nd order circuits will be discussed in details.

Next, natural response with excitation being switched off, i.e. no forced response case, $x = 0$, will be considered.

Finally, the general case, with both responses present will be discussed.

It is assumed, that a circuit to be analyzed is modeled by its diagram, built of ideal elements. Then, taking into account element $i-u$ relationships (3.1.3), (3.1.8), (3.1.16) and Kirchhoff's laws (3.1.1), (3.1.2), the circuit can be described by the system of differential or integro-differential equations. This system may be solved

- in time-domain or
- Laplace transforms may be used.

In the latter case, the equation or equations are first Laplace transformed, and then, solved by straightforward algebraic means. The inverse transform of the solution is the last step of circuit transient analysis.

The order of the highest-order derivative of differential equations describing a circuit, denoted by *n*, determines the **circuit order**. The *n*-th order circuit can be also identified by the presence of *n* energy storage elements (after series-parallel simplification of homogeneous two-terminal subcircuit(s) built of coils or capacitors, if present).

FORCED RESPONSE

After throwing a switch, the dc voltage source *E* may be described by the unit step function (3.2.3a) and the current source *J* may be described by the unit step function (3.2.3b).

$$e(t) = e = E\mathbf{1}(t) \tag{3.2.3a}$$

$$j(t) = j = J\mathbf{1}(t) \tag{3.2.3b}$$

Three different methods of transient analysis can be distinguished:

- Time-domain method,
- Laplace Transform or Operator method,
- Method based on boundary values determination.

First two are applicable to both 1^{st} order circuit and higher-order circuit, the last one is applicable only to the 1^{st} order circuit. Use of all three methods will be presented on exemplary circuits.

<u>1^{st} order circuit – time-domain method</u>

<u>Example 3.2.1</u>

The simple one-loop *RL* circuit is presented in Fig. 3.2.2. Find the coil current and voltage after closing the switch, by means of the time-domain method.

÷

There is no energy initially stored in the coil, $I_0 = 0$. Then, only forced response should be considered. The dc circuits at boundary, initial and steady state, conditions are presented in Fig. 3.2.3. The boundary values of the circuit responses are collected in Table 3.2.1.

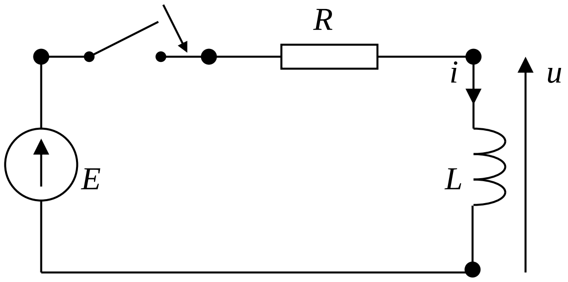


Fig. 3.2.2 *RL* circuit for Example 3.2.1

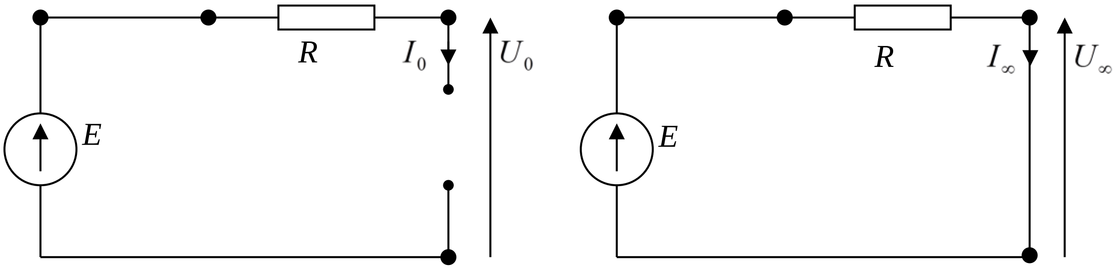


Fig. 3.2.3 Example 3.2.1 circuit at boundary conditions

Table 3.2.1

Boundary conditions for Example 3.2.1

	$t=0$	$t=\infty$
I_t	0	E/R
U_t	E	0

For $t \geq 0$, the circuit equations are

$$u + u_R = E \tag{3.2.4a}$$

$$u = L\frac{di}{dt} \tag{3.2.4b}$$

$$u_R = Ri \tag{3.2.4c}$$

From these equations, the circuit equation (KVL equation) can be formulated

$$L\frac{di}{dt} + Ri = E \tag{3.2.5}$$

The 1st order equation has been obtained, its solution consists of two components:

- Particular solution, the steady state response (3.2.6a)

$$I_\infty = E/R\,. \tag{3.2.6a}$$

- Solution of the homogeneous equation (3.2.5a), the transient exponential response (3.2.6b).

$$\frac{di}{dt} + \alpha i = 0 \quad , \ \alpha = R/L = 1/T \tag{3.2.5a}$$

$$i_{\approx} = B\exp(-\alpha t) = B\exp(-t/T) \tag{3.2.6b}$$

Then, the total solution is

$$i = I_{\infty} + i_{\approx} = E/R + B\exp(-t/T) \tag{3.2.6c}$$

Constant B can be calculated from the second boundary condition

$$I_0 = E/R + B \Rightarrow B = -E/R \tag{3.2.6d}$$

and the final obtained solution of (3.2.5) is

$$i = \frac{E}{R} - \frac{E}{R}\exp(-t/T) \tag{3.2.7}$$

where,

$$T = L/R \tag{3.2.8}$$

is the **time constant for the *RL* circuit**.

The coil voltage can be calculated from (3.2.4b)

$$u = E\exp(-t/T) \tag{3.2.9}$$

Please note, that both responses (3.2.7) and (3.2.9) start at $t = 0$. They are not multiplied by the unit step for simplicity of description. The responses are graphed in Fig. 3.2.4.

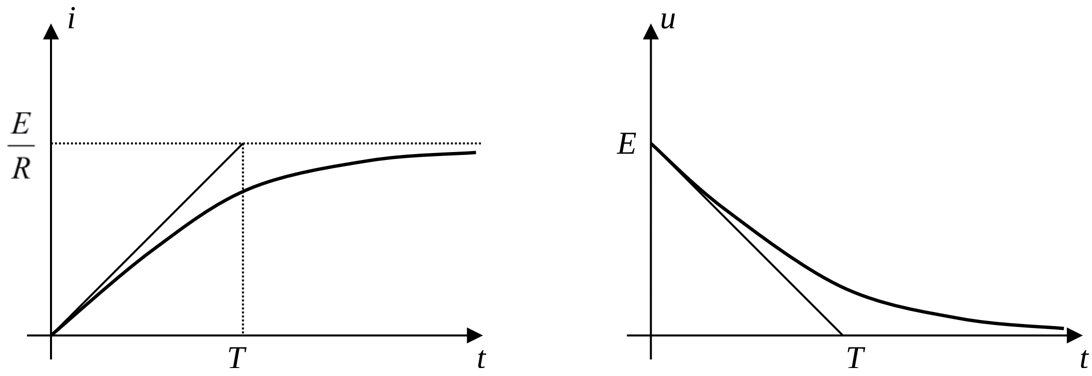


Fig. 3.2.4 Responses for Example 3.2.1

From (3.1.17b), the total energy stored is

$$W_{\infty} = \frac{L(E/R)^2}{2} \tag{3.2.10a}$$

The total energy supplied/dissipated at $t \geq t_{\infty}$ is

$$w \cong \frac{E^2}{R}t \tag{3.2.10b}$$

♦

1st order circuit – *s*-domain method

The following algorithm of the *s*-domain method can be formulated.

Algorithm 3.2.1 – *s*-domain method

Step 1.	Predict the response(s): a) evaluate the circuit order, b) designate initial condition(s), if present, c) designate boundary values Y_0, Y_∞. To find initial condition(s) and boundary values, perform dc analysis three times with each energy storage element replaced by a short-circuit or an open-circuit or an ideal source, as presented in Table 3.1.1. In the zero initial condition case: dc analysis at $t = 0_-$ is omitted, at $t = 0$, coil is replaced by an open-circuit and capacitor by a short-circuit.
Step 2.	Build the circuit diagram at *s*-domain, formulate the analysis equations – generalized Kirchhoff's analysis or nodal analysis can be utilized.
Step 3.	Solve the equations to find the response in *s*-domain, $Y(s)$.
Step 4	Find the inverse transformation $y(t) = y$ – dictionary approach or Heaviside's formula can be utilized.
Step 5.	Plot the response(s). Check whether the obtained boundary values match the predicted ones.

Example 3.2.2

The simple one-loop *RC* circuit is presented in Fig. 3.2.5. Find the capacitor current and voltage after closing the switch, by means of the *s*-domain method. Then, find total energy supplied, stored and dissipated.

÷

There is no energy initially stored in the capacitor, $U_0 = 0$. Then, only forced response should be considered. The dc circuits at boundary, initial and steady state, conditions are presented in Fig. 3.2.6. The boundary values of the circuit responses are collected in Table 3.2.2.

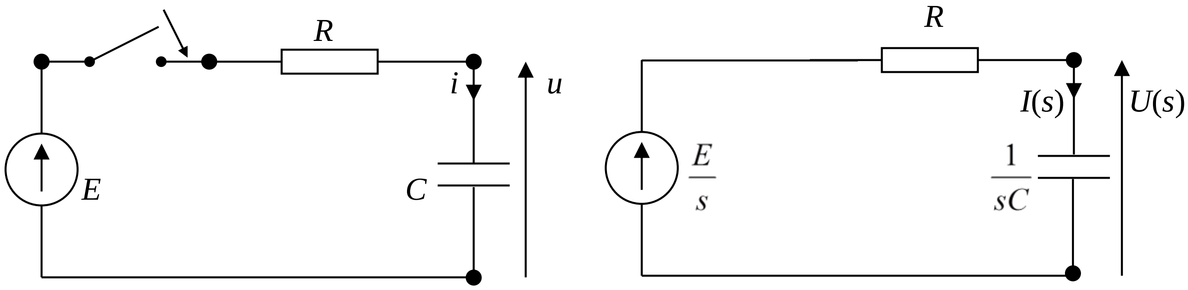


Fig. 3.2.5 *RC* circuit for Example 3.2.2 in time-domain and *s*-domain

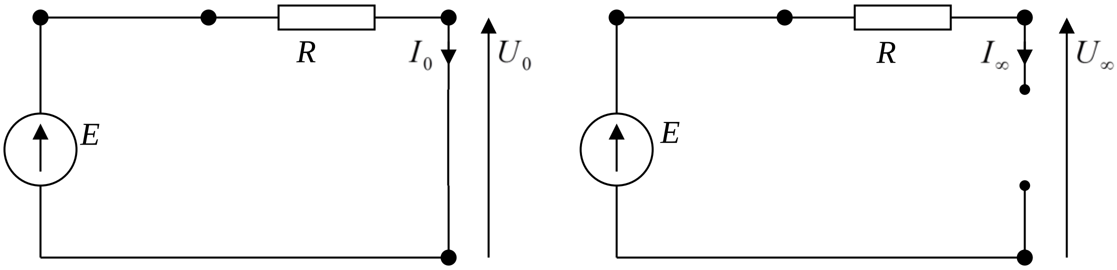


Fig. 3.2.6 Example 3.2.2 circuit at boundary conditions

Table 3.2.2
Boundary conditions for Example 3.2.2

	$t=0$	$t=\infty$
I_t	E/R	0
U_t	0	E

From Fig. 3.2.5 circuit, the mesh current is

$$I(s)=\frac{\frac{E}{s}}{R+\frac{1}{sC}}=\frac{E}{R}\frac{T}{1+sT} \tag{3.2.11}$$

The current inverse transform can be found in the dictionary (A10a):

$$i=\frac{E}{R}\exp(-t/T) \tag{3.2.12}$$

where

$$T=RC \tag{3.2.13}$$

is the **time constant for the *RC* circuit**.

The capacitor voltage is

$$u=\frac{1}{C}\int_0^t idt=E-E\exp(-t/T) \tag{3.2.14}$$

After closing the switch, the capacitor is charged, through the resistance *R*, the value equal to the source value of E is reached after $t_\infty=5T=5RC$. The smaller is the resistance, the shorter is the charging period $t\in<0,t_\infty>$, the larger is the current initial value $I_0=E/R$.

Large resistance case and *small resistance* case, at $C=const$, are depicted in Fig. 3.2.7.

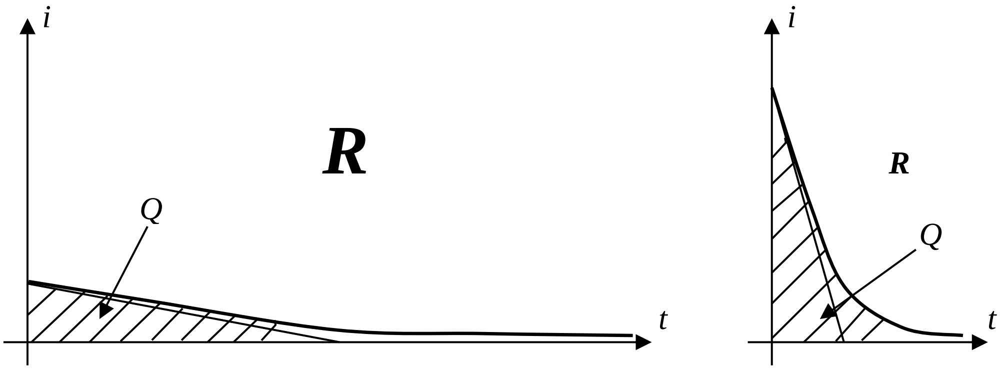


Fig. 3.2.7 *RC* circuit response, *large resistance* and *small resistance* case

In the ideal case, when $R = 0$, then

$$I(s) = \frac{\frac{E}{s}}{\frac{1}{sC}} = EC \tag{3.2.11a}$$

and the time-domain response is the impulse

$$i = EC\delta(t) \ , \tag{3.2.12a}$$

an infinitely tall and infinitely narrow pulse of *EC* area.

Regardless the value of resistance, total charge that flows in the circuit (hatched area) is the same

$$Q = \int_0^\infty i\,dt = EC = const \tag{3.2.15}$$

Total energy supplied by the source is:

$$W_{E\infty} = E\int_0^\infty i\,dt = E^2C \tag{3.2.16}$$

Half of this energy, total energy stored on the capacitor, can be fully recovered. From (3.1.9b), this energy is:

$$W_\infty = \frac{CE^2}{2} \tag{3.2.16a}$$

The other half is dissipated as a heat, on the resistor:

$$W_{R\infty} = R\int_0^\infty i^2\,dt = \frac{CE^2}{2} \tag{3.2.16b}$$

♦

1^{st} order circuit – boundary values based method

Single energy storage element may be extracted from the 1^{st} order circuit, and then such circuit can be considered as a resistive subcircuit loaded with an energy storage element. Thevenin's theorem can be utilized and an arbitrary 1^{st} order circuit can be reduced to the one-loop *RL* or *RC* circuit. The circuits described in the *s*-domain are presented in Fig. 3.2.8, where, R_t is the Thevenin resistance - the resistance seen from terminals of the energy storage element after deactivating all sources.

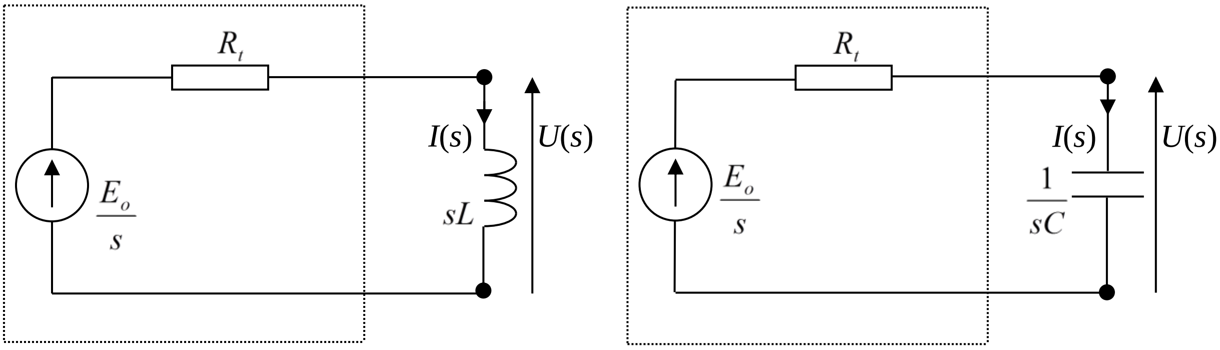


Fig. 3.2.8 The 1^{st} order circuit reduced to one-loop circuit

The obtained circuits have been already considered (Examples 3.2.1 and 3.2.2).Their time constants are given by equations (3.2.8) and (3.2.13).

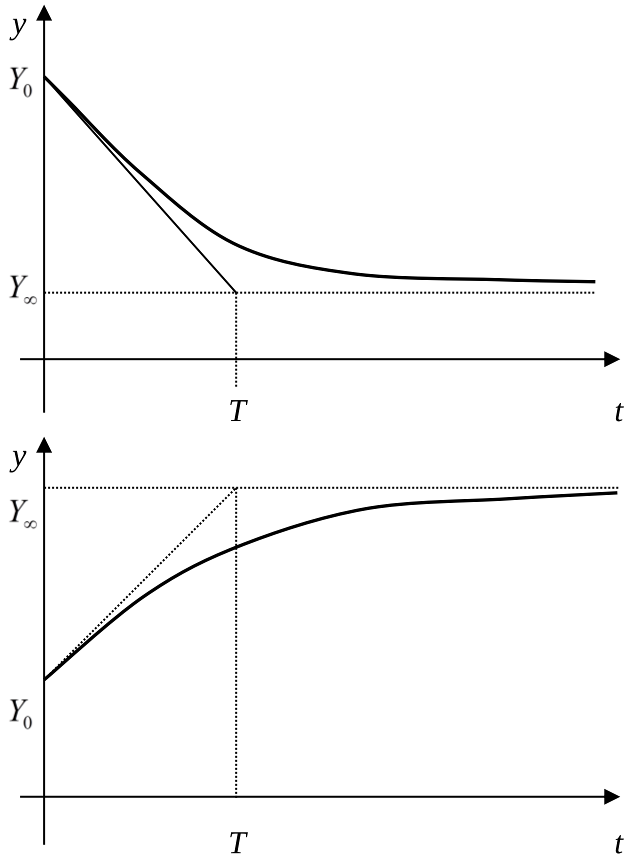


Fig. 3.2.9 Decaying and rising exponential responses

The 1st order circuit arbitrary response $y(t) = y$, voltage or current, is the exponential function (A7b) uniquely described by the time constant *T* and boundary values Y_0, Y_∞:

$$y = Y_\infty + (Y_0 - Y_\infty)\exp(-t/T)\,,\quad t \ge 0 \qquad (3.2.17)$$

For $Y_0 > Y_\infty$, decaying function is obtained, for $Y_0 < Y_\infty$, the rising one, as presented in Fig. 3.2.9. The special case, $Y_\infty = Y_0$ is not considered. Then, algorithm of the boundary values method can be formulated.

Algorithm 3.2.2 - Boundary values based method

Step 1. Predict the response(s):
- a) evaluate the circuit order,
- b) designate initial condition(s), if present,
- c) designate boundary values Y_0, Y_∞.

To find initial condition(s) and boundary values, perform dc analysis three times with energy storage element replaced by a short-circuit or an open-circuit or an ideal source, as presented in Table 3.1.1.
In the zero initial condition case:
- dc analysis at $t = 0_-$ is omitted,
- at $t = 0$, coil is replaced by an open-circuit and capacitor by a short-circuit.

Step 2. Find R_t, equivalent resistance of the resistive part, with all sources deactivated. Then, the time constant is given by (3.2.8) or (3.2.13).

Step 3. Plot the response – connect the boundary values Y_0, Y_∞ by the exponential curve with time constant *T* (Fig.3.2.9).

Step 4. Express the response algebraically (3.2.17).

Example 3.2.3

The circuit, presented in Fig. 3.2.10, consists of the series connection of a dc practical voltage source that is switched at time $t = 0$ across a practical coil. Find the response, voltage *u* across the practical coil.

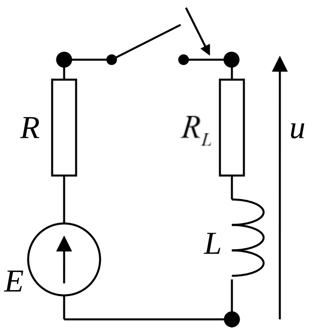


Fig. 3.2.10 Circuit for Example 3.2.3

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It is the 1st order circuit with no energy initially stored. The dc circuits at boundary, initial and steady state, conditions are presented in Fig. 3.2.11. Then, the response boundary values are:

$$U_0 = E,\quad U_\infty = \frac{R_L}{R + R_L}E \qquad (3.2.18)$$

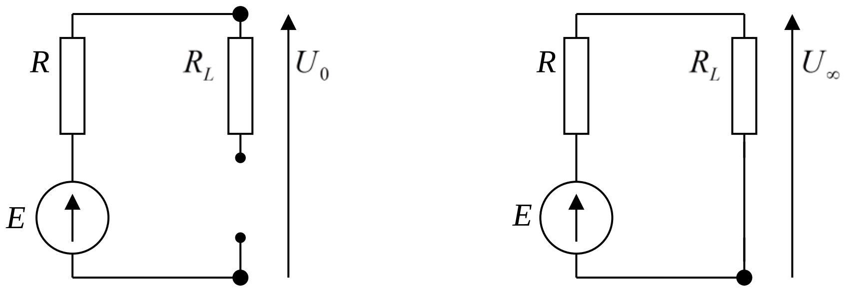

Fig. 3.2.11 Example 3.2.3 circuit at boundary conditions

The total resistance, seen from the ideal coil terminals is

$$R_t = R + R_L \tag{3.2.19}$$

and the circuit time constant is

$$T = \frac{L}{R_t} = \frac{L}{R + R_L} \tag{3.2.20}$$

The response is presented in Fig. 3.2.12.

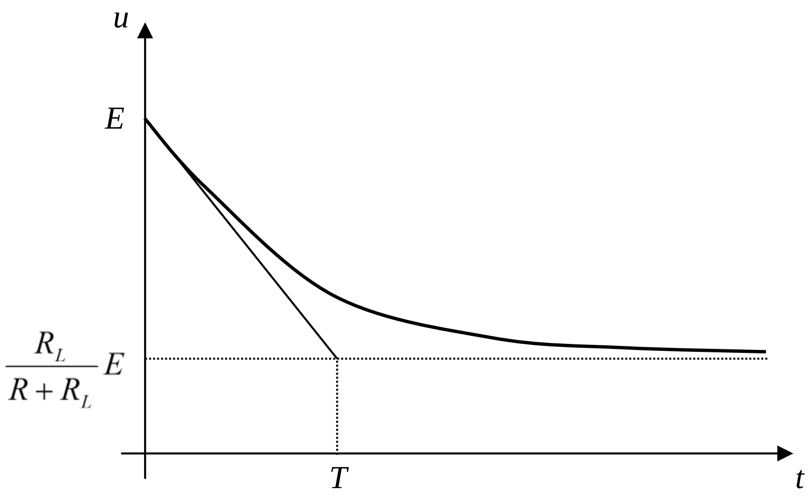

Fig. 3.2.12 Example 3.2.3 response, practical coil voltage

Its algebraic expression is

$$u = \frac{R_L}{R + R_L} E + \frac{R}{R + R_L} E \exp(-t/T) \tag{3.2.21}$$

♦

Example 3.2.4

Find the source current i and the capacitor current i_C for $t \geq 0$.

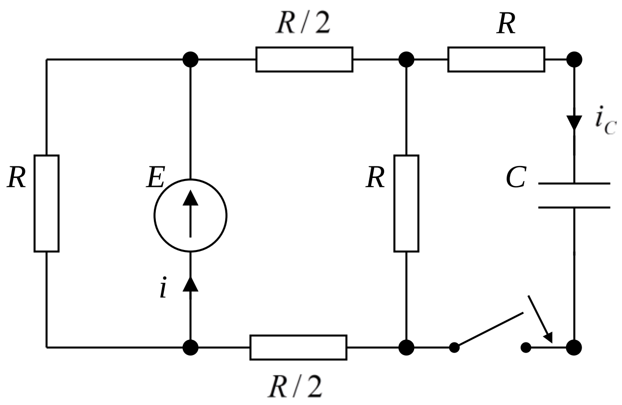

Fig. 3.2.13 Circuit for Example 3.2.4

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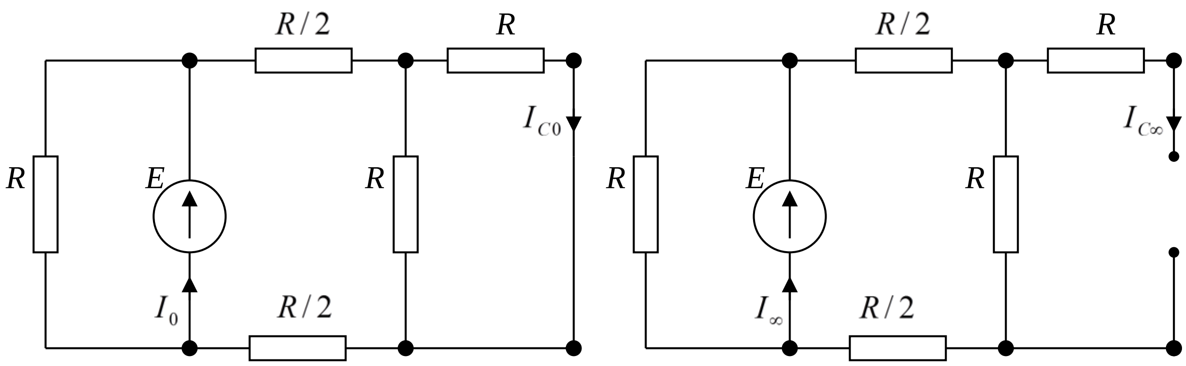

Fig. 3.2.14 Example 3.2.4 circuit at boundary conditions

It is the 1st order circuit with no energy initially stored. The dc circuits at boundary conditions are presented in Fig. 3.2.14. From dc analyses of these circuits the boundary values are designated. They are collected in Table 3.2.3.

Table 3.2.3

Boundary conditions for Example 3.2.4

	$t=0$	$t=\infty$
I_t	$\frac{E}{R}+\frac{E}{1.5R}=\frac{5}{3}\frac{E}{R}$	$\frac{E}{R}+\frac{E}{2R}=\frac{3}{2}\frac{E}{R}$
I_{Ct}	$\frac{E}{3R}$	0

Equivalent (Thevenin) resistance the resistive part is:

$$R_t = ((R/2+R/2)\|R)+R = 1.5R \tag{3.2.22}$$

Then, the time constant is:

$$T = 1.5RC \tag{3.2.23}$$

and responses can be plotted and expressed algebraically.

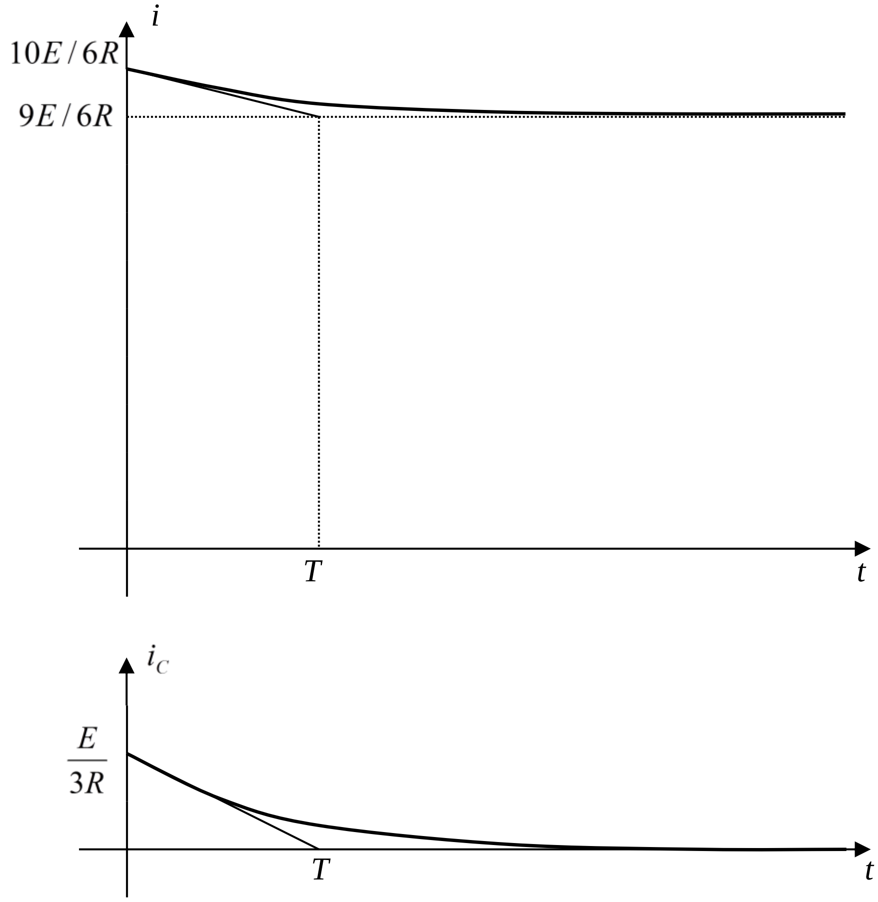

Fig. 3.2.15 Example 3.4.4 responses

$$i = \frac{3}{2}\frac{E}{R} + \frac{1}{6}\frac{E}{R}\exp(-t/T) \tag{3.2.24a}$$

$$i_C = \frac{1}{3}\frac{E}{R}\exp(-t/T) \tag{3.2.24b}$$

♦

2nd order circuit – *s*-domain method

The 2nd order circuit contains two energy storage elements, after an optional series-parallel simplification of homogeneous two-terminal circuit(s). Such circuit, in time-domain is described by the 2nd order differential equations. In the *s*-domain, denominator of the obtained solution is of the 2nd order. Use of Algorithm 3.2.1 to series *RLC* circuit analysis will be discussed in details.

Example 3.2.5

Series *RLC* circuit is presented in Fig. 3.2.16a. Find the capacitor voltage and current after closing the switch at $t=0$.

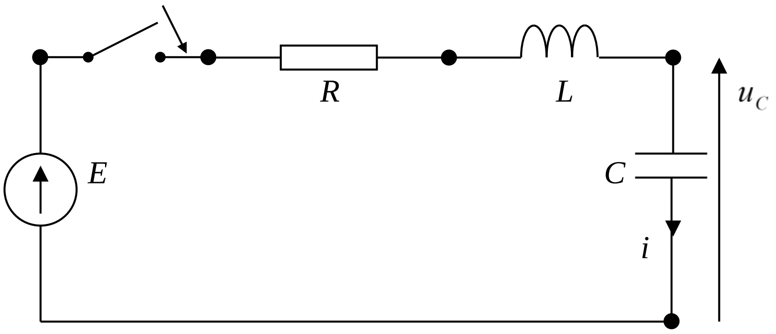

Fig. 3.2.16a Series *RLC* circuit (Example 3.2.5)

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There is no energy stored initially in the circuit, $I_0 = 0, U_{C0} = 0$. At $t=0$ coil is an open-circuit, at $t=\infty$ capacitor is an open-circuit. The boundary values of the responses are collected in Table 3.2.4.

Table 3.2.4
Boundary conditions for Example 3.2.4

	$t=0$	$t=\infty$
I_t	0	0
U_{Ct}	0	E

The circuit *s*-domain diagram is presented in Fig. 3.2.16b. From element equations and KVL equation, the mesh current is

$$I(s) = \frac{\frac{E}{s}}{R + sL + \frac{1}{sC}} = \frac{EC}{1 + s^2LC + sCR} = \frac{E}{L}\frac{1}{s^2 + \frac{R}{L}s + \frac{1}{LC}} = K\frac{1}{(s-s_1)(s-s_2)} \quad (3.2.25)$$

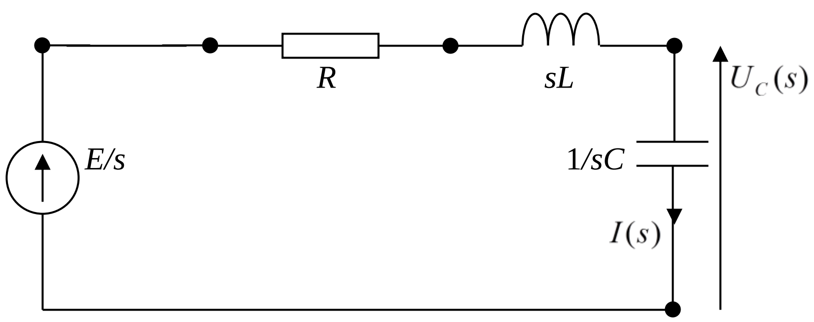

Fig. 3.2.16b Series *RLC* circuit (Example 3.2.5) in Laplace-domain

The current gain is

$$K = \frac{E}{L} \tag{3.2.25a}$$

and its poles are

$$s_{1,2} = -\frac{R}{2L} \pm \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} = -\alpha \pm \beta \tag{3.2.25b}$$

where,

$$\alpha = \frac{R}{2L} \tag{3.2.25c}$$

is the so called **damping coefficient**,

$$\beta = \sqrt{\alpha^2 - \omega_0^2} = j\sqrt{\omega_0^2 - \alpha^2} = j\omega$$

and

$$\omega = \omega_d = \frac{2\pi}{T}, \quad \omega_0 = \frac{1}{\sqrt{LC}} = \frac{2\pi}{T_0} \tag{3.2.25d}$$

are the so called **damped resonant frequency** and undamped resonant frequency or **natural frequency**, respectively.

The inverse transform of this function is discussed in Appendix A - equations (A15) and (A16). Three different cases have to be considered, subjected by the character of poles.

1. two simple real poles - response with two terms exponentially decaying to zero, the so called overdamped response,
2. simple pair of complex conjugate poles – damping is accompanied by oscillations, the so called underdamped response,
3. two repeated poles – dividing line between overdamped and underdamped case, the so called critically-damped response.

1. Overdamped response

$$\alpha > \sqrt{\frac{1}{LC}} \equiv R > 2\sqrt{\frac{L}{C}} = R_c \ , \ \beta > 0 \tag{3.2.26a}$$

where, R_c is called the **critical resistance**. Then,

$$i = K\frac{1}{2\beta}[\exp(-t/T_1) - \exp(-t/T_2)]; \quad T_1 = -1/s_1, T_2 = -1/s_2 \tag{3.2.27a}$$

2. Underdamped response

$$\alpha < \sqrt{\frac{1}{LC}} \equiv R < 2\sqrt{\frac{L}{C}} = R_c \ , \ \beta = j\omega \tag{3.2.26b}$$

Then,

$$i = K\frac{1}{\omega}\exp(-\alpha t)\sin\omega t \tag{3.2.27b}$$

2a. Undamped response

$$\alpha = 0 \equiv R = 0 \;,\; \beta = j\omega_0 = j\sqrt{\frac{1}{LC}} \tag{3.2.26c}$$

Then,

$$i = E\sqrt{\frac{C}{L}}\sin\omega_0 t \tag{3.2.27c}$$

3. Critically-damped response

$$\alpha = \sqrt{\frac{1}{LC}} \equiv R = 2\sqrt{\frac{L}{C}} = R_c \;,\; \beta = 0 \tag{3.2.26d}$$

Then,

$$i = Kt\exp(-\alpha t) \tag{3.2.27d}$$

The nonoscillatory, overdamped and critically-damped responses are presented in Fig. 3.2.17a and b. The oscillatory, underdamped and undamped responses are presented in Fig. 3.2.17c and d.

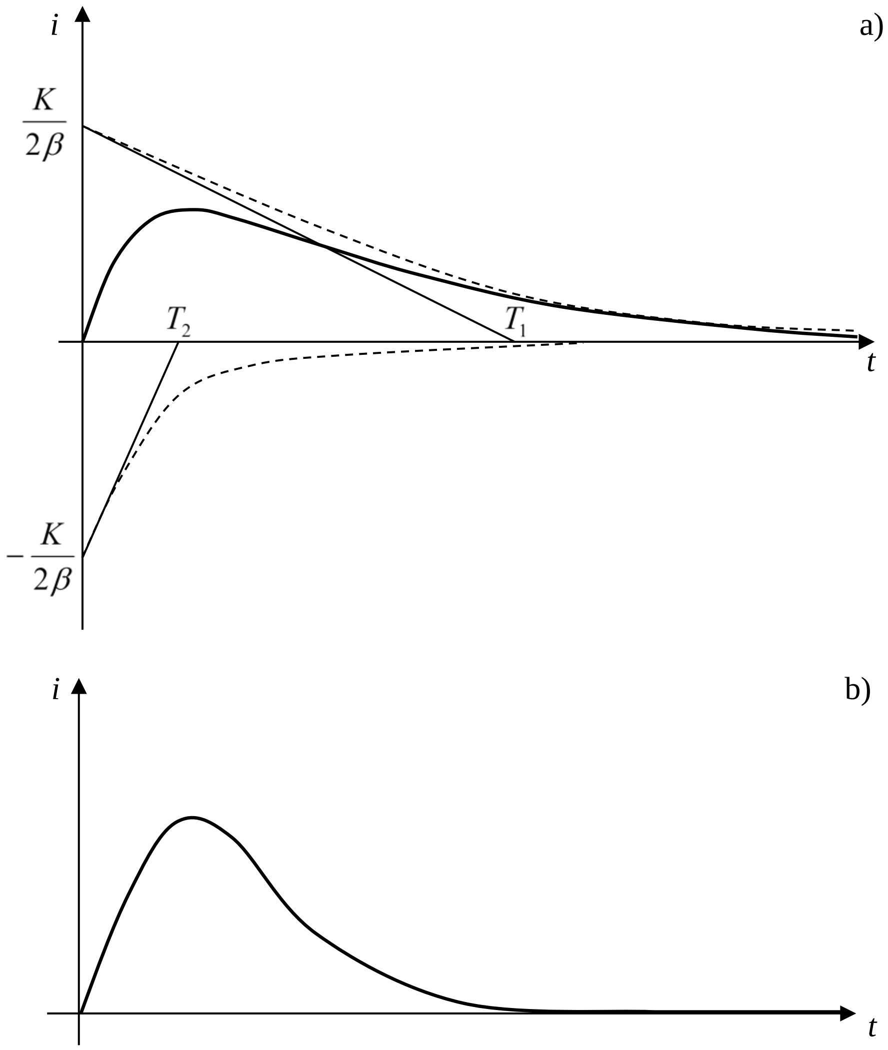

Fig. 3.2.17 Nonoscillatory responses in the series *RLC* circuit

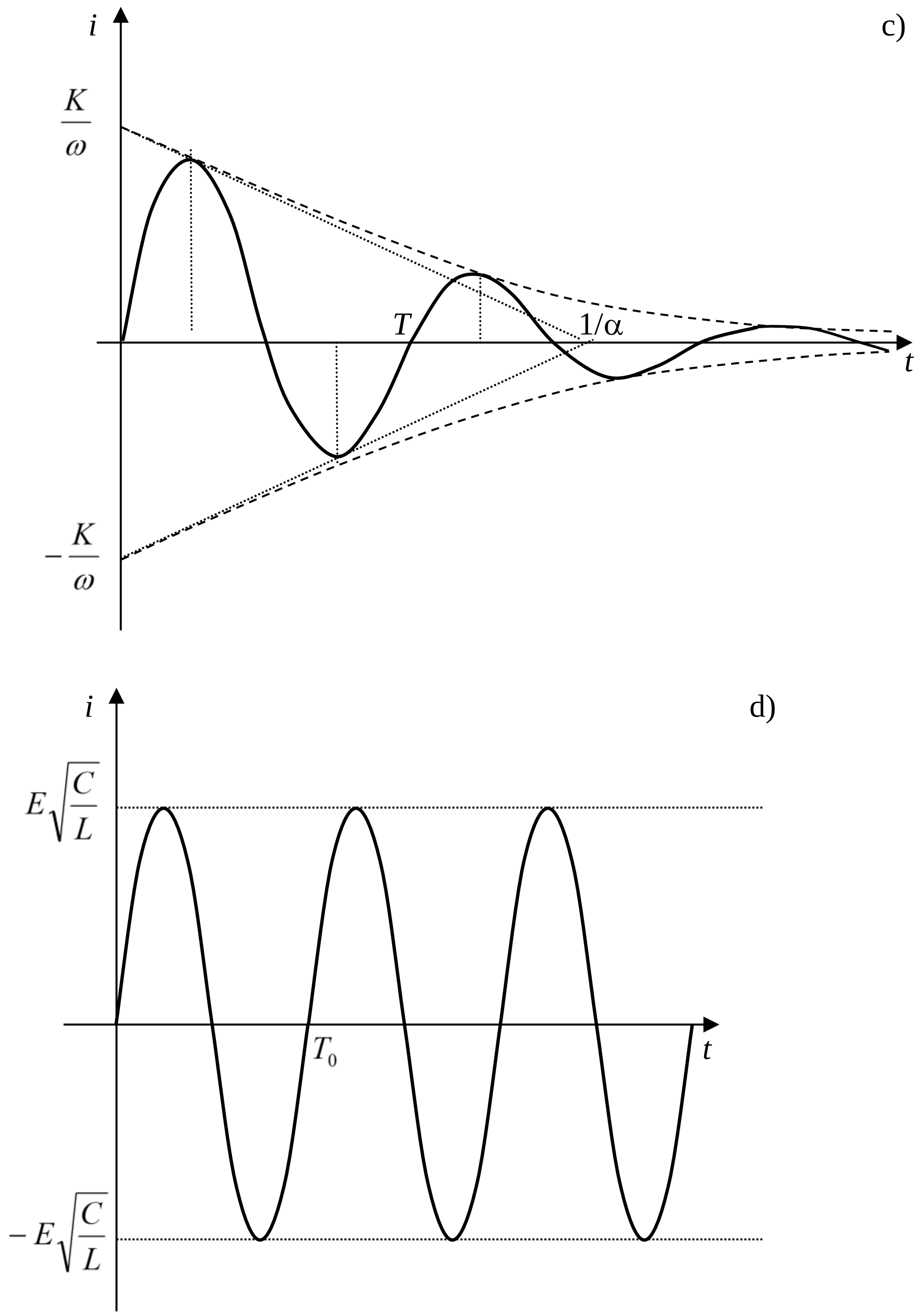

Fig. 3.2.17 cont. Oscillatory responses in the series *RLC* circuit

The capacitor voltage can be found from the integral formula (3.1.8b). The overdamped, critically-damped and undamped voltages are presented in Fig. 3.2.18.

For the undamped case ($R = 0$), the mesh current and the capacitor voltage are oscillating with the constant amplitude. Period of this oscillations T_0 is given by (3.2.25d) and the maximum values reached are

$$I_{\max} = E\sqrt{\frac{C}{L}}\,, \quad U_{\max} = 2E \tag{3.2.28}$$

The energy is oscillating between the source and energy storage elements.

- At $t=(2n+1)\dfrac{T_0}{4}$; $n=1,2,...$: $W_{L\max}=\dfrac{I_{\max}^2 L}{2}=\dfrac{E^2 C}{2}=W_C$.
- At $t=(2n+1)\dfrac{T_0}{2}$; $n=1,2,...$: $W_{C\max}=\dfrac{U_{C\max}^2 C}{2}=4\dfrac{E^2 C}{2}$, $W_L=0$.
- After $t=nT_0$; $n=1,2,...$: total energy supplied=total energy stored=zero.

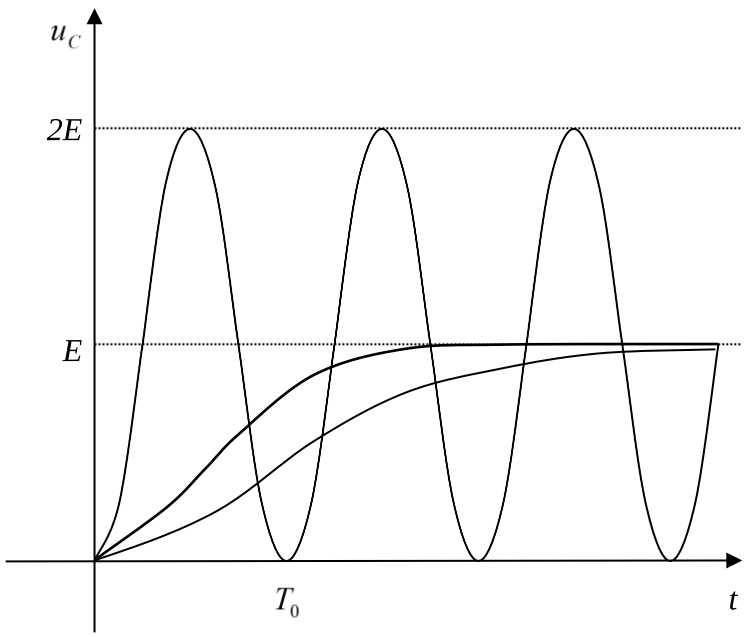

Fig. 3.2.18 Transient capacitor voltage in the series *RLC* circuit; overdapmed, critically-damped (bold curve) and undamped case

It should be observed, that the critically-damped circuit demonstrates the fastest convergence to the steady state. Assume $L=1\text{H}$, $C=1/4\,\text{F}$ and consider overdamped case, critically-damped case, underdamped case and undamped case.

- Overdamped case, $R=1.25R_c=5\Omega$:

  $T_1=1\text{s}, T_2=1/4\text{s}$ and the steady state time $t_\infty=5T_1=5\text{s}$.
- Critically-damped case, $R=R_c=4\Omega$:

  $1/\alpha=1/2\text{s}$ and the steady state time $t_\infty=5/\alpha=2.5\text{s}$.
- Underdamped case, $R=0.5R_c=2\Omega$:

  $1/\alpha=1\text{s}$, $\omega=\sqrt{3}=1.7\,\text{rad/s}$ and the steady state time $t_\infty=5T_1=5\text{s}$.
- Undamped case, $R=0\,\Omega$:

  $\alpha=0$, $\omega_0=2\,\text{rad/s}$ and oscillations are not vanishing.

As can be observed, for two resistances, $R=5\Omega$ and $R=2\Omega$, the steady state condition is reached after the same period of time, however in the latter case the transient is oscillatory.

♦

NATURAL RESPONSE

The transient state after switching off all the excitations is considered. Then, the energy initially stored is dissipated on resistors, as a heat. All the circuit variables decay to zero, i.e. all steady state values are zero. Two exemplary circuits, source free *RC* circuit and source free *RL* circuit, will be considered.

<u>Example 3.2.6</u>
Find the mesh current after throwing the switch.

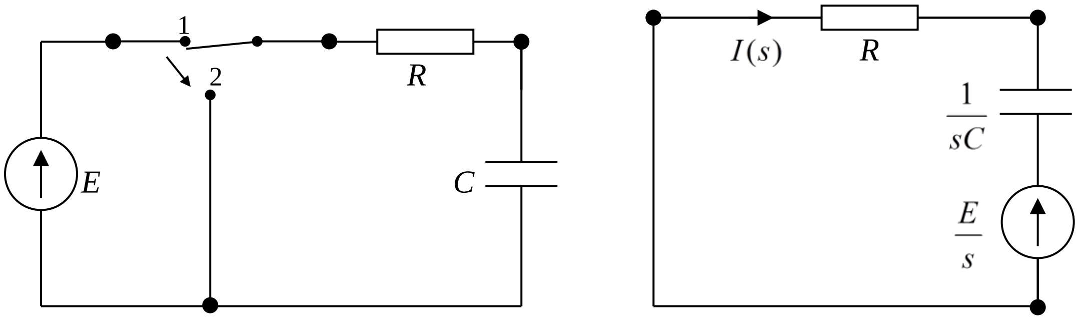

Fig. 3.2.19 Circuit for Example 3.2.6 in time-domain and in s-domain

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The dc circuits for $t = 0_{-}$ and $t = 0$ are presented in Fig. 3.2.19a.

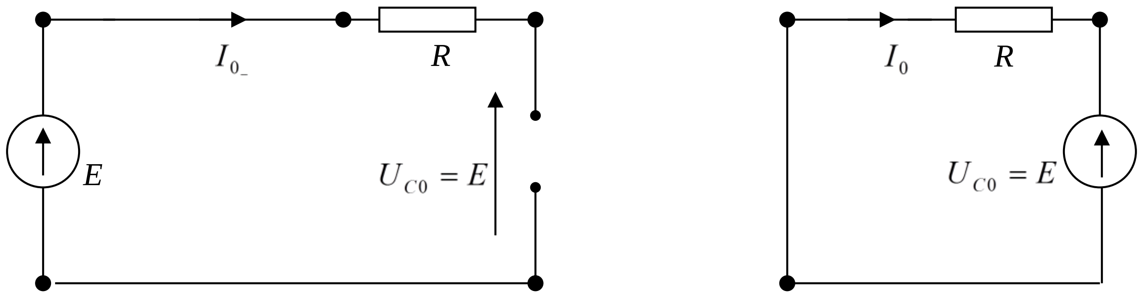

Fig. 3.2.19a Example 3.2.6 dc circuits at $t = 0_{-}$ and $t = 0$

The energy initially stored is

$$W_0 = \frac{E^2 C}{2} \tag{3.2.29}$$

After changing position of the switch, this energy is dissipated on the resistor, as a heat. The mesh current initial value is

$$I_0 = -\frac{E}{R} \tag{3.2.30}$$

Its steady state value is zero and the transient response is

$$i = -\frac{E}{R}\exp(-t/T), \;\; T = RC \tag{3.2.31}$$

This transient response is presented in Fig. 3.2.20, for small and large value of resistance R ($E = const,\ C = const$).

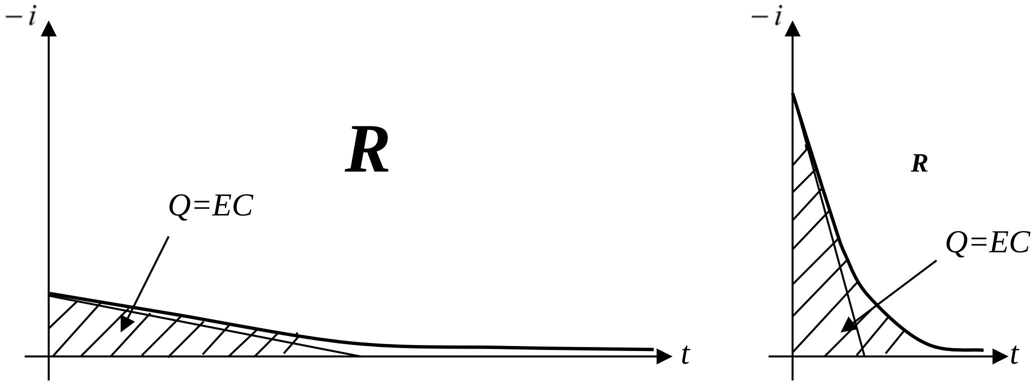

Fig. 3.2.20 Example 3.2.6 mesh current, *large resistance* and *small resistance* case

From the *s*-domain circuit (Fig. 3.2.19) analysis, the mesh current for the resistanceless case, $R = 0$, can be found

$$I(s) = -\frac{\frac{E}{s}}{\frac{1}{sC}} = -EC \tag{3.2.31a}$$

and the time-domain response is the impulse

$$i = -EC\delta(t)\,, \tag{3.2.31b}$$

an infinitely tall and infinitely narrow pulse of EC area. In practice, if resistance is very small, then the absolute initial value of the current is very large. For example:

$$\text{if } C = 1\mu\text{F},\ E = 10\,\text{V},\ R = 0.1\Omega,\ \text{ then } |I_0| = 100\,\text{A}\ \text{ !!!},\ T = 0.1\mu\text{s}\,.$$

This phenomenon, called overcurrent, can be utilized in welding of thin wires. In an electric (electronic) circuit, when short-circuiting the charged capacitor by a switch (relay), its terminals can be welded !!! ♦

Example 3.2.6 – cont.

Find the mesh current after changing position of the switch, taking into account the capacitor residual inductance L_C .

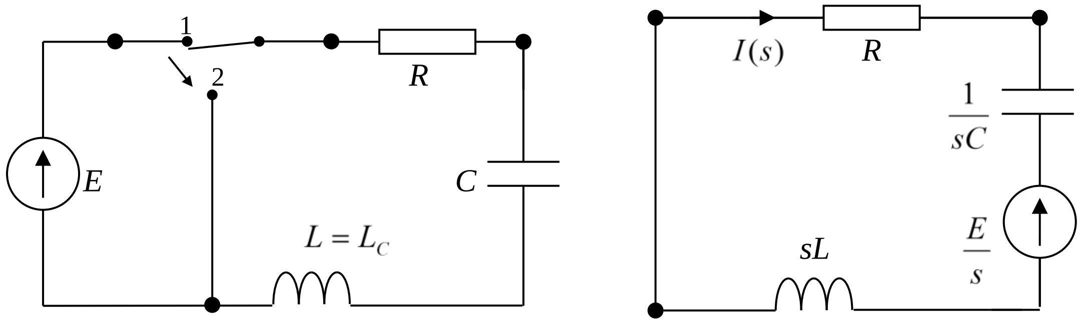

Fig. 3.2.21 Circuit for Example 3.2.6, after taking into account the residual inductance

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The initial conditions are: $I_{L0} = I_0 = 0$, $U_{C0} = E$. Then, the *s*-domain circuit presented in Fig. 3.2.21 is obtained. This circuit is identical with the circuit of Example 3.2.5, presented in Fig. 3.2.16b, the source arrowhead direction is the only difference. Then, for a very small resistance (underdamped case), from (3.2.27b), the mesh current is

$$i = -\frac{E}{\omega L}\exp(-\alpha t)\sin\omega t \tag{3.2.31c}$$

$$\alpha = \frac{R}{2L},\ \omega \cong \omega_0 = \sqrt{\frac{1}{LC}}$$

This current (multiplied by −1) is presented in Fig. 3.2.17d. Its value starts from zero, as the coil current may not change abruptly. After $t = T_0/4$, the maximum current, given by (3.2.28), is reached. For the same exemplary values of E, C, R and the residual inductance of $L = 0.1$ mH, the maximum current and the time constant are

$|I_{\max}| \cong 1$ A, $1/\alpha = 2$ ms.

In the practical circuit, the overcurrent is much less than in the ideal circuit, however it is still not acceptable for electric (electronic) circuit. The overcurrent can be limited by series connection of resistance.

♦

Example 3.2.7

Find the coil voltage after moving the switch, the circuit is presented in Fig. 3.2.22

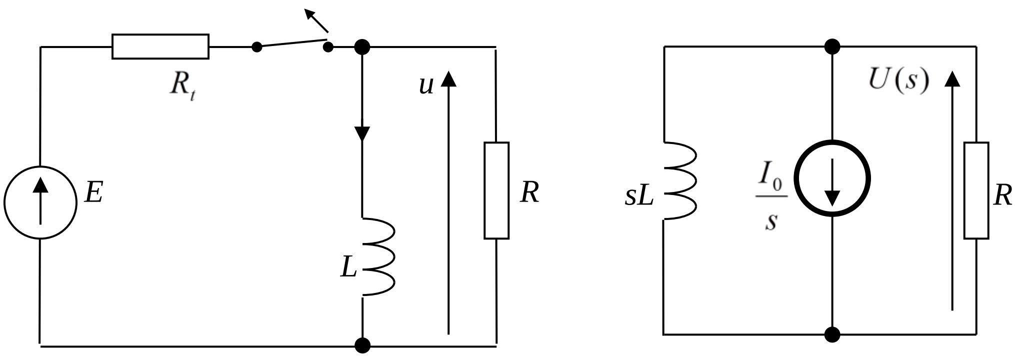

Fig. 3.2.22 Circuit for Example 3.2.7 in time-domain and in s-domain

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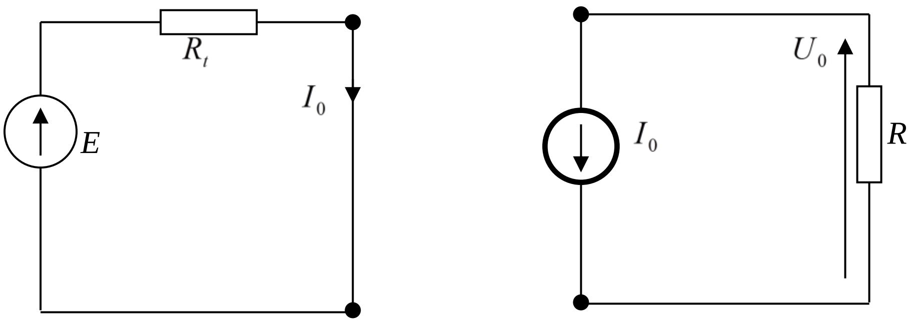

Fig. 3.2.22a Example 3.2.7 dc circuits for $t = 0_-$ and $t = 0$

The dc circuits for $t = 0_-$ and $t = 0$ are presented in Fig. 3.2.22a. The coil current initial value is

$$I_{L0} = I_0 = \frac{E}{R_t} \tag{3.2.32a}$$

The coil voltage initial value is

$$U_0 = -I_0 R = -\frac{ER}{R_t} \tag{3.2.32b}$$

Its steady state value is zero and the transient response is

$$u = -\frac{ER}{R_t}\exp(-t/T),\ T = L/R \tag{3.2.33}$$

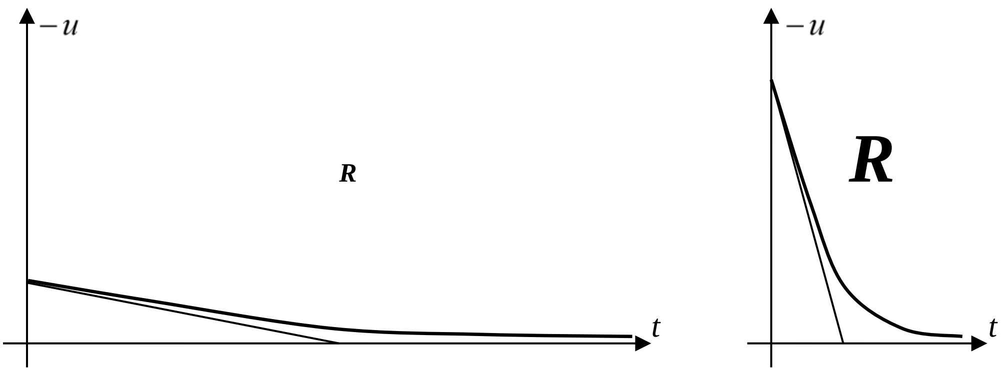

Fig. 3.2.23 Example 3.2.7 mesh current, *large resistance* and *small resistance* case

This transient response is presented in Fig. 3.2.23, for small and large value of resistance *R* ($E = const,\ R_t = const,\ L = const$).

From the *s*-domain circuit (Fig. 3.2.22) analysis, the coil voltage at conductanceless case, $G = 1/R = 0$, can be found

$$U(s) = -sL\frac{I_0}{s} = -\frac{E}{R_t}L \tag{3.2.34a}$$

and the time-domain response is the impulse

$$u = -\frac{E}{R_t}L\delta(t), \tag{3.2.34b}$$

an infinitely tall and infinitely narrow pulse. In practice, if resistance is very large, then the absolute initial value of voltage is very large. For example:

if $L = 100\,\text{mH}$, $E = 10\,\text{V}$, $R_t = 100\,\Omega$, $R = 100\,\text{k}\Omega$, then $|U_0| = 10\,\text{kV}$!!!, $T = 0.1\,\mu\text{sec}$.

This phenomenon, called overvoltage, may cause damage to the coil loading circuit represented by the resistance *R*. Methods of overvoltage protection will be presented further on.

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Example 3.2.7 – cont.

Find the coil voltage after moving the switch, taking into account the coil residual capacitance (the circuit is presented in Fig. 3.2.24).

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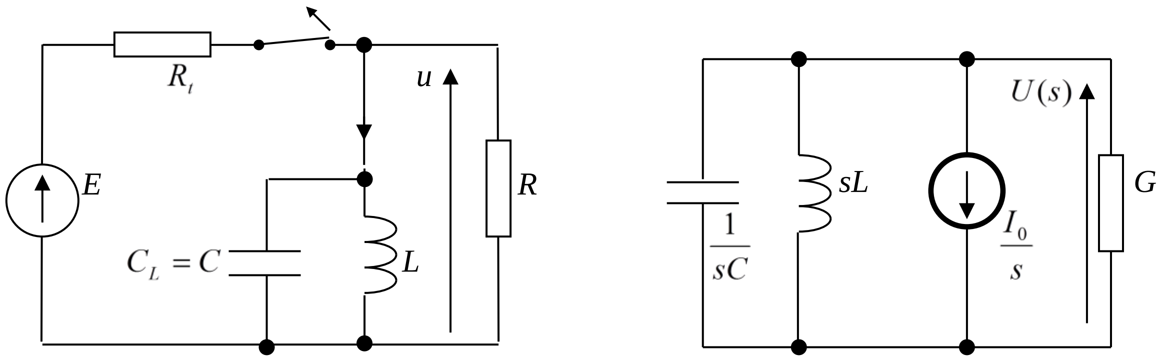

Fig. 3.2.24 Circuit for Example 3.2.7, after taking into account the residual capacitance

The initial conditions are: $I_{L0} = I_0 = E/R_t$, $U_{C0} = U_0 = 0$. Then, the *s*-domain circuit presented in Fig. 3.2.24 is obtained. The coil voltage is

$$U(s) = -\frac{I_0}{C}\frac{1}{s^2 + \frac{G}{C}s + \frac{1}{LC}} = -K\frac{1}{(s-s_1)(s-s_2)} \tag{3.2.35a}$$

This equation is identical as equation (3.2.25), with *R* replaced by *G*, E replaced by $I_0 = E/R_t$ and *L* swapped with *C*. Then, for a very small conductance (the underdamped case - equation (3.2.27b)) the coil time-voltage is

$$u = -\frac{E}{R_t C\omega}\exp(-\alpha t)\sin\omega t \tag{3.2.35b}$$

$$\alpha = \frac{G}{2C}, \quad \omega \cong \omega_0 = \sqrt{\frac{1}{LC}}$$

This curve (multiplied by −1) is presented in Fig. 3.2.17c. The voltage value starts from zero, as the capacitor voltage may not change abruptly. After $t = T_0/4$, the maximum, given by (3.2.28) with *E* replaced by I_0, is reached. For the same exemplary values of E, R_t, L, R and the residual capacitance of $C = 100$ nF the maximum voltage and the time constant are

$$|U_{\max}| \cong 100\,\text{V},\ 1/\alpha = 20\ \text{ms}$$

In the practical circuit of Fig. 3.2.24, the overvoltage is much less than in the ideal circuit of Fig. 3.2.22, however its value is still not acceptable for electric (electronic) circuit. The overvoltage can be limited by parallel connection (to the coil) of capacitance C_p. Then, total capacitance $C_t = C + C_p$. Even better effect can be achieved by connecting a diode in parallel to the coil, as presented in Fig. 3.2.25. For the position 1 of the switch, the diode is inversely polarized and can be replaced by an open-circuit. After changing the switch position, from 1 to 2, energy stored in the coil is dissipated through a diode, practically immediately.

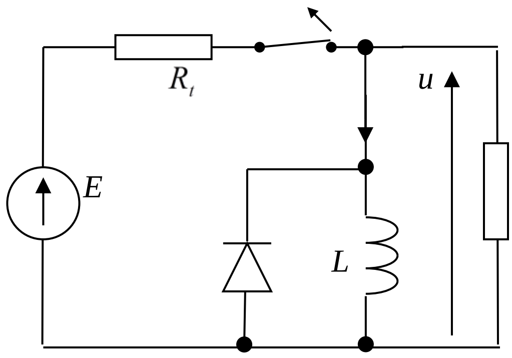

Fig. 3.2.25 Example 3.2.7, overvoltage protection

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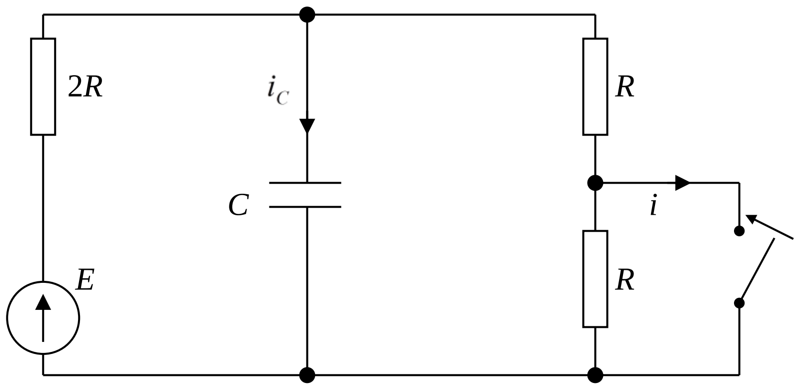
COMPLETE RESPONSE: NATURAL RESPONSE + FORCED RESPONSE

In general (complete response) case, superposition principle can be utilized. However, for the 1st order circuit analysis, boundary values based approach is suggested. Use of Algorithm 3.2.2 to a general case will be illustrated by two examples.

Example 3.2.8

Find currents i_C and i after closing the switch.

Fig. 3.2.26
Circuit for Example 3.2.8



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Following Step 1 of Algorithm 3.2.2, three dc circuits are constructed, as presented in Fig. 3.2.27. The calculated boundary values are collected in Table 3.2.5.

Table 3.2.5

Boundary values for Example 3.2.8

	$t = 0_-$	$t = 0$	$t = \infty$
U_C	$E/2$	$E/2$	$E/3$
I_C	0	$\frac{E}{2R} - \frac{E/2}{2R/3} = -\frac{E}{4R}$	0
I	0	$\frac{E}{2R}$	$\frac{E}{3R}$

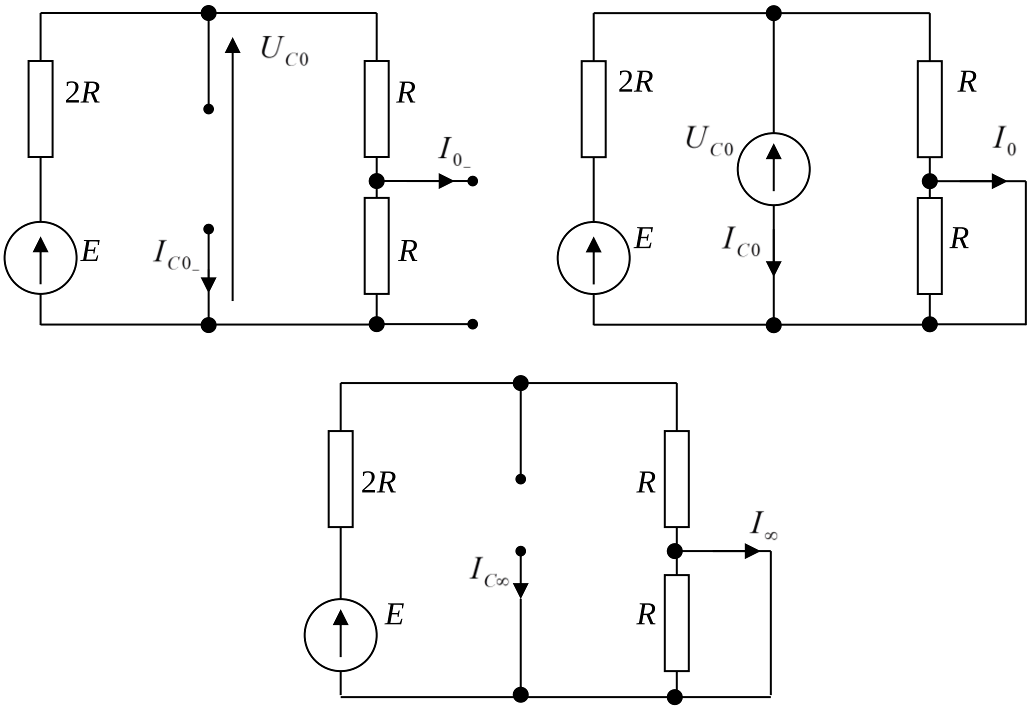


Fig. 3.2.27 dc circuits for Example 3.2.8

Then, the total resistance of the resistive part is

$$R_t = 2R \| R = \frac{2}{3}R \tag{3.2.36}$$

and the time constant is

$$T = \frac{2}{3}RC \tag{3.2.37}$$

Finally, the following algebraic form of transient responses is obtained:

$$u_C = \frac{E}{3} + \frac{E}{6}\exp(-t/T) \tag{3.2.38a}$$

$$i_C = -\frac{E}{4R}\exp(-t/T) \tag{3.2.38b}$$

$$i = \frac{E}{3R} + \frac{E}{6R}\exp(-t/T) \tag{3.2.38c}$$

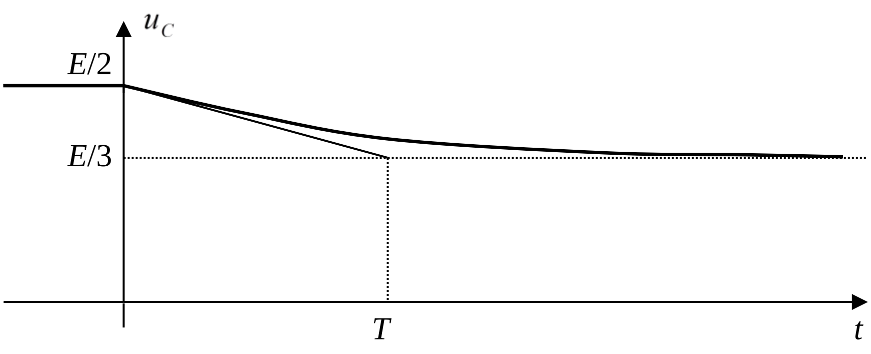


Fig. 3.2.28 Capacitor voltage in Example 3.2.8

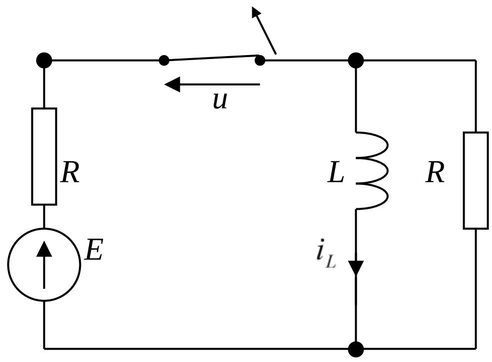
The capacitor is discharging from $E/2$ to $E/3$, as presented in Fig. 3.2.28. The lost energy is equal to $\Delta W = \frac{5}{72} E^2 C$.

♦

Example 3.2.9

Find voltage across the switch, after its opening.

Fig. 3.2.29 Circuit for Example 3.2.9



The circuit initial condition is $I_{L0} = E/R$. The dc circuits at boundary conditions are presented in Fig. 3.2.30.

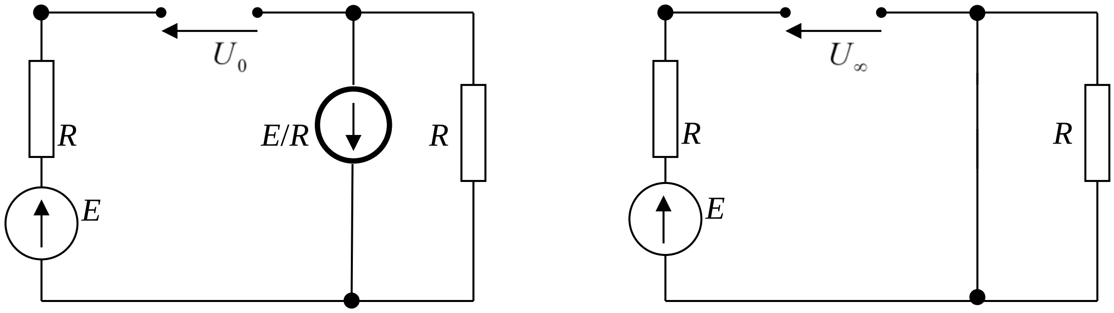


Fig. 3.2.30 The dc circuits for Example 3.2.9

Then,

$$U_0 = E + (E/R)R = 2E \tag{3.2.39a}$$

$$U_\infty = E$$

and the voltage across the switch, for $t \geq 0$, is

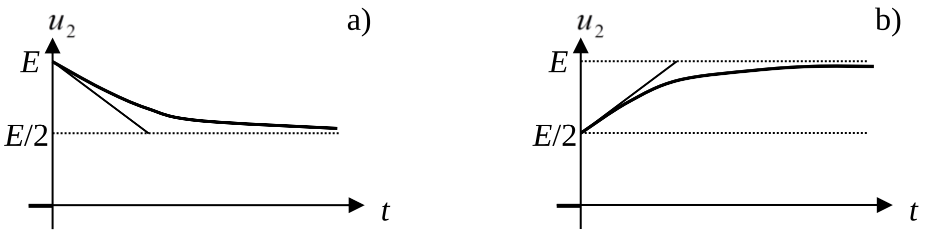
$$u = E + E\exp(-t/T), \; T = L/R \tag{3.2.39b}$$

♦

Drill problems 3.2

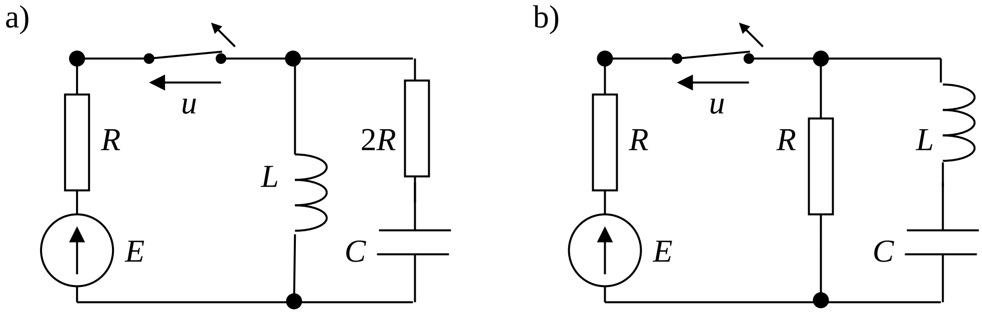
1. For the step input $u_1 = E \cdot \mathbf{1}(t)$ of two-port and the given output voltage (exponential function), draw its simplest structure.

Fig. P.3.2.1



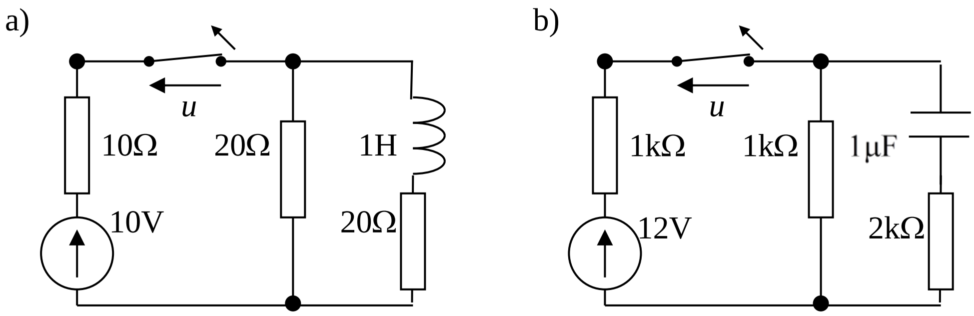
2. Sketch voltage across the switch after its opening. Assume: a) overdamped, b) underdamped case.

Fig. P. 3.2.2



3. Plot voltage across the switch after its opening.

Fig. P.3.2.3



4. Plot current that flows through the switch after its closing.

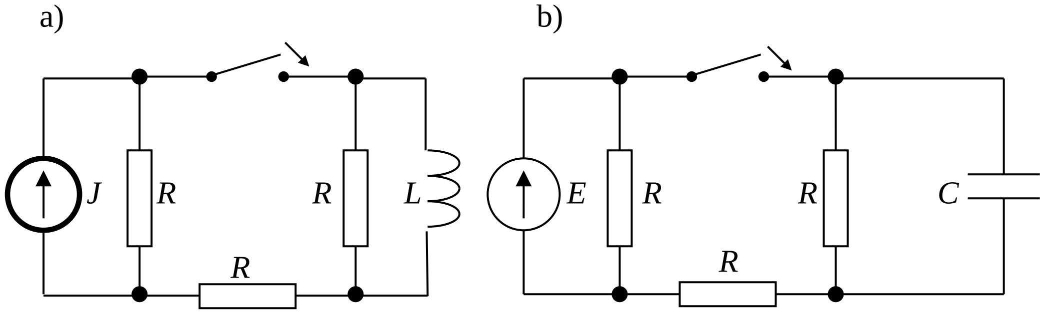


Fig. P.3.2.4

5. Plot the capacitor voltage after closing the switch

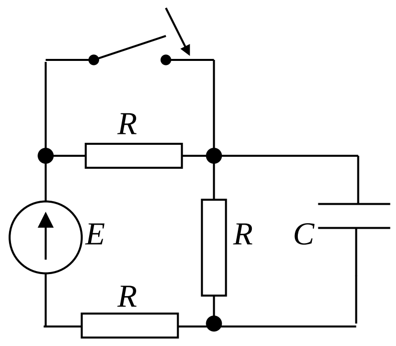


Fig. P.3.2.5

6. Find resistance R such that no transient response is present in the source current. Plot this current.

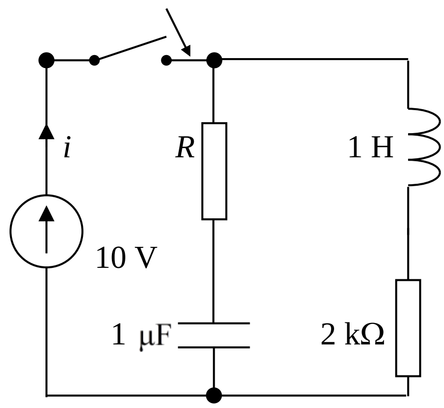


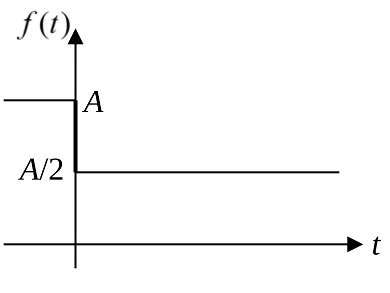
Fig. P.3.2.6

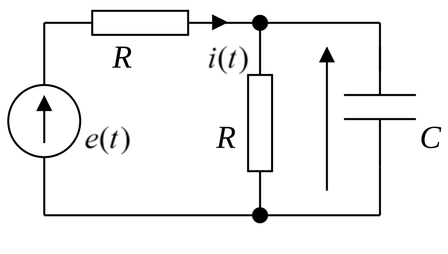
7. In a circuit of Fig 3.2.24 the switch is thrown at $t = 0$. Sketch the element currents and the voltage if $R = \infty$, $L = 4\,\mathrm{mH}$, $C_L = 1\,\mathrm{pF}$. Find the maximum energy stored in each element.

8. In a circuit of Fig. 3.2.10 the practical coil voltage reaches the steady state value of $U_\infty = U_0 / 2$ after $t = 5T = 10$ ms. Find the values of L and R_L.

9. Sketch $i(t)$ and $u_C(t)$ for the excitation graphed: $e(t) = f(t)$; $A = E$.

Fig. P.3.2.9





10. In a circuit of Fig. 3.2.16 the switch closes at $t = 0$. Sketch the element voltages and the mesh current if $R = 0$, $L = 4\,\mathrm{mH}$, $C = 1\,\mu\mathrm{F}$. Find the maximum energy stored in C.

11. Sketch, with no calculations, voltage u and current i after closing the switch. Assume: a) underdamped case, b) overdamped case.

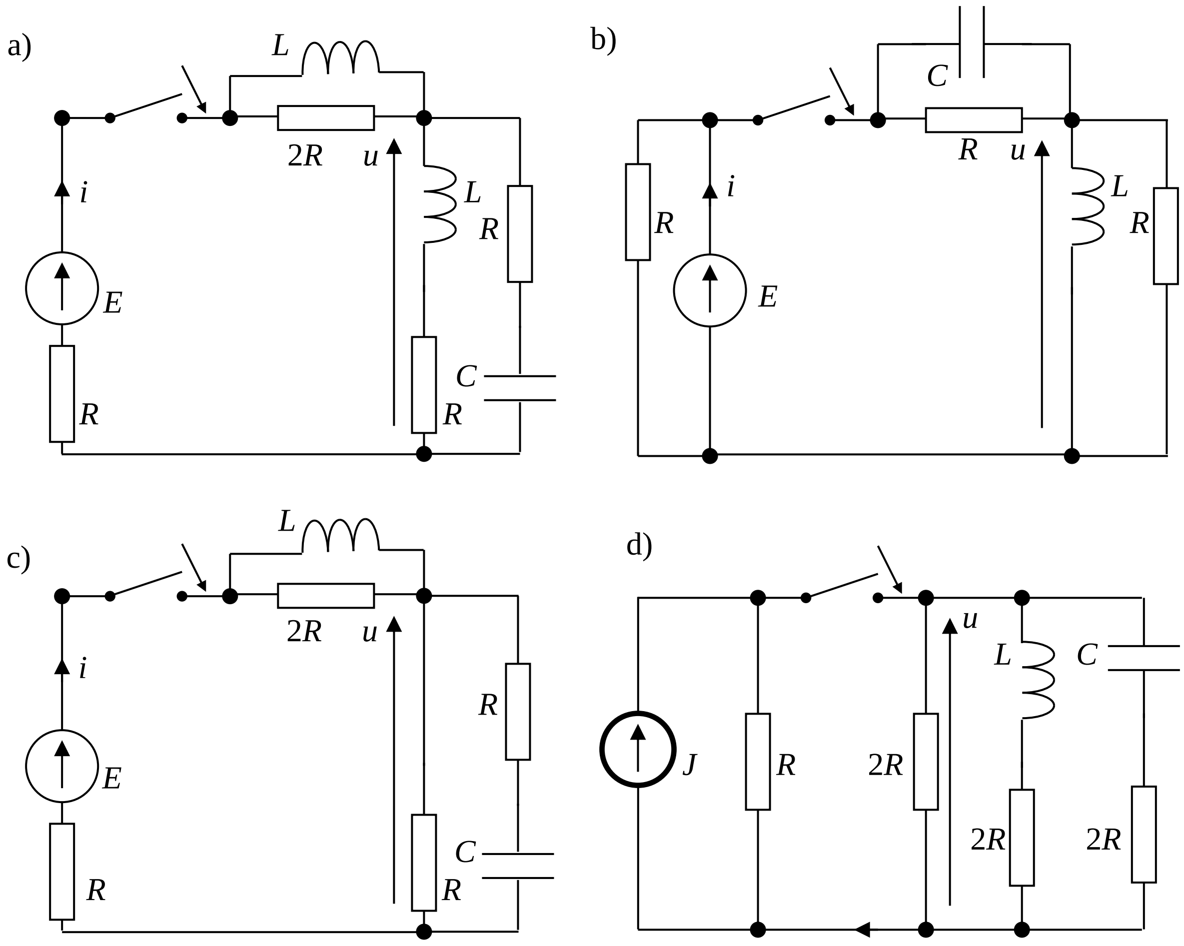
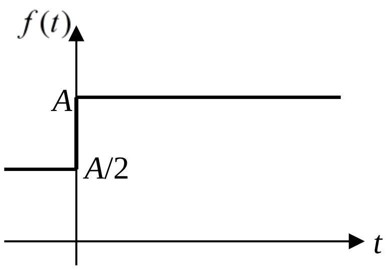


Fig. P.3.2.11

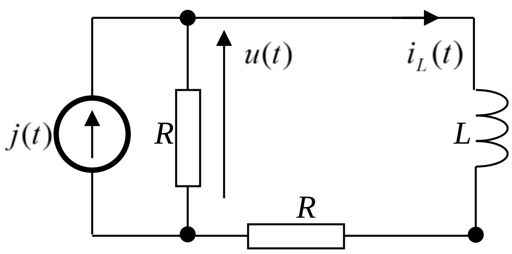
12. In Problem 3.2.9 circuit, sketch $i(t)$ and $u_C(t)$, for the new excitation graphed: $e(t) = f(t);\ A = E$.

Fig. P.3.2.12



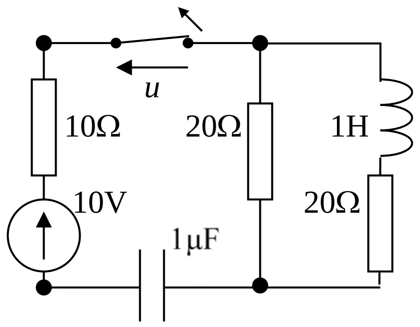
13. Sketch $i_L(t)$ and $u(t)$, for a circuit shown and the excitations graphed in Problems 3.2.9 and 3.2.12: $j(t) = f(t);\ A = J$.

Fig. P.3.2.13



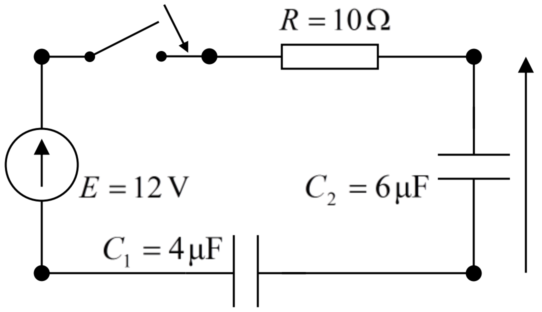
14. The switch opens at $t = 0$. Sketch the voltage *u*.

Fig. P.3.2.14



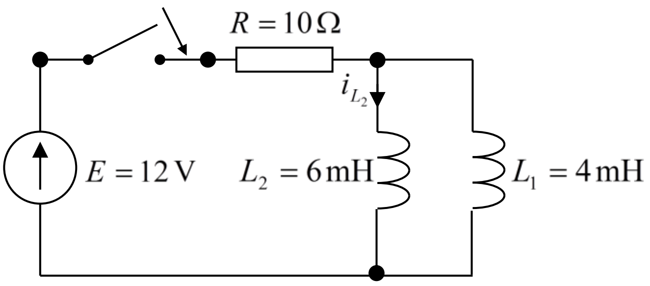
15. The switch is thrown at $t = 0$. Compute the energy stored in each capacitor at $t = \infty$. Sketch $u_{C_2}(t)$.

Fig. P.3.2.15



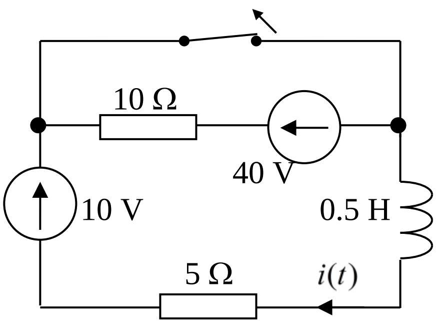
16. The switch is thrown at $t = 0$. Compute the energy stored in each inductor at $t = \infty$. Sketch $i_{L_2}(t)$.

Fig. P.3.2.16



17. Find $i(t)$ after the switch opens at $t = 0$.

Fig. P.3.2.17



18. Find $u(t)$ after the switch opens at $t=0$.

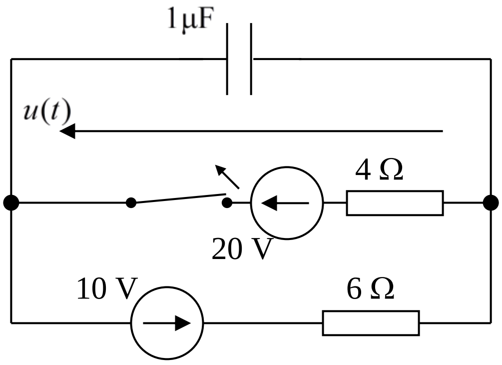


Fig. P.3.2.18

19. The switch opens at $t=0$. Sketch $u(t)$ and $u_C(t)$ for $J=2\,\text{A}$, $C=1\mu\text{F}$, $L=0.5$ H, and: a) $R_x=R=1\text{k}\Omega$, b) $R_x=2R$.

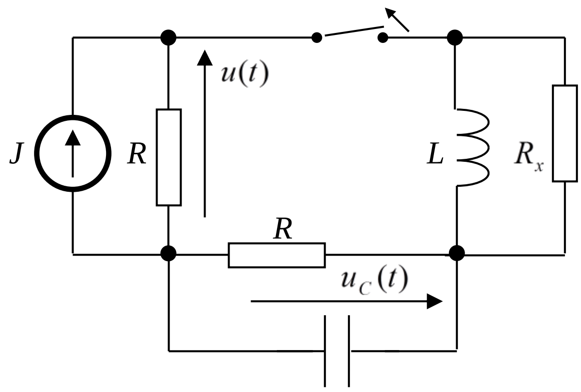


Fig. P.3.2.19

20. The switch is moved from 1 to 2 at $t=0$. Sketch $u_C(t)$ for $E_2=2E_1=20$ V, $R=6\Omega$, $R_1=4\Omega$ and two values of R_2, a) 4 Ω, b) 14 Ω.

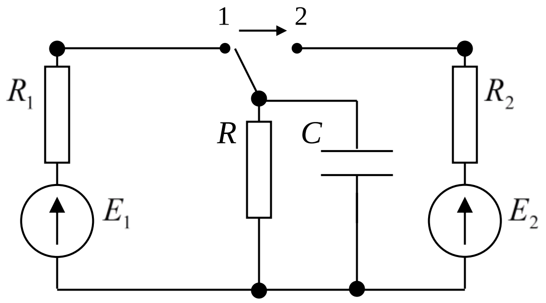


Fig. P.3.2.20

21. The switch is moved from 1 to 2 at $t=0$. Sketch $i(t), u_{C_1}(t), u_{C_2}(t)$ if $R_1=R_2=2\,\text{k}\Omega$, $E=10\,\text{V}$, $C_1=4\mu\text{F}$, $C_2=6\mu\text{F}$.

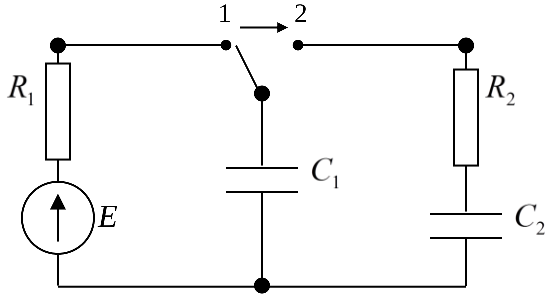
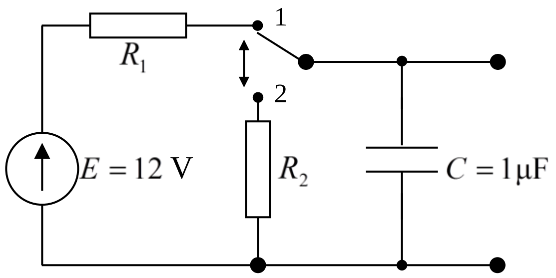


Fig. P.3.2.21

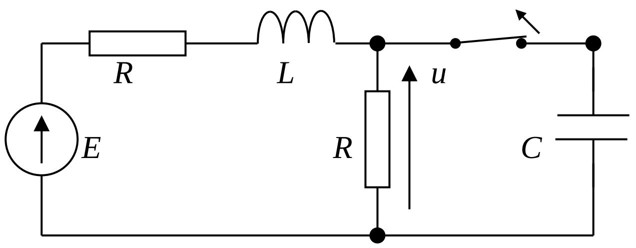
22. An approximate sawtooth waveform is produced by charging and discharging a capacitor with widely different time constants. Select the value of R_1 such that charging lasts one time constant $T_1 = 2$ ms, and the value of R_2 such that discharging lasts $5T_2 = 0.1T_1$.

Fig. P.3.2.22



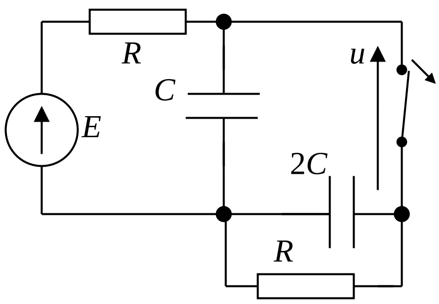
23. Find $u(0_-)$ and sketch $u(t)$ after the switch opens at $t = 0$. Assume $E = 10$ V, $R = 1\text{k}\Omega$, $C = 1\mu\text{F}$, $L = 0.5$ H.

Fig. P.3.2.23



24. Sketch $u(t)$ after the switch opens at $t = 0$. Assume $E = 10$ V, $R = 1\text{k}\Omega$, $C = 1\mu\text{F}$.

Fig. P.3.2.24



3.3 TRANSIENT ANALYSIS in CIRCUITS with ARBITRARY EXCITATION

Transient response caused by arbitrary aperiodic excitation will be considered. In most cases, circuit analysis is much more narrowly defined than that of finding all responses to all excitations. Very often it is limited to Single-Input-Single–Output (SISO) analysis. Then, transfer function approach is preferred, as presented in Chapter 2.7 for dc circuits. The same approach is preferred in transient analysis in circuits with arbitrary aperiodic excitation. Transfer function in *s*-domain will be defined at first. Then, its use to circuit transient analysis will be presented.

TRANSFER FUNCTION – PROPERTIES and SELECTED EXAMPLES

Transient response may be considered as sum of two components: natural response and forced response, as presented in Fig. 3.2.1 for time-domain signals. After setting initial conditions to zero and *t*-domain to *s*-domain transformation, the SISO system described in the *s*-domain is obtained, as presented in Fig. 3.3.1.

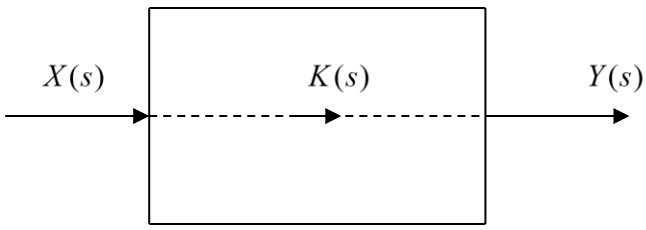


Fig. 3.3.1 SISO linear passive *s*-domain circuit

In the *s*-domain, a linear circuit input-output pair is related by the system of linear equations. Then, transfer function in the *s*-domain can be defined.

Laplace transfer function of a circuit is defined as the ratio of the response of the circuit to its excitation, expressed in the s-domain, with the assumption that all initial conditions are set to zero:

$$K(s) = \frac{Y(s)}{X(s)} \tag{3.3.1}$$

Thus, problem of finding the transient response can be solved using the concept of Laplace transfer function, the algorithm is as follows.

Algorithm 3.3.1 – Laplace transfer function method of transient analysis

Step 1. Predict the response:
- a) evaluate the circuit order,
- b) designate boundary values Y_0, Y_∞,
- c) predict shape of the response, if possible.

Step 2. Find algebraic expression of the excitation, if given by a graph, $x = x(t)$.

Step 3. Find the Laplace transform of excitation, $X(s)$.
Step 4. Find the circuit transfer function, $K(s)$.
Step 5. Find the response in the *s*-domain,

$$Y(s) = K(s)X(s) \tag{3.3.1a}$$

Step 6. Find the inverse transformation $y(t) = y$.
Step 7. Plot the response. Check whether the obtained solution matches the predicted one.

In the MIMO system, arbitrary forced *s*-domain output (response) due to all excitations (sources) is the superposition of the separate transfer functions for this output and each input times the corresponding *s*-domain inputs:

$$Y_j(s) = \sum_{i=1}^{M} K_{ij}(s) X_i(s); \;\; j = 1,\ldots,N \tag{3.3.1b}$$

Transfer function $K_{ij}(s)$ uniquely defines a circuit (system) with respect to one input signal $X_i(s)$ and one output signal $Y_j(s)$, impedance and admittance are the special cases. Before presenting use of this Laplace transfer function approach to exemplary circuit analysis, properties of the function will be discussed. Then, circuits that perform basic signal transformations in time-domain, integration and differentiation, will be studied.

<u>Properties</u>

1. Transfer function is the ratio of two polynomials:

$$K(s) = K\frac{L(s)}{M(s)} = K\frac{s^l + a_1 s^{l-1} + \cdots + a_{l-1}s + a_l}{s^m + b_1 s^{m-1} + \cdots + b_{m-1}s + b_m} = K\frac{\prod_{j=1}^{l}(s - q_j)}{\prod_{k=1}^{m}(s - s_k)} \tag{3.3.2}$$

   where,

   q_j, s_k are roots of numerator and denominator polynomials, zeroes and poles of $K(s)$, respectively

   $K = const$ is gain.

   Roots and poles can be real or complex numbers and they may be expressed graphically, in the complex plane (*s*-plane), as the **pole-zero plot**, where the zeros are flagged by • and the poles by ♦.

   For practical circuits:
   a) poles lie in the left half of the complex plane,
   b) degree of the numerator polynomial can not be greater than degree of the denominator polynomial ($l \le m$).

   Degree of the denominator polynomial designates the circuit order, $n = m$.

2. Inverse transform of the Laplace transfer function, transfer function in the *t*-domain, $k(t) = \boldsymbol{L}^{-1}\{K(s)\}$, is the unit **impulse response**.

   If $X(s) = \boldsymbol{L}\{\delta(t)\} = 1$, then $Y(s) = K(s)$ and $y(t) = k(t)$ (3.3.3)

3. The **step response** is the integral of the impulse response:

If $X(s) = \boldsymbol{L}\{\mathbf{1}(t)\} = \dfrac{1}{s}$, then $Y(s) = \dfrac{1}{s}K(s)$ and $y(t) = \int_0^t k(t)dt$ (3.3.4)

4. Time-domain response equals the **convolution** of the impulse response and the input:

$$y(t) = k(t) * x(t) = \int_{-\infty}^{\infty} k(\tau)x(t-\tau)\,d\tau \tag{3.3.5}$$

where, $k(t) * x(t) = \boldsymbol{L}^{-1}\{K(s)\cdot X(s)\}$ (3.3.5a)

is the so called convolution of two time functions.

<u>Transfer functions of selected circuits</u>

Integrator

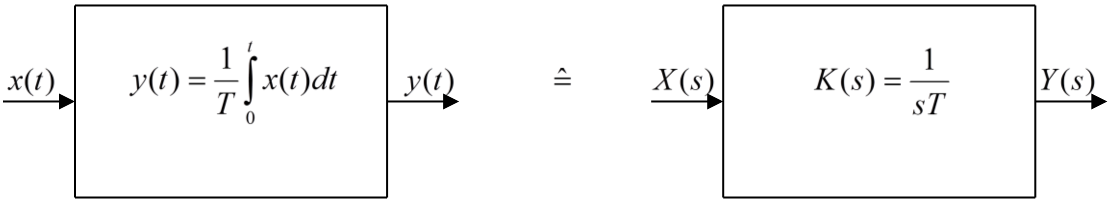


Fig. 3.3.2 Block diagram of an ideal integrator

Following Property 2 of the Laplace transformation (A4)

Laplace transfer function of the **ideal integrator** is:

$$K(s) = \frac{1}{sT} \tag{3.3.6}$$

where, T is the **integration constant**.

RC circuit realization of voltage integrator

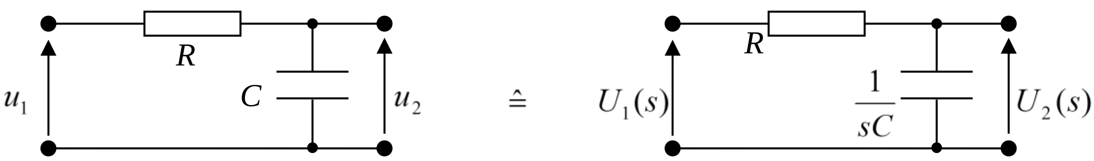


Fig. 3.3.2a *RC* voltage integrator

The *RC* integrator output voltage is

$$U_2(s) = \frac{\frac{1}{sC}}{R+\frac{1}{sC}} U_1(s) = \frac{1}{1+sT} U_1(s) \tag{3.3.7}$$

Then, its transfer function is

$$K(s) = \frac{1}{1+sT},\ T = RC \tag{3.3.7a}$$

The pole-zero plots of the transfer function, for the ideal and the practical *RC* integrators are presented in Fig. 3.3.2b.

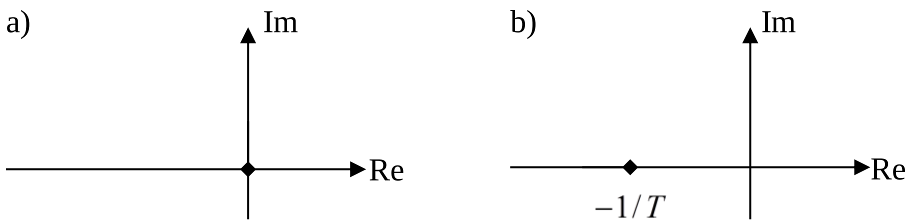


Fig. 3.3.2b Pole-zero plot of integrator transfer function, a) ideal, b) *RC*

The unit step input, $u_1 = U_1\mathbf{1}(t)$, will be used to compare responses of the ideal and the practical integrators. The integrator input in the *s*-domain:

$$U_1(s) = \boldsymbol{L}\{U_1 \cdot \mathbf{1}(t)\} = \frac{U_1}{s} \tag{3.3.8}$$

Then, the outputs are given by equations (3.3.9) and they are graphed in Fig. 3.3.3

- ideal integrator:

$$U_2(s) = \frac{1}{sT}\frac{U_1}{s} = \frac{U_1}{T}\frac{1}{s^2} \mathrel{\hat{=}} u_2(t) = \frac{U_1}{T}t \tag{3.3.9a}$$

- *RC* integrator:

$$U_2(s) = \frac{1}{1+sT}\frac{U_1}{s} = U_1\frac{1}{s(1+sT)} \mathrel{\hat{=}} u_2(t) = U_1[1-\exp(-t/T)] \tag{3.3.9b}$$

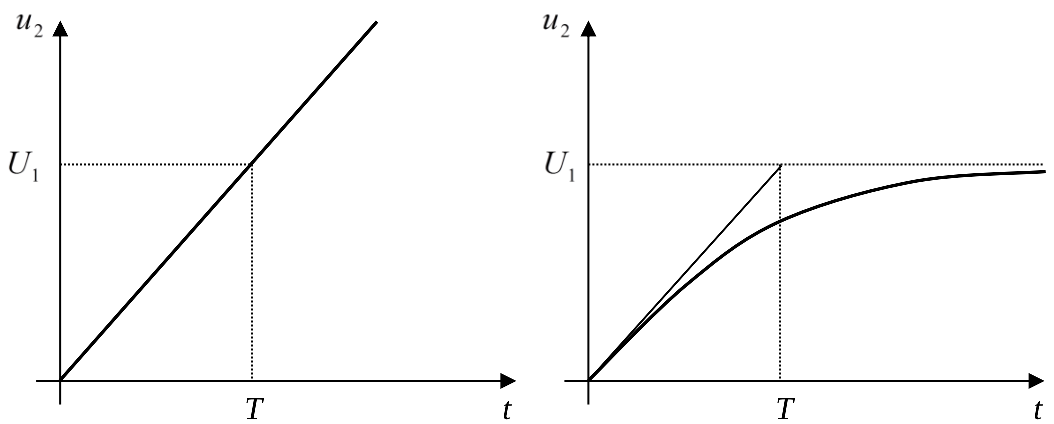


Fig. 3.3.3 Step input response of ideal and *RC* integrator

As can be observed,

- the ideal integrator step response never attains steady state (the response pole is located in the origin of the complex plane),
- the *RC* integrator exhibits "good" integration for $t \le T$, then the integration decays to zero, steady state is reached after $t_\infty \cong 5T$.

Differentiator

Fig. 3.3.4 Block diagram of an ideal differentiator

Following Property 3 of Laplace transformation (A5)

Laplace transfer function of the **ideal differentiator** is:

$$K(s) = sT \tag{3.3.10}$$

where, *T* is the **differentiation constant**.

***RC* circuit realization of voltage differentiator**

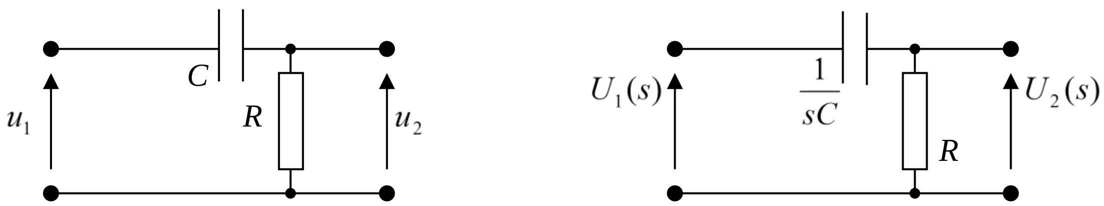


Fig. 3.3.4a The *RC* voltage differentiator

The *RC* differentiator output voltage is

$$U_2(s) = \frac{R}{R + \dfrac{1}{sC}} U_1(s) = \frac{sT}{1+sT} U_1(s) \tag{3.3.11}$$

Then, its transfer function is

$$K(s) = \frac{sT}{1+sT}, \ T = RC \tag{3.3.11a}$$

The unit step input, $u_1 = U_1 \cdot \mathbf{1}(t)$, will be used to compare responses of the ideal and the practical differentiator. The differentiator input in the *s*-domain is given by equation (3.3.8). Then, the output are given by equations (3.3.12) and they are graphed in Fig. 3.3.5.

- ideal differentiator:

$$U_2(s) = sT\frac{U_1}{s} = U_1 T \hat{=} u_2(t) = U_1 T\delta(t) \tag{3.3.12a}$$

- *RC* differentiator:

$$U_2(s) = U_1\frac{T}{1+sT} \hat{=} u_2(t) = U_1 \exp(-t/T) \tag{3.3.12b}$$

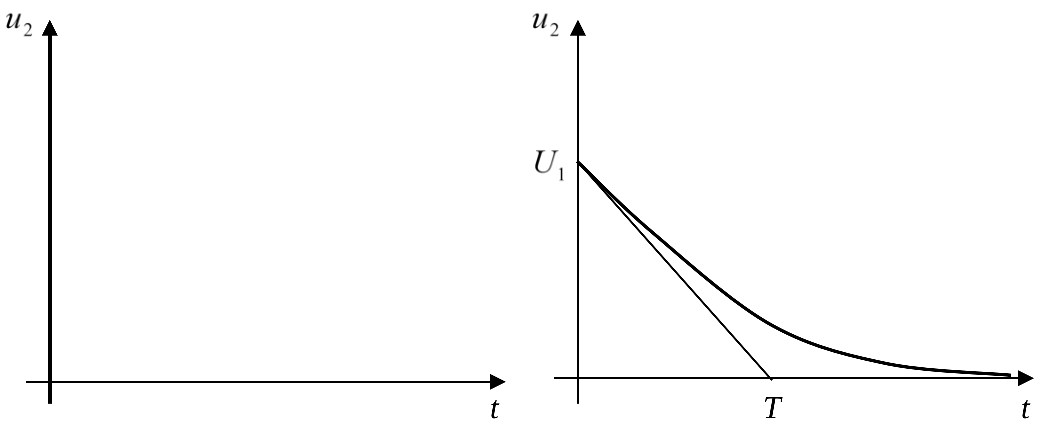


Fig. 3.3.5 Step input response of ideal and *RC* differentiator

As can be observed,

- the ideal integrator step response is an infinitely tall and infinitely narrow pulse of the TU_1 area,
- the *RC* integrator step response is the exponential decay that starts at the step value and lasts after $t_\infty \cong 5T$.

TRANSFER FUNCTION BASED TRANSIENT ANALYSIS - EXAMPLES

Transfer function approach to transient analysis will be illustrated by exemplary circuits. Use of Algorithm 3.3.1 and its modification will be presented. In this modification, input signal is divided into linear segments, each segment described by step or/and ramp function, and sequence of analyses is performed.

Algorithm 3.3.1a – Laplace transfer function method of transient analysis – sequence of analyses

Step 1. Predict the response, if possible.

Step 2. Divide the input signal into linear segments: $x^1,\dots,x^N \;\hat{=}\; X^1(s),\dots,X^N(s)$.

$$x^i = \pm[X^{i_0} + \frac{X^{i_\tau} - X^{i_0}}{\tau^i} t^i]\cdot\mathbf{1}(t^i) \;\hat{=}\; X^i(s) = \pm\left[X^{i_0}\frac{1}{s} + \frac{X^{i_\tau} - X^{i_0}}{\tau^i}\frac{1}{s^2}\right] \quad (3.3.13)$$

where, τ^i - duration of the i-th segment, as presented in Fig. 3.3.6,

X^{i_0}, X^{i_τ} - initial and terminal values of the i-th segment,

$t^i = t - \sum_{j=1}^{i} \tau^{j-1};\ \tau^0 = 0$ - time that starts at the beginning of the i-th segment.

Set $i = 1$.

Step 3. Find the circuit transfer functions $K(s)$ and $K_0(s)$, where

$K(s)$ is transfer function for the input signal, $X^i(s)$,

$K_0(s)$ is transfer function for the initial condition (designated by the preceding segment), $X_0^i(s) = X_0^i / s$.

Step 4. For the i-th segment, find the output signal: $y^i = y_f^i + y_n^i = \boldsymbol{L}^{-1}\{Y_f^i(s) + Y_n^i(s)\}$, where

y_f^i is the forced response, caused by x^i,

y_n^i is the natural response, caused by the i-th segment initial condition X_0^i.

Step 5. If $i < N$,

then set $i = i + 1$, find initial condition for the next segment, X_0^i and GO TO Step 3,

GO TO Step 5, otherwise.

Step 6. Plot the total response. Check whether the obtained solution matches the predicted one.

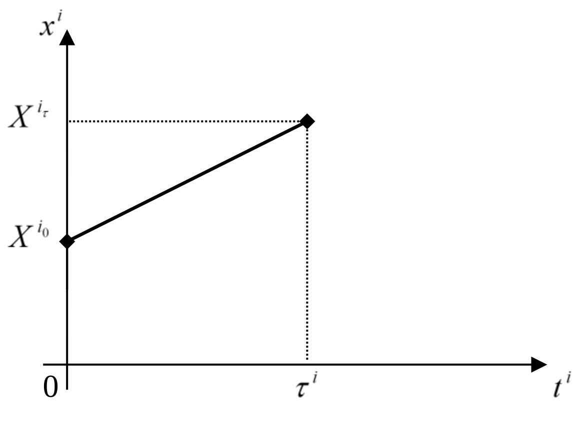


Fig. 3.3.6
The i-th segment of the input signal

In Steps 2 and 3 of the non-modified Algorithm 3.3.1, the algebraic expression of the input signal (normally given by a graph) is designated and transformed into the s-domain. Step and pulse are the most common input signals. Laplace transforms of ideal signals are presented in Appendix A, practical step and practical pulse will be discussed hereafter.

Practical step

A practical step, i.e. step with nonzero rise time τ_r, can be considered as addition of two ramps, as presented in Fig. 3.3.7. Then, the practical step Laplace transform is

$$X(s) = X^1(s) + X^2(s) = \frac{X}{\tau_r}\frac{1}{s^2} - \frac{X}{\tau_r}\frac{1}{s^2}\exp(-s\tau_r) \qquad (3.3.14)$$

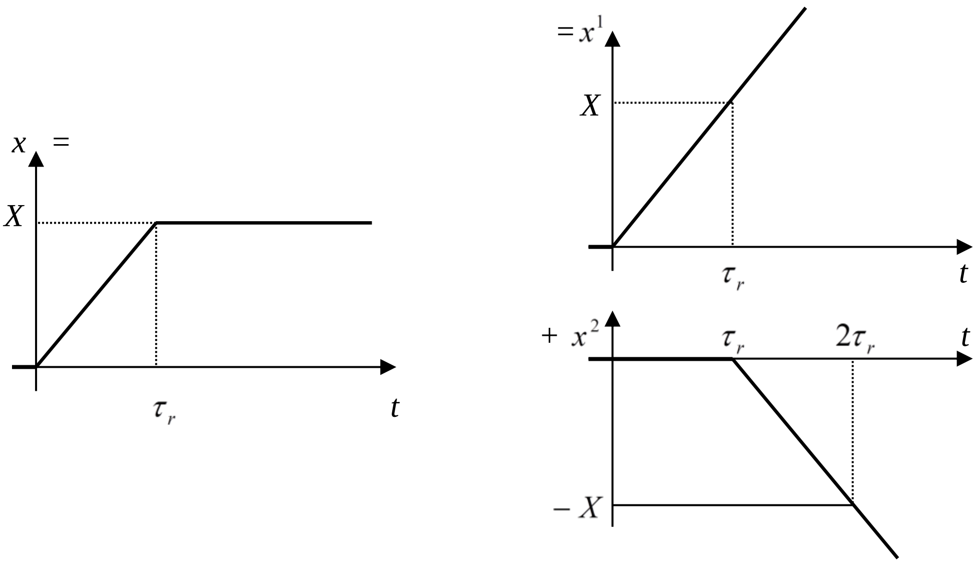


Fig. 3.3.7 Practical step as addition of two ramps

Practical pulse

A practical pulse, i.e. pulse with nonzero rise and fall times, $\tau_r = \tau_1$, $\tau_f = \tau_3 - \tau_2$, can be considered as addition of four ramps, two of them are presented in Fig. 3.3.7, two other in Fig. 3.3.8.

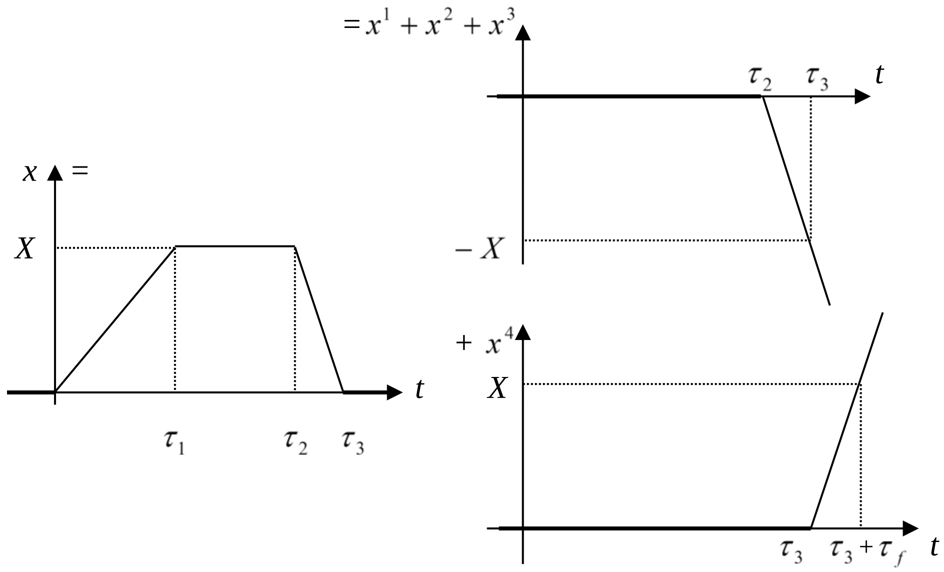


Fig. 3.3.8 Practical pulse as addition of four ramps

Then, the practical pulse Laplace transform is

$$X(s)=\sum_{i=1}^{4}X^{i}(s)=X\frac{1}{s^{2}}-X\frac{1}{s^{2}}\exp(-s\tau_{1})-X\frac{1}{s^{2}}\exp(-s\tau_{2})+X\frac{1}{s^{2}}\exp(-s\tau_{3})\quad(3.3.15)$$

<u>Example 3.3.1</u>

Find a practical step response of the *RC* differentiator.

÷

Algorithm 3.3.1

The input voltage is described by equation (3.3.14), transfer function by equation (3.3.11a). Then, the output voltage is

$$U_{2}(s)=U_{1}\frac{T}{\tau_{r}}\frac{1}{s(1+sT)}-U_{1}\frac{T}{\tau_{r}}\frac{1}{s(1+sT)}\exp(-s\tau_{r})\quad(3.3.16a)$$

$$u_{2}=U_{1}\frac{T}{\tau_{r}}[1-\exp(-t/T)]\cdot\mathbf{1}(t)-U_{1}\frac{T}{\tau_{r}}\left(1-\exp[-(t-\tau_{r})/T]\right)\cdot\mathbf{1}(t-\tau_{r})\quad(3.3.16b)$$

This time-voltage, for the assumed $T=\tau_{r}$, is presented in Fig. 3.3.9.

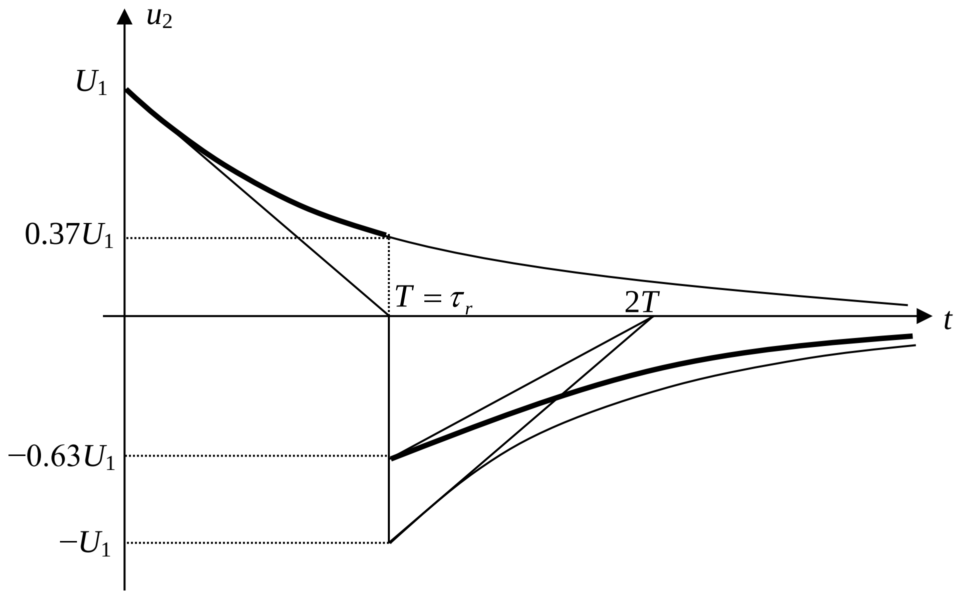


Fig. 3.3.9 Output voltage of *RC* differentiator for practical step input and $T=\tau_{r}$

Algorithm 3.3.1a

The input voltage is divided into two segments:

i=1: $0\le t\le\tau_{r}$

$$t^{1}=t\,,\; u_{1}^{1}=\frac{U_{1}}{\tau_{r}}t\mathbf{1}(t)\,,\; U_{1}^{1}(s)=\frac{U_{1}}{\tau_{r}}\frac{1}{s^{2}}\,,\; U_{C0}^{1}=0\quad(3.3.17a)$$

i=2: $t\ge\tau_{r}$

$$t^{2}=t-\tau_{r}\,,\; u_{1}^{2}=U_{1}\mathbf{1}(t^{2})\,,\; U_{1}^{2}(s)=U_{1}\frac{1}{s}\quad(3.3.17b)$$

Transfer function for the initial condition (capacitor voltage) can be designated from Fig. 3.3.10.

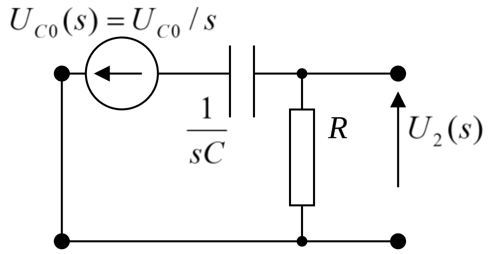


Fig. 3.3.10 Circuit for calculation of $K_0(s)$ in *RC* differentiator

The output voltage, and then, the transfer function are

$$U_2(s) = -\frac{R}{R + \frac{1}{sC}} U_{C0}(s) \Rightarrow K_0(s) = -\frac{sT}{1+sT} \tag{3.3.18}$$

where the initial condition is calculated from equation (3.3.19)

$$u_C = u_1 - u_2 \tag{3.3.19}$$

For *i*=1

$$U_2^1(s) = U_1 \frac{T}{\tau_r} \frac{1}{s(1+sT)} \;\hat{=}\; u_2^1 = U_1 \frac{T}{\tau_r}[1 - \exp(-t/T)] \tag{3.3.20}$$

$$u_C^1 = \frac{U_1}{\tau_r} t - U_1 \frac{T}{\tau_r}[1 - \exp(-t/T)] \Rightarrow u_C^1(\tau_r) = U_{C0}^2 = U_1 - U_1 \frac{T}{\tau_r}[1 - \exp(-\tau_r/T)]$$

(3.3.20a)

For *i*=2

$$U_2^2(s) = (U_1 - U_{C0}^2) \frac{T}{1+sT} \;\hat{=}\; u_2^2 = (U_1 - U_{C0}^2) \exp(-t^2/T) \tag{3.3.21}$$

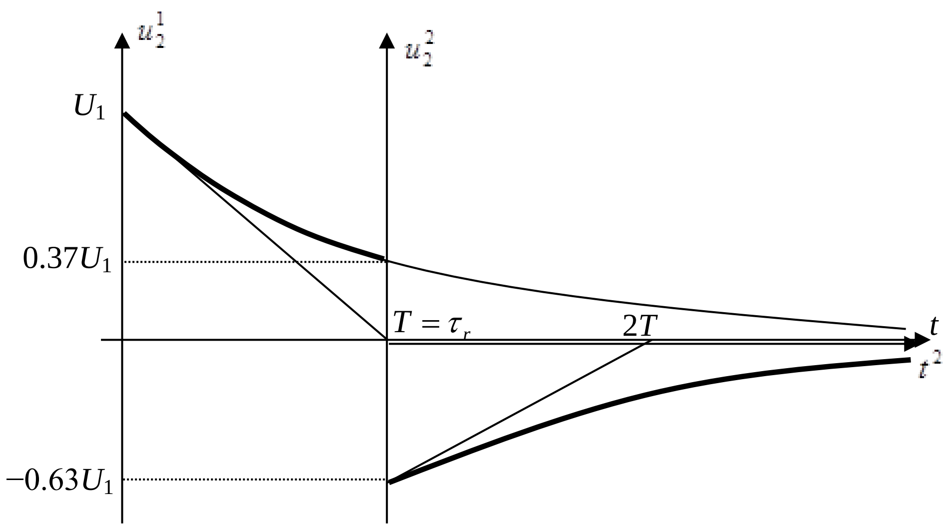


Fig. 3.3.11 Output voltage of *RC* differentiator for practical step input and $T = \tau_r$

For the assumed $T=\tau_r$, $U_{C0}^2 = 0.37U_1$. The output voltage is presented in Fig. 3.3.11.

♦

Example 3.3.2

Find ideal pulse response of the *RC* integrator.

÷

Algorithm 3.3.1

The input voltage is described by equation (A.12a), transfer function by equation (3.3.7a). Then, the output voltage is

$$U_2(s) = U_1 \frac{1}{s(1+sT)} - U_1 \frac{1}{s(1+sT)} \exp(-s\tau) \tag{3.3.22a}$$

$$u_2 = U_1[1-\exp(-t/T)]\cdot \mathbf{1}(t) - U_1\left(1-\exp[-(t-\tau)/T]\right)\cdot \mathbf{1}(t-\tau) \tag{3.3.22b}$$

This time-voltage, for the assumed $T=\tau$, is presented in Fig. 3.3.12.

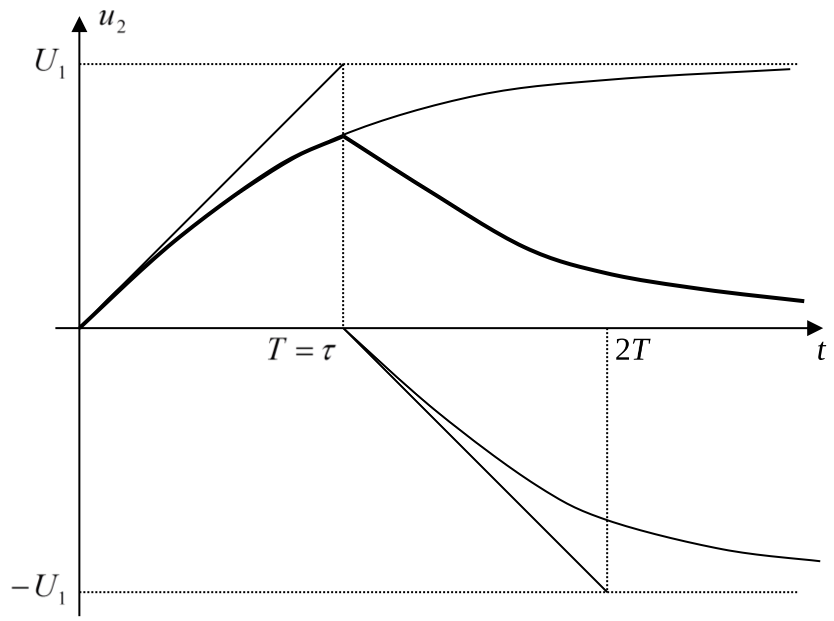


Fig. 3.3.12 Output voltage of *RC* integrator for practical step input and $T=\tau$

Algorithm 3.3.1a

The input voltage is divided into two segments:

i=1: $0 \le t \le \tau$

$$t^1 = t\ ,\ u_1^1 = U_1\mathbf{1}(t)\ ,\ U_1^1(s) = U_1\frac{1}{s}\ ,\ U_{C0}^1 = 0 \tag{3.3.23a}$$

i=2: $t \ge \tau$

$$t^2 = t-\tau\ ,\ u_1^2 = 0 \tag{3.3.23b}$$

Transfer function for the initial condition (capacitor voltage) can be designated from Fig. 3.3.13.

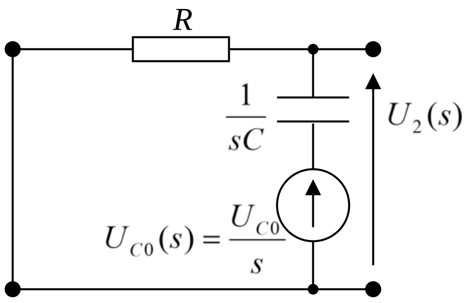


Fig. 3.3.13 Circuit for calculation of $K_0(s)$, in *RC* integrator

The output voltage, and then, the transfer function are

$$U_2(s)=\frac{R}{R+\frac{1}{sC}}U_{C0}(s)\Rightarrow K_0(s)=\frac{sT}{1+sT} \tag{3.3.24}$$

where the initial condition is calculated from equation (3.3.25)

$$u_C=u_2 \tag{3.3.25}$$

For i=1

$$U_2(s)=U_1\frac{1}{s(1+sT)}\,\hat{=}\,u_2^1=U_1[1-\exp(-t/T)] \tag{3.3.26}$$

$$u_C^1(\tau)=U_{C0}^2=U_1[1-\exp(-\tau T)] \tag{3.3.26a}$$

For i=2

$$U_2(s)=U_{C0}^2\frac{T}{1+sT}\,\hat{=}\,u_2^1=U_{C0}\exp(-t^2/T) \tag{3.3.27}$$

This voltage, for the assumed $T=\tau$, is presented in Fig. 3.3.14.

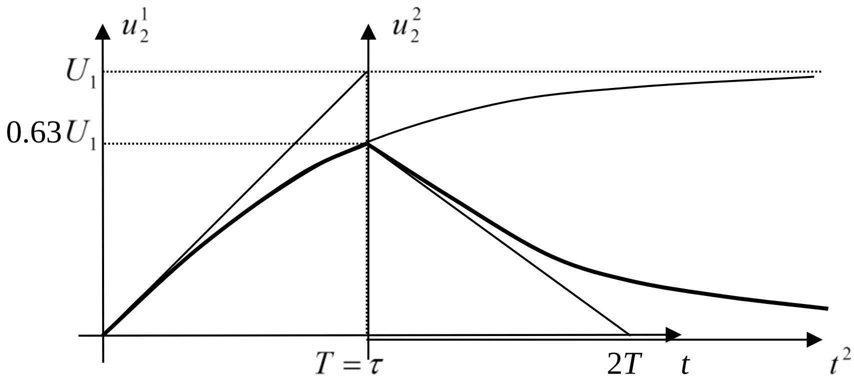


Fig. 3.3.14 Output voltage of *RC* integrator for practical step input and $T=\tau_r$

♦

Drill problems 3.3

1. For the $E = 1\,\mathrm{V}$ step input, draw the *RC* differentiator (integrator) output, for $R = 1\mathrm{k}\Omega$; $C = 1\mu\mathrm{F}$. Assume a) the ideal step, b) the practical step of $\tau_r = 1\,\mathrm{s}$.
2. For the ideal pulse input of $E = 1\,\mathrm{V}$; $\tau = 1\,\mathrm{s}$, draw the *RC* differentiator (integrator) output, for $R = 1\mathrm{k}\Omega$; $C = 1\mu\mathrm{F}$.
3. For the practical step input of $E = 10\,\mathrm{V}$; $\tau_r = 10\,\mathrm{ms}$, draw the ideal differentiator (integrator) output Assume the differentiation (integration) constant $T = \tau_r$.
4. For the ideal pulse input: $E = 10\,\mathrm{V}$; $\tau = 10\,\mathrm{ms}$, draw the ideal integrator output. Assume the integration constant $T = \tau$.
5. For the practical pulse input of $E = 10\,\mathrm{V}$; $\tau = 10\,\mathrm{m}$; $\tau_r = 2\,\mathrm{ms}$; $\tau_f = 1\,\mathrm{ms}$, draw the ideal differentiator output. Assume the differentiation constant $T = \tau$.
6. Find the ideal integrator ($T = 2$ ms) output, $u_2(5\,\mathrm{ms})$, if 20 V step is inputted.
7. For the given voltage waveform of the *RC* integrator input, plot the output voltage. Assume: $RC << \tau$

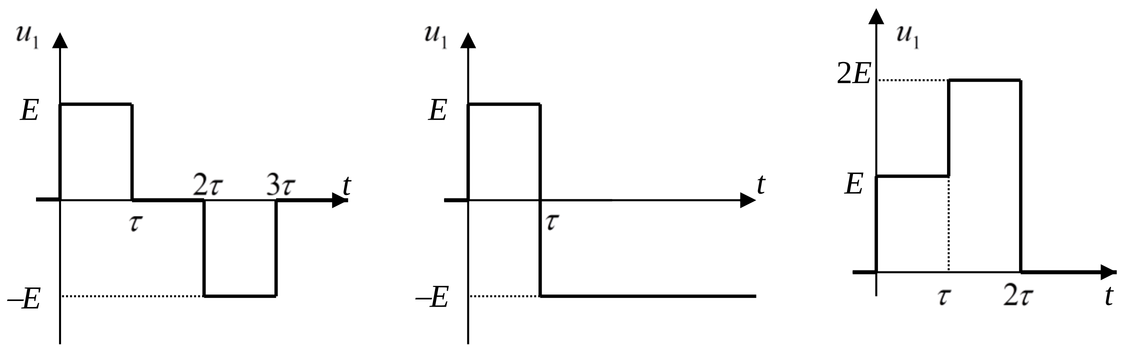
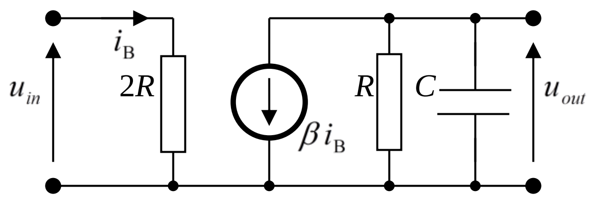


Fig. P.3.3.7

8. Repeat Problem 3.3.7 for the *RC* differentiator.
9. For the circuit shown, compute the transfer function $K(s) = U_{out}(s)/U_{in}(s)$. Sketch $u_{out}(t)$ if $u_{in}(t) = E\mathbf{1}(t)$.

Fig. P.3.3.9

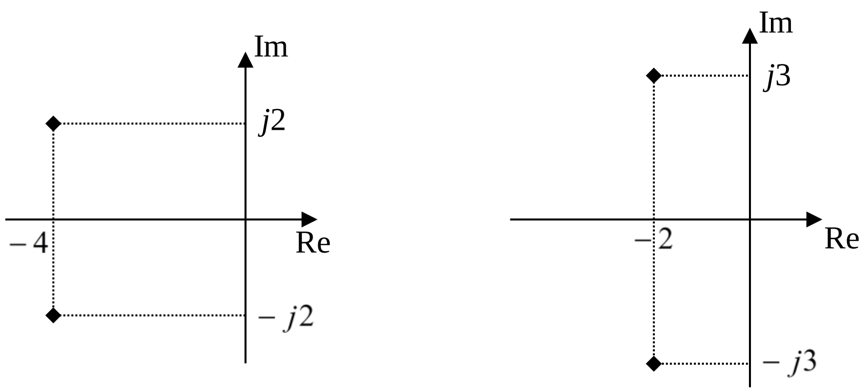


10. Find the current response of the series combination of *R* and *L* to an applied voltage impulse of 2 Vs.
11. Repeat Problem 3.3.10 for the series combination of *R* and *C*.
12. Draw the pole-zero plot of differentiator transfer function, for both the ideal and *RC* differentiators.
13. Draw the pole-zero plot of admittance of the series connection of *R*, *L* and *C*. Assume: $C = 1\mu\mathrm{F}$, $L = 1\,\mathrm{H}$, $R =$ a) 1 kΩ, b) 2 kΩ, c) 3 kΩ.

14. Draw the pole-zero plot of impedance of the parallel connection of G, L and C. Assume: $C = 1\mu\text{F}$, $L = 1\,\text{H}$, $G =$ a) 1 mS, b) 2 mS, c) 3 mS.

15. The pole-zero plots of two transfer functions are shown. If their gain is $K = 2$, find $K(s)$. Which function has the greater dumping coefficient, which has the greater damped resonant frequency ?

Fig. P.3.3.15



4. AC STEADY-STATE ANALYSIS

4.1 ALTERNATING CURRENT – RMS VALUE, PHASOR NOTATION

An **alternating current**, ac in short, is by definition a sinusoidal current:

$$i = I_m \sin(\omega t + \alpha_i) \tag{4.1.1}$$

where,

I_m is the amplitude, in [A]

$$\omega = \frac{2\pi}{T} = 2\pi f \tag{4.1.1a}$$

is the radian or angular frequency, in [rad/s],

T is the period, in [s]; f is the frequency, in hertz [Hz],

α_i is the initial phase angle, or simply the phase, in radians.

Two sinusoidal currents with different phases are presented in Fig. 4.1.1

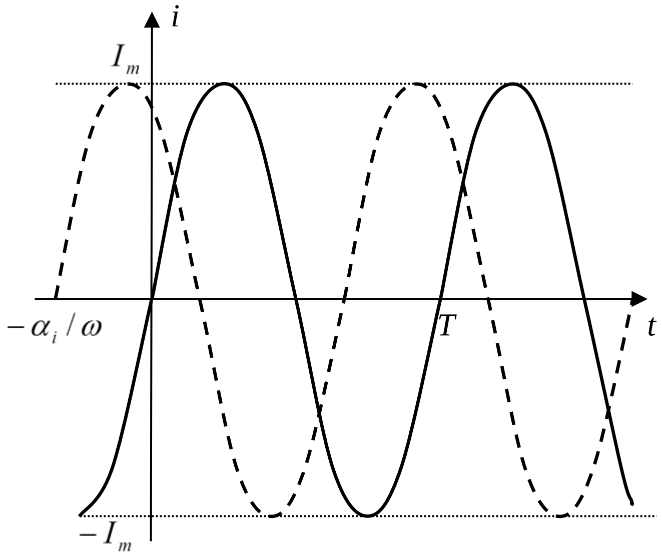


Fig. 4.1.1 Two sinusoids with different phase

The solid curve phase is zero while the dashed curve phase is α_i radians. It can be said, that the dashed sinusoid leads the solid one by the angle of α_i radians, or the solid one lags the dashed one by the same angle.

To describe the energy delivered by a periodic current or voltage to a resistive load, its root-mean-square value, rms in short, or effective value is defined.

The **rms value** of a periodic current (voltage) is a constant that is equal to the dc current (voltage) that delivers the same power to a resistance *R*.

The energy delivered by the dc current within the time of one complete cycle *T* should be equal to the energy delivered by a periodic current during the same time:

$$I_{\text{rms}}^2 RT = \int_0^T i^2 R\, dt \tag{4.1.2}$$

Then, the rms current is

$$I_{\text{rms}} = I = \sqrt{\frac{1}{T}\int_0^T i^2 dt} \tag{4.1.3a}$$

In a similar manner, the rms voltage is

$$U_{\text{rms}} = U = \sqrt{\frac{1}{T}\int_0^T u^2 dt} \tag{4.1.3b}$$

For a sinusoidal current (voltage), rms value is equal to

$$I = \frac{I_m}{\sqrt{2}};\ U = \frac{U_m}{\sqrt{2}} \tag{4.1.4}$$

Thus, for a sinusoidal waveform, the effective or rms value is 0.707 times the maximum value. For example, the household ac voltage is 230 V, with a maximum voltage of 325 V. Any sinusoidal current or voltage, at a given radian frequency ω is uniquely characterized by its effective value and phase, so called **phasor**, as described in Appendix B.

$$I(j\omega) = I(\omega)\exp[j\alpha_i(\omega)] \mathrel{\hat{=}} i = I(\omega)\sqrt{2}\sin[\omega t + \alpha_i(\omega)] \tag{4.1.5a}$$

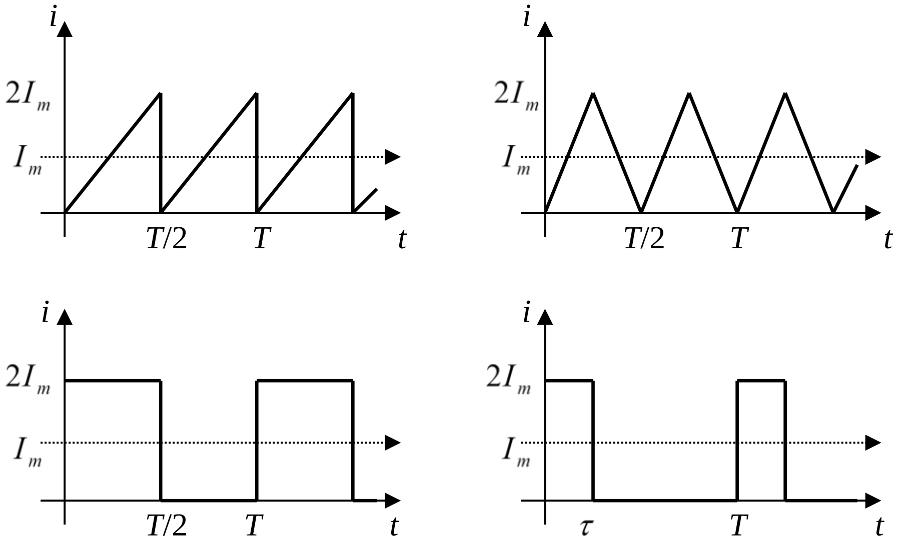
$$U(j\omega) = U(\omega)\exp[j\alpha_u(\omega)] \mathrel{\hat{=}} u = U(\omega)\sqrt{2}\sin[\omega t + \alpha_u(\omega)] \tag{4.1.5b}$$

Then, element equations and Kirchhhoff's laws can be transformed from the time-domain into the frequency (phasor)-domain and the problem of ac steady-state analysis can be carried out, as presented in the next Chapter. For simplicity of notation, if the frequency is fixed, i.e. circuit frequency characteristics are not considered, then: $I(\omega) = I,\ \alpha_i(\omega) = \alpha_i$, etc.

Drill problems 4.1

1. Find the effective value of periodic current for sawtooth, triangular and rectangular waveforms. Repeat calculations for $i^* = i + I_m$, as denoted by the dotted time axis.

Fig. P.4.1.1



4.2 PHASOR ANALYSIS

KIRCHHOFF'S LAWS

Kirchhoff's Current Law

If sinusoidal excitation is applied to a circuit, then sinusoidal currents (4.1.1) flow through the elements. If cosinusoidal excitation is applied to a circuit, then cosinusoidal currents, say $i' = I\sqrt{2}\cos(\omega t + \alpha_i)$, flow through the elements. KCL holds for both excitations, then it also holds for the complex excitation (defined in Appendix B):

$$\sum_{\bullet i}[i'(t) + ji(t)] = \sum_{\bullet i}\hat{i}(t) = \sum_{\bullet i}\sqrt{2}I(j\omega)\exp(j\omega t) = 0$$

Dividing out the common factor $\sqrt{2}\exp(j\omega t)$, KCL for phasors is obtained.

The phasor algebraic sum of all currents at a node (cutset) is equal to zero

$$\sum_{\bullet i} I(j\omega) = 0 \qquad (4.2.1)$$

Kirchhoff's Voltage Law

A similar development will also establish KVL.

The phasor algebraic sum of all voltages around a loop (closed path) is equal to zero

$$\sum_{\circ i} U(j\omega) = 0 \qquad (4.2.2)$$

ELEMENT LAWS

Resistor

Setting sinusoids into the Ohm's law, the following equation is obtained:

$$U\sqrt{2}\sin(\omega t + \alpha_u) = RI\sqrt{2}\sin(\omega t + \alpha_i) \qquad (4.2.3)$$

Thus, the resistor rms voltage may be expressed by its rms current, the voltage phase by the current phase

$$U = RI \qquad (4.2.4a)$$

$$\alpha_u = \alpha_i \qquad (4.2.4b)$$

Setting these equations in the voltage phasor (4.1.5b), the following equation is obtained

$$U(j\omega) = U\exp(j\alpha_u) = RI\exp(j\alpha_i)$$

Finally, current-voltage law for phasors can be formulated.

$$U(j\omega) = RI(j\omega) \tag{4.2.5a}$$
$$I(j\omega) = GU(j\omega) \tag{4.2.5b}$$

Circuit symbols for a resistor described in time and frequency domains are presented in Fig. 4.2.1.

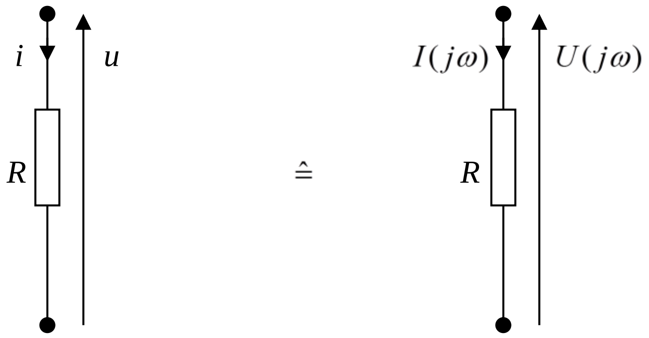


Fig. 4.2.1 Circuit symbols for a resistor described in time and frequency domain

Inductor

Setting sinusoids into the inductor law (3.1.16a)

$$U\sqrt{2}\sin(\omega t + \alpha_u) = \omega LI\sqrt{2}\sin(\omega t + \alpha_i + 90^\circ) \tag{4.2.6}$$

Then,

$$U = \omega LI \tag{4.2.7a}$$
$$\alpha_u = \alpha_i + 90^\circ \tag{4.2.7b}$$

Next,

$$U(j\omega) = U\exp(j\alpha_u) = \omega LI\exp(j\alpha_i)\exp(j90^\circ)$$

Finally, current-voltage law for phasors can be formulated.

$$U(j\omega) = j\omega LI(j\omega) \tag{4.2.8a}$$
$$I(j\omega) = \frac{1}{j\omega L}U(j\omega) \tag{4.2.8b}$$

Circuit symbols for a coil described in time and frequency domains are presented in Fig. 4.2.2.

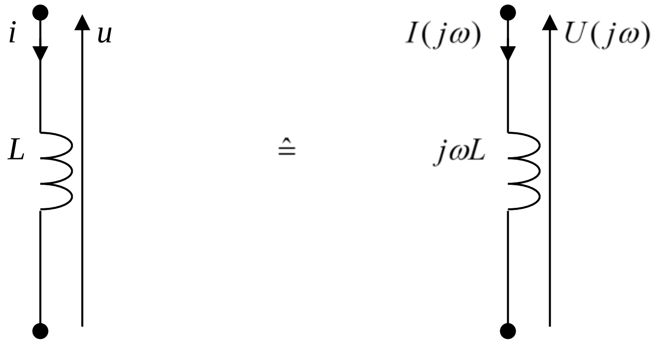


Fig. 4.2.2 Circuit symbols for a coil described in time and frequency domain

Capacitor

Setting sinusoids into the capacitor law (3.1.8a), the following equation is obtained:

$$I\sqrt{2}\sin(\omega t + \alpha_i) = \omega CU\sqrt{2}\sin(\omega t + \alpha_u + 90^\circ) \tag{4.2.9}$$

Thus,

$$I = \omega CU \tag{4.2.10a}$$

$$\alpha_i = \alpha_u + 90^\circ \tag{4.2.10b}$$

Next,

$$I(j\omega) = I\exp(j\alpha_i) = \omega CU\exp(j\alpha_u)\exp(j90^\circ)$$

Finally, current-voltage law for phasors can be formulated.

$$I(j\omega) = j\omega CU(j\omega) \tag{4.2.11a}$$

$$U(j\omega) = \frac{1}{j\omega C}I(j\omega) \tag{4.2.11b}$$

Circuit symbols for a capacitor described in time and frequency domains are shown in Fig. 4.2.3.

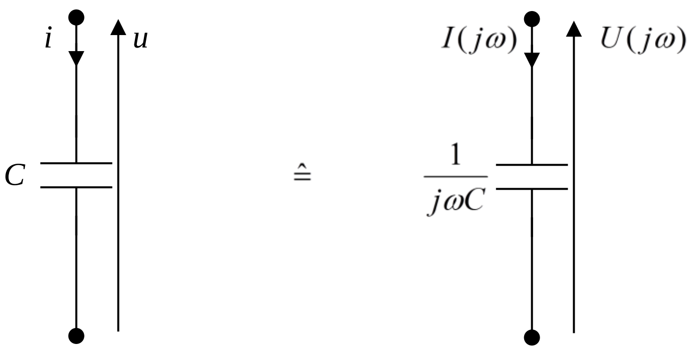


Fig. 4.2.3 Circuit symbols for a capacitor described in time and frequency domains

GENERAL TWO-TERMINAL PHASOR CIRCUIT, PHASOR IMPEDANCE

Element equations (4.2.5), (4.2.8) and (4.2.11) can be expressed in the general form

$$U(j\omega) = Z(j\omega)I(j\omega) \tag{4.2.12a}$$

$$I(j\omega) = Y(j\omega)U(j\omega) \tag{4.2.12b}$$

where

$$Z(j\omega) = 1/Y(j\omega) = \begin{cases} R & \text{resistor impedance} \\ j\omega L & \text{inductor impedance} \\ 1/j\omega C & \text{capacitor impedance} \end{cases} \tag{4.2.13}$$

is called the element **complex impedance**, $Y(j\omega)$ is called the **complex admittance**. As can be observed, these impedances (admittances) are the *s*-domain impedances (admittances), collected in Table 3.1.2, with $s = j\omega$.

Consider a general phasor subcircuit with two accessible terminals, as presented in Fig. 4.2.4.

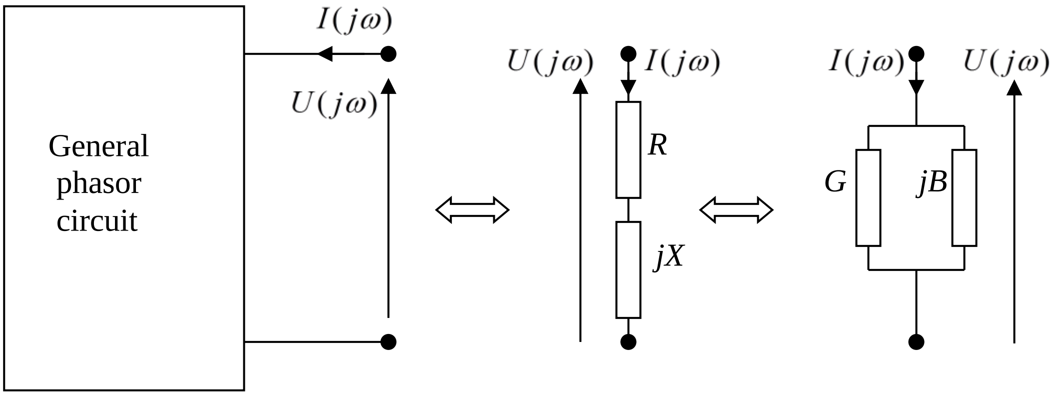


Fig. 4.2.4 General phasor two-terminal subcircuit and its equivalents

The equivalent impedance of such subcircuit can be defined, as the ratio of the phasor voltage to the phasor current:

$$Z(j\omega) = Z(s)\big|_{s=j\omega} = \frac{U(j\omega)}{I(j\omega)} = Z(\omega)\exp[j\varphi(\omega)] = R(\omega) + jX(\omega) \tag{4.2.14a}$$

The reciprocal of impedance, the ratio of the phasor current to the phasor voltage is the equivalent admittance:

$$Y(j\omega) = Y(s)\big|_{s=j\omega} = \frac{I(j\omega)}{U(j\omega)} = Y(\omega)\exp[-j\varphi(\omega)] = G(\omega) + jB(\omega) \tag{4.2.14a}$$

It is important to stress that impedance (admittance) is a complex number that scales one phasor to produce another, but it is not a phasor. The modulus (magnitude) of impedance is the ratio of effective values of the voltage and the current and the angle is the difference of the voltage and the current angles, as presented in equations (4.2.15) – for simplicity of description argument ω is omitted (fixed frequency is assumed).

$$Z = \frac{U}{I}, \ \varphi = \alpha_u - \alpha_i \tag{4.2.15}$$

The impedance (admittance) can be expressed in a rectangular form, as presented in equations (4.2.14), where:

- $R = \mathrm{Re}\{Z(j\omega)\}$ is the resistive component of *Z*, or simply **resistance**,
- $X = \mathrm{Im}\{Z(j\omega)\}$ is the reactive component of *Z*, or simply **reactance**,
- $G = \mathrm{Re}\{Y(j\omega)\}$ is the conductive component of *Y*, or simply **conductance**,
- $B = \mathrm{Im}\{Y(j\omega)\}$ is the susceptive component of *Y*, or simply **susceptance**.

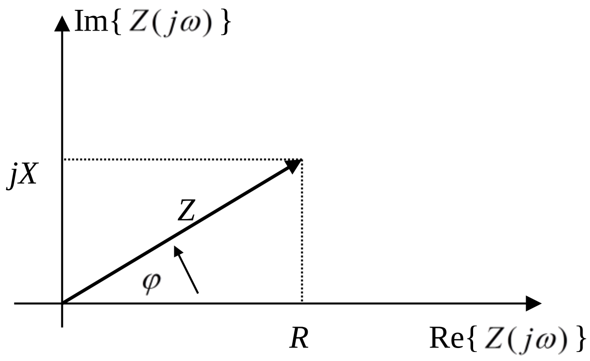
For the given components of impedance (admittance) its magnitude and angle can be determined, or vice-versa:

$$Z = \sqrt{R^2 + X^2}, \ \varphi = \arctan\frac{X}{R} \tag{4.2.16a}$$

$$R = Z\cos\varphi, \ X = Z\sin\varphi \tag{4.2.16b}$$

These relationships are graphically expressed in Fig. 4.2.5.

Fig. 4.2.5 Graphical representation of impedance



For the given components of impedance, components of admittance can be determined, or vice-versa:

$$Y = G + jB = \frac{1}{R + jX} = \frac{R - jX}{R^2 + X^2} \Rightarrow G = \frac{R}{R^2 + X^2}, \ B = -\frac{X}{R^2 + X^2} \tag{4.2.17a}$$

$$R = \frac{G}{G^2 + B^2}, \ X = -\frac{B}{G^2 + B^2} \tag{4.2.17b}$$

For the fixed frequency ω, taking into account rectangular form of the impedance, its series equivalent circuit can be found, taking into account rectangular form of the admittance, its parallel equivalent circuit can be found, as presented in Fig. 4.2.4.

The resistance is always nonnegative while reactance can be positive (inductive) or negative (capacitive), and correspondingly the impedance angel can be positive or negative

$$\varphi \in < -90^\circ, +90^\circ > \tag{4.2.18}$$

All possible cases are considered next.

- $\varphi = 90^{\circ}$

  The voltage leads the current by 90 degrees, as presented in Fig. 4.2.6 ($I(j\omega)$ is assumed as the reference phasor, $\alpha_i = 0^{\circ}$). The resistance is equal to zero and subcircuit has pure inductive character, its equivalent consists of one element, inductor, as presented in the same Fig. 4.2.6.

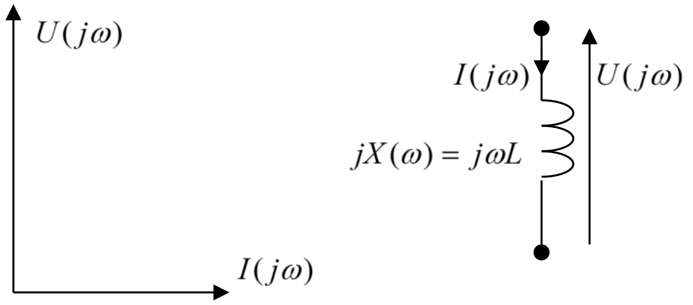


Fig. 4.2.6 Voltage and current phasors and circuit equivalent,

- $0^{\circ} < \varphi < 90^{\circ}$

  The voltage leads the current by the angle less than 90 degrees, as presented in Fig. 4.2.7 ($\alpha_i = 0^{\circ}$). The resistance is greater than zero, reactance is positive, $X = \omega L_s$. Then, the circuit has inductive character. Its equivalent consists of two elements, resistor and inductor, as presented in the same Fig. 4.2.7.

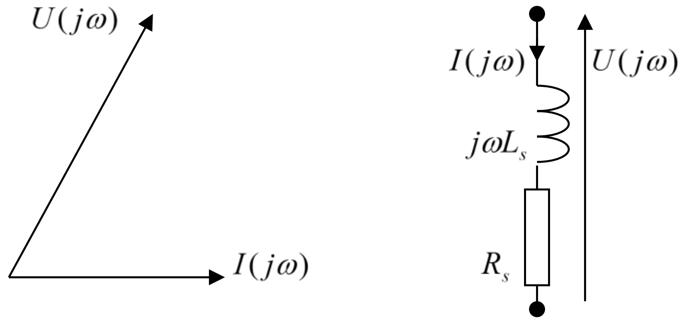


Fig. 4.2.7 Voltage and current phasors and circuit equivalent, 0°< φ < 90°

- $\varphi = 0^{\circ}$

  There is no shift between the voltage and the current, as presented in Fig. 4.2.8 ($\alpha_i = 0^{\circ}$). The resistance is greater than zero, reactance is equal to zero. Then, the subcircuit has resistive character, its equivalent consists of one element, resistor, as presented in the same Fig. 4.2.8.

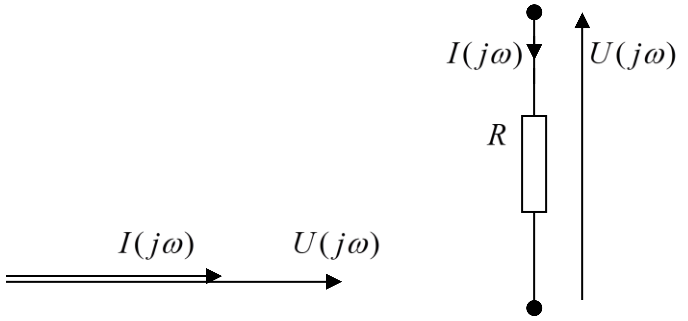


Fig. 4.2.8 Voltage and current phasors and circuit equivalent, $\varphi = 0°$

- $-90^{\circ} < \varphi < 0^{\circ}$
  The current leads the voltage by the angle less than 90 degrees, as presented in Fig. 4.2.9 ($\alpha_i = 0^{\circ}$). The resistance is greater than zero, reactance is negative, $X = -1/\omega C_s$. Then, the circuit has capacitive character. Its equivalent consists of two elements, resistor and capacitor, as presented in the same Fig. 4.2.9.

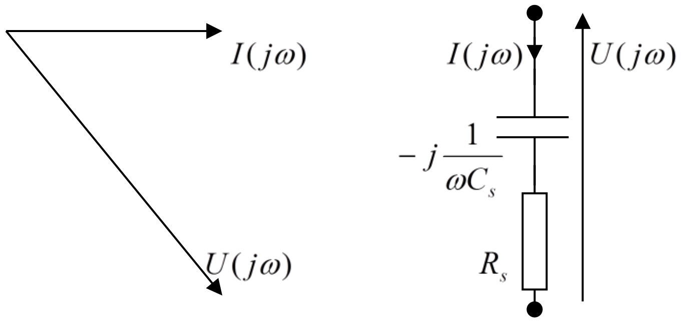


Fig. 4.2.9 Voltage and current phasors and circuit equivalent, −90°< φ < 0°

- $\varphi = -90^{\circ}$
  The current leads the voltage by the angle of 90 degrees, as presented in Fig. 4.2.10 ($\alpha_i = 0^{\circ}$). The resistance is equal to zero, reactance is negative, $X = -1/\omega C$. Then, the subcircuit has pure capacitive character, its equivalent consists of one element, capacitor, as presented in the same Fig. 4.2.10.

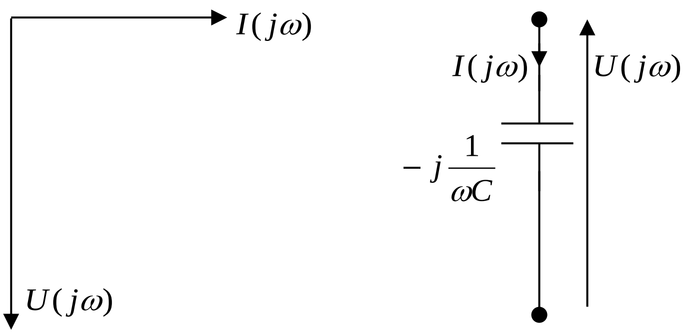


Fig. 4.2.10 Voltage and current phasors and circuit equivalent

Example 4.2.1

Find the series and parallel equivalents of the circuit presented in Fig. 4.2.11, $R_1 = 5\Omega,\ R_2 = 10\Omega,\ \omega L = 15\Omega,\ 1/\omega C = 10\Omega,\ \omega = 1000$ rad/s.

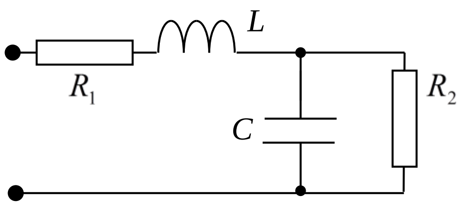


Fig. 4.2.11 Circuit for Example 4.2.1

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The subcircuit impedance is

$$Z(j\omega) = R_1 + j\omega L + \frac{R_2 \frac{1}{j\omega C}}{R_2 + \frac{1}{j\omega C}} = 5 + j15 + \frac{10(-j10)}{10 - j10} = 5 + j15 + 5 - j5 = 10 + j10\ \Omega \tag{4.2.19}$$

As can be observed, for the given frequency of 1000 rad/s, the subcircuit has inductive character.

The series equivalent consists of $R_s = 10\,\Omega$ resistance and $X = \omega L_s = 10\,\Omega$ reactance. Then, the series inductance is $L_s = 10$ mH.

From (4.2.17a), parameters of the parallel equivalent can be calculated: $G = 0.05$ S, $B = -1/\omega L_p = -0.05$ S. Then, the parallel resistance is equal to $R_p = 1/G = 20\,\Omega$ and the parallel inductance is equal to $L_p = 20$ mH.

Both equivalent circuits are presented in Fig. 4.2.12.

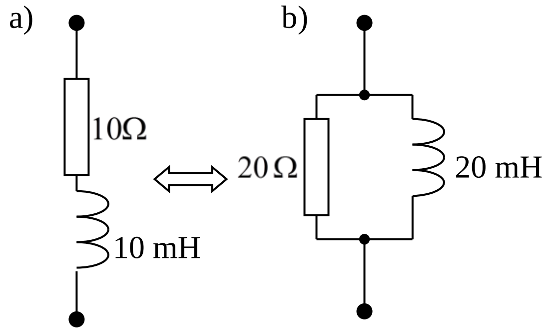

Fig. 4.2.12 Equivalent circuits for Example 4.2.1

ALGORITHM OF AC STEADY-STATE ANALYSIS

The Kirchhoff's laws (4.2.1) and (4.2.2), together with element laws (4.2.5), (4.2.8) and (4.2.11), can be used to formulate circuit equations in the phasor-domain. The analysis is therefore identical to the resistive circuit analysis, with impedances replacing resistances and phasors replacing dc currents and voltages, nodal analysis can be applied. Then, algorithm of ac steady-state analysis can be formulated.

Algorithm 4.2.1 Phasor method of ac steady-state analysis

Step 1. Built a phasor circuit.
Step 2. Formulate phasor equations, nodal method can be applied.
Step 3. Solve the equations to find phasors describing currents and voltages.
Step 4. Express the solution graphically, by means of the phasor diagram.

Transformation of the solution to the time-domain is trivial. Once phasors are found, they can be converted immediately to the time-domain sinusoidal answers. Phasor diagram is helpful in checking correctness of the solution, also allows to read phase shifts between phasors.

Example 4.2.2

Find the mesh current, element voltages and voltage $U_{\mathrm{CA}}(j\omega)$; draw the phasor diagram:
$e = 10\sin(314t + 45^\circ)$ V, $X_L = \omega L = 20\,\Omega$, $|X_C| = 1/\omega C = 10\,\Omega$, $R = 10\,\Omega$.

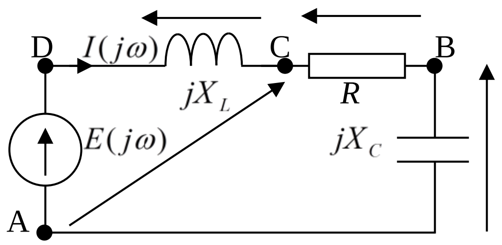

Fig. 4.2.13 Phasor circuit of Example 4.2.2

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The mesh current is:

$$I(j\omega) = \frac{E(j\omega)}{R + j(\omega L - 1/\omega C)} = \frac{10/\sqrt{2}\exp(j45^\circ)}{10 + j10} = 0.5\exp(j0^\circ) = 0.5\,\mathrm{A} \qquad (4.2.20)$$

It lags the supply voltage, what means that, for $\omega = 314$ rad/s, *RLC* series circuit has inductive character.
Next, element voltages can be calculated

$$U_R(j\omega) = 5\exp(j0^\circ) = 5\,\mathrm{V} \qquad (4.2.21a)$$

$$U_L(j\omega) = 10\exp(j90^\circ) = j10\,\mathrm{V} \qquad (4.2.21a)$$

$$U_C(j\omega) = 5\exp(-j90^\circ) = -j5\,\mathrm{V} \qquad (4.2.21a)$$

These voltages satisfy KVL equation

$$U(j\omega) = U_L(j\omega) + U_R(j\omega) + U_C(j\omega) \qquad (4.2.22)$$

This solution can be expressed graphically. Fig. 4.2.14 presents three phasor diagrams:

a) all phasors are anchored in the origin of the complex plane,

b) voltage phasors are shifted, following KVL equation,

c) voltage phasors are shifted, following KVL equation, such that the circuit topology is mapped.

In this latter case, location of the circuit nodes is uniquely defined and thus, all other voltages may be read directly from the diagram. The voltage between nodes C and A is

$$U_{CA}(j\omega) = 10/\sqrt{2}\exp(-j45^\circ)\,\mathrm{V} \mathrel{\hat{=}} u_{CA} = 10\sin(314t - 45^\circ)\,\mathrm{V} \qquad (4.2.23)$$

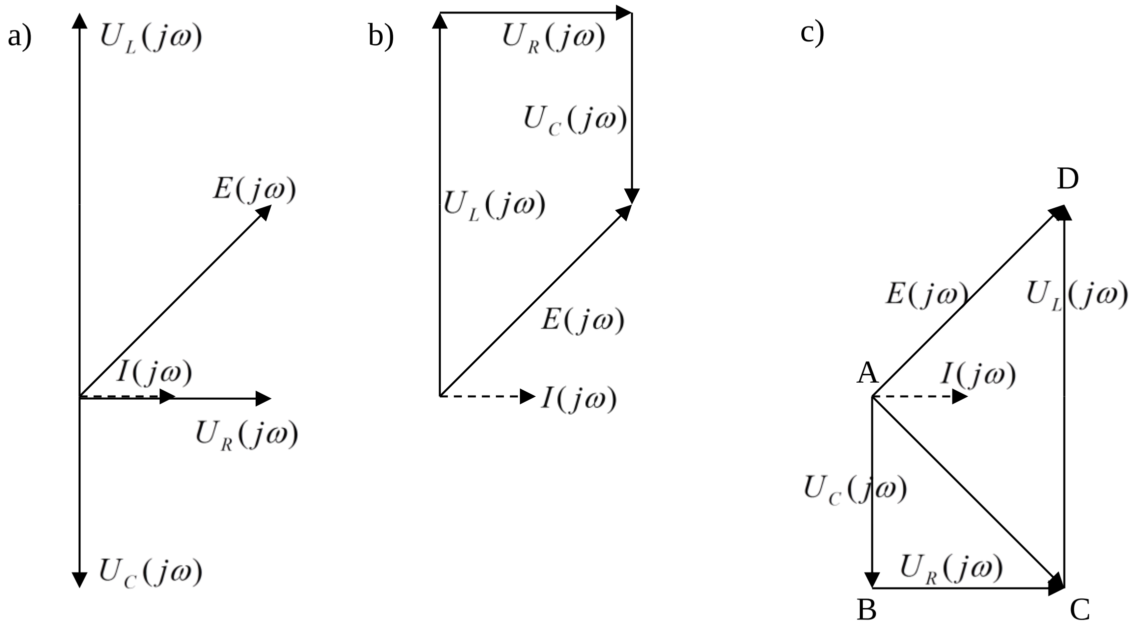

Fig. 4.2.14 Phasor diagrams for RLC series circuit (Example 4.2.2)

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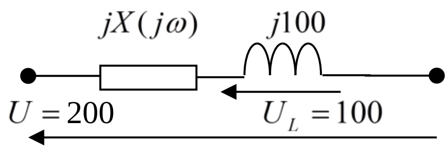
Drill problems 4.2

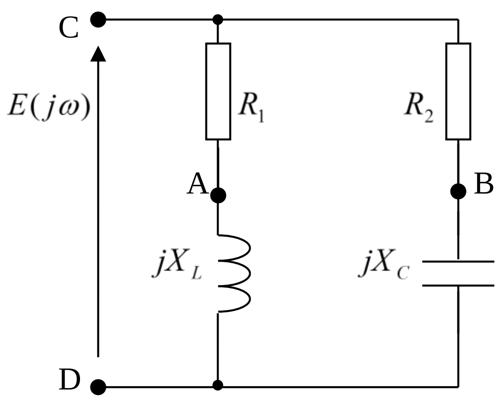
1. What reactance of: a) inductive character, b) capacitive character, should be connected in series with $j100\ \Omega$ coil such that at $U = 200\,\mathrm{V}$ supply, coil voltage drops by 50%, i.e. down to 100 V?

Fig. P.4.2.1



2. Sketch the phasor diagram and read the voltage $U_{BA}(j\omega)$: $R_1 = R_2 = X_L = |X_C| = 10\,\Omega$, $E(j\omega) = 10$ V. Repeat calculations (drawing) for: a) $R_1 = 0$, b) $R_2 = 0$.

Fig. P.4.2.2



3. Find the effective value of the mesh current and the coil voltage.

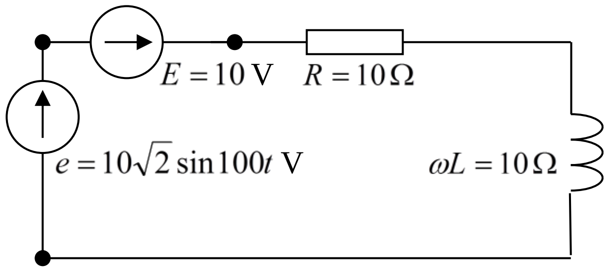


Fig. P.4.2.3

4. For the parallel *RL* circuit, find the total rms current if rms currents of elements are: $I_R = 4\,\mathrm{A}$, $I_L = 3\,\mathrm{A}$.
5. For the series *RL* circuit, find the total rms voltage if rms voltages of elements are: $U_R = 4\,\mathrm{V}$, $U_L = 3\,\mathrm{V}$.
6. Repeat Problems 4.2.4 and 4.2.5 with *L* replaced by *C*.
7. Find the effective value of the mesh current and the capacitor voltage.

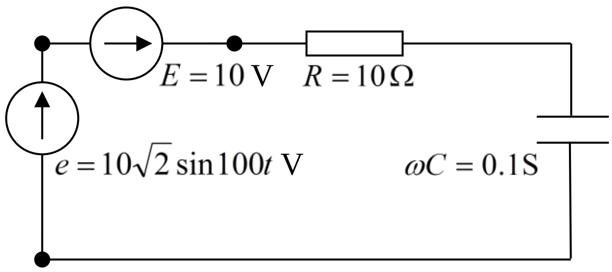


Fig. P.4.2.7

8. The rms voltages of the *RLC* series circuit are: $U_R = 10\,\mathrm{V}$, $U_C = 4\,\mathrm{V}$, $U_L = 3\,\mathrm{V}$. Find the supply rms voltage.
9. A two-terminal circuit is a series connection of resistor and energy storage element. Identify character of this element and find both constants *R* and *C* or *L* if the circuit voltage and current are: $u = 12\sqrt{2}\sin 100t\ \mathrm{V}$, $i = 3\sin(100t + 45^{\mathrm{o}})\ \mathrm{A}$.
10. Sketch the phasor diagram mapping the topology. Assume: $E(j\omega) = E$, $X_L = R$, $|X_C| =$ a) X_L, b) $2X_L$, c) $0.5X_L$.

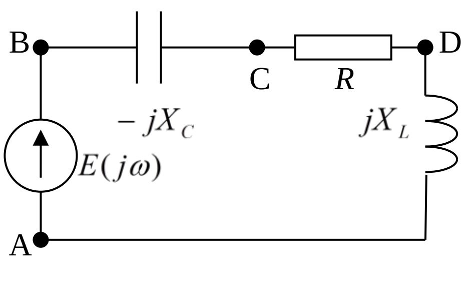
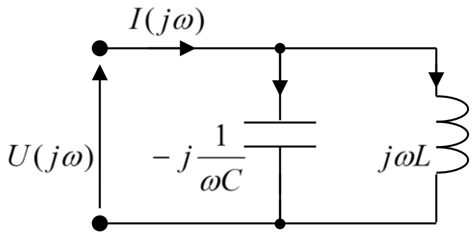


Fig. P.4.2.10

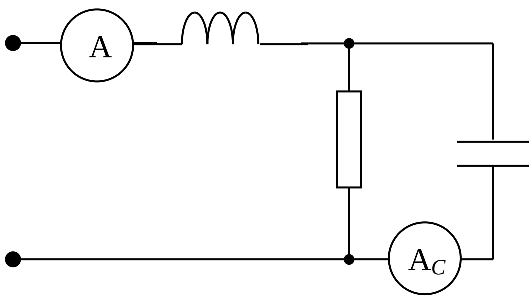
11. At $\omega_1 = 100$ rad/s the rms currents are: $I_L(\omega_1) = 16\,\text{A}$, $I_C(\omega_1) = 4\,\text{A}$. Find the rms currents $I(jk\omega_1), I_L(jk\omega_1), I_C(jk\omega_1)$ for: a) $k = 2$, b) $k = 1/2$ and $U(\omega) = const$.

Fig. P.4.2.11



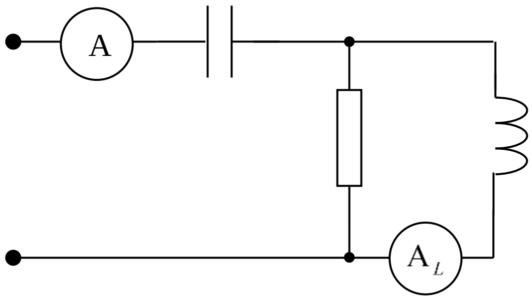
12. Find the coil ammeter indication, if the capacitor ammeter indication is 2 A and $\omega L = R = 1/\omega C = 10\,\Omega$.

Fig. P.4.2.12



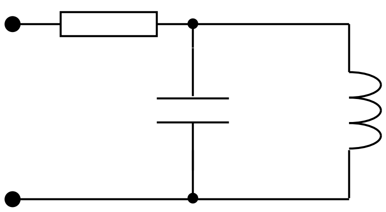
13. Find the capacitor ammeter indication, if the coil ammeter indication is 2 A and $\omega L = R = 1/\omega C = 10\,\Omega$.

Fig. P.4.2.13



14. Find resistance and reactance of the series equivalent at $\omega = 5000$ rad/s, for $R = 10\,\Omega$, $L = 1\text{mH}$, $C = 20\,\mu\text{F}$.

Fig. P.4.2.14

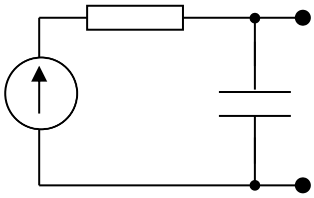


15. The rms current taken from a voltage source $e = 20\sqrt{2}\sin 10t$ V by a series combination of $R = 3\,\Omega$ and $L = ?$ is 4 A. Find the inductance.

16. Find the rms current taken from a voltage source $e = 20\sqrt{2}\sin 10^3 t$ V by a capacitor of 100 nF in series with a resistance of 10 kΩ.

17. Find the Thevenin equivalent, $e = 10\sqrt{2}\sin 100t$ V, $R = 1\text{k}\Omega$, $C = 0.1\mu\text{F}$.

Fig. P.4.2.17

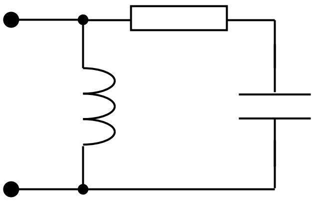


18. If the current that flows through the RLC branch is $i = 10\sqrt{2}\sin 10t$ V and $R = 2\Omega$, $L = 0.6$ H, $C = 0.05$ F, find the branch voltage.

19. In the circuit of Problem 4.2.12 parameters are not known but it is known that the resistor rms current is 6 A and the capacitor rms current is 8 A. Find the coil rms current.

20. It is known that the resistor rms voltage is 8 V and the capacitor rms voltage is 4 V. Find the coil rms voltage.

Fig. P.4.2.20



21. A series combination of a resistance and a capacitance produces a 2 A rms current that leads the applied voltage by 45° . If the amplitude of this voltage is 200 V (50 Hz), what are the resistance and capacitance ?

4.3 AC STEADY-STATE POWER

Few different measures of ac steady-state power are used, and they all are presented. The special attention is laid on the average power. Methods of its calculation and maximum power transfer condition are discussed.

MEASURES OF POWER

Instantaneous power

In a linear circuit with periodic excitation, the steady-state currents and voltages are also periodic, each having identical period. Then, instantaneous power absorbed by two-terminal element (subcircuit) is also periodic. For the considered sinusoidal excitation, instantaneous power is also a sinusoid.

$$p = ui = U\sqrt{2}\sin(\omega t + \alpha_u) I\sqrt{2}\sin(\omega t + \alpha_i) = UI\cos(\alpha_u - \alpha_i) - UI\cos(2\omega t + \alpha_u + \alpha_i) \tag{4.3.1}$$

The first term in this equation is independent of time, the second term varies periodically over time at twice the angular frequency. Proper operation of electrical devices limits the maximum instantaneous power. This power, the so called **peak power**, is a commonly used specification for characterizing elements or devices.

$$P_{\text{peak}} = UI\cos(\alpha_u - \alpha_i) + UI \tag{4.3.1a}$$

Average or real power

Mathematically, the first term of instantaneous power is its average value. This term is called the average power or real power, and it represents the power delivered by a source or absorbed by a two-terminal element or device (subcircuit).

$$P = \frac{1}{T}\int_{t_0}^{t_0+T} p\,dt = UI\cos(\alpha_u - \alpha_i) = UI\cos\varphi \tag{4.3.2}$$

The average power, in watts [W], is always nonnegative and satisfies the power balance principle. It is the product of the rms voltage, the rms current and the cosine of the angle between them. This cosine is called the **power factor**, *pf*.

$$pf = \cos(\alpha_u - \alpha_i) = \cos\varphi \tag{4.3.2a}$$

If a two-terminal element is a resistor *R*, then $\varphi = 0^{\circ}$, $pf = 1$, and the real power absorbed is

$$P_R = I^2 R = \frac{U^2}{R} \tag{4.3.2b}$$

For capacitor or coil $pf = 0$, because the angle between the voltage and the current is $\varphi = -90^{\circ}$ or $\varphi = 90^{\circ}$, respectively. Consequently, the capacitor or coil real power is equal to zero,

$$P_C = P_L = 0 \tag{4.3.2c}$$

In identifying a load character, the *pf* is characterized as leading or lagging by the phase of current with respect to that of the voltage. Then, a capacitive load has a leading *pf* and an inductive load has a lagging *pf*.

The average power designates the energy absorbed by two-terminal element or subcircuit. The energy absorbed in time interval from $t=0$ to $t=nT$, where n is a positive integer, is designated by the following equation:

$$W_{nT} = \int_0^{nT} p\,dt = \int_0^{nT} P\,dt - \int_0^{nT} UI\cos(2\omega t + \alpha_u + \alpha_i)\,dt = nTP \qquad (4.3.3)$$

Consider the circuit of Fig. 4.3.1, consisting of a practical source (sinusoidal generator), modeled by the Thevenin equivalent, connected to a load subcircuit.

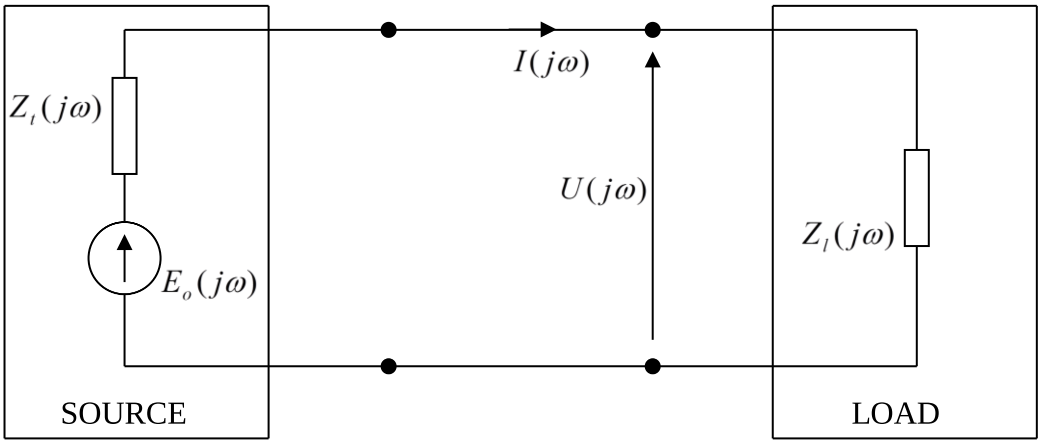


Fig. 4.3.1 Load impedance connected to a source

The real power transferred from the generator to the load can be designated in three different ways.

1. Phasors $U(j\omega)$ and $I(j\omega)$ are designated at first, then equation (4.3.2) is utilized to find the real power transferred.
2. Effective currents of load resistors are calculated at first. Then, the power balance principle (4.3.4) is utilized, where *N* is number of load resistors.

$$P = \sum_{i=1}^{N} P_{R_i} = \sum_{i=1}^{N} I_{R_i}^2 R_i \qquad (4.3.4)$$

3. The effective current *I* and the equivalent series resistance of the load R_l are calculated at first, then the real power transferred is designated from equation (4.3.2b).

<u>Example 4.3.1</u>

Find the real power transferred to the subcircuit presented in Fig. 4.2.11, if the voltage on its terminals is $U(j\omega) = 10\,\text{V}$.

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1. $Z_l(j\omega) = 10 + j10\ \Omega$ (4.3.5a)

$$I(j\omega)=\frac{10}{10+j10}=\frac{1}{1+j}=\frac{1}{\sqrt{2}}\exp(-j45^{\circ})\ \text{A} \qquad (4.3.5b)$$

$$P=10\frac{1}{\sqrt{2}}\cos 45^{\circ}=5\ \text{W} \qquad (4.3.5c)$$

2. $$I_{R_1}=I,\ P_{R_1}=\left(\frac{1}{\sqrt{2}}\right)^2 5=2.5\,\text{W} \qquad (4.3.6a)$$

$$I_{R_2}(j\omega)=I(j\omega)\frac{-jX_C}{R_2-jX_C}=\frac{1}{1+j}\frac{-j10}{10-j10}=-j0.5\,\text{A},\ P_{R_2}=(0.5)^2 10=2.5\ \text{W} \qquad (4.3.6b)$$

$$P=P_{R_1}+P_{R_2}=5\,\text{W} \qquad (4.3.6c)$$

3. $$P=I^2R_l=\left(\frac{1}{\sqrt{2}}\right)^2 10=5\ \text{W} \qquad (4.3.7)$$

♦

Apparent power

For the given line voltage $U(j\omega)$, the real power consumed by the load is strongly related to its power factor. It may vary from 0 to the maximum of

$$S=P\big|_{\varphi=0}=UI \qquad (4.3.8)$$

This product is called the apparent power. To avoid confusing with the unit of average power, the watt, the apparent power unit is **volt-ampere** [VA] and obviously, the apparent power does not satisfy the power balance principle. It simply defines the maximum capacity of a source (power plant).

The power factor of a load has a very important practical meaning. Power company is very interested in having customer keep this factor as close to unity as possible, to minimize the power line losses

$$P_{\text{line}}=I^2R_{\text{line}} \qquad (4.3.9)$$

Example 4.3.2

Suppose, that a mill consumes $P=1\,\text{kW}$ from a $U=200$ V line at a lagging power factor of $pf=\cos 60^{\circ}=1/2$. Then, the required current is

$$I=\frac{P}{U\cos\varphi}=\frac{1000}{200\cos 60^{\circ}}=10\,\text{A} \qquad (4.3.10)$$

For $pf=1$, the required current would be only 5 A. As can be seen, the power plant must generate a larger current in the case of lower *pf* and it causes larger line losses. For instance, if the transmission line resistance is $R_{\text{line}}=5\Omega$, then the line losses increase from 125 to 500 W!

♦

Most industrial and many residential loads are inductive (lagging power factor). Although it is not possible to change the inductive nature of a load itself, it is possible to connect a capacitive load in parallel with this load, and correct the power factor to unity that way. The circuit for power factor correction is presented in Fig. 4.3.2, together with the phasor diagram.

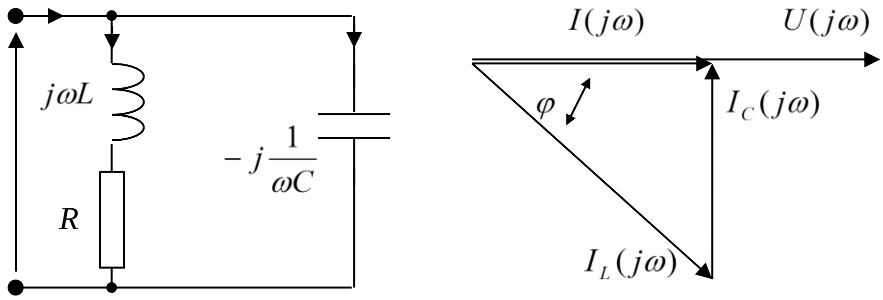


Fig. 4.3.2 Circuit for power factor correction and corresponding phasor diagram

<u>Example 4.3.3</u>

A load operating at a lagging power factor of $pf = \cos 45^{\circ} = 0.7$ dissipates 2 kW when connected to a 220 V, 50 Hz power line. What value of capacitance is needed to correct the power factor to unity?

÷

The inductive load rms current is:

$$I_L = \frac{P}{U\cos\varphi} \tag{4.3.11a}$$

From the current triangle (Fig. 4.3.2), the capacitor rms current is

$$I_C = I_L \sin\varphi = \frac{P}{U}\tan\varphi \tag{4.3.11b}$$

Then, the capacitance is:

$$C = \frac{I_C}{\omega U} = \frac{P\tan\varphi}{U^2\omega} = \frac{2000}{220^2 \cdot 314} = 131.6\,\mu\text{F} \tag{4.3.11c}$$

♦

<u>Reactive power</u>

Energy storage elements, capacitors and coils, neither supply nor dissipate power on average, but rather exchange it back and forth with the rest of the circuit. To measure the amount of periodic energy exchange taking place between a given subcircuit and the rest of the circuit, reactive power Q is introduced.

$$Q = UI\sin\varphi$$

The sign of Q is positive for inductive loads and negative for capacitive ones. For the pure capacitive or pure inductive load, $Q = -UI$ or $Q = UI$, respectively. For pure resistive load, $Q = 0$.

The unit of reactive power is **var** or **volt-ampere reactive**, [VAr]. The reactive power satisfies the power balance principle.

Example 4.3.1 – cont.

Find the reactive power transferred to the subcircuit.

÷

From (4.3.5) repeated

$$Z_l(j\omega) = 10 + j10\ \Omega,\ \ I(j\omega) = 1/\sqrt{2}\exp(-j45^{\circ})\ \text{A} \tag{4.3.12}$$

Then, the reactive power is

$$Q = I^2 X_l = 5\ \text{VAr} \tag{4.3.13}$$

♦

Complex power

To extend phasor analysis to the study of power in ac steady-state circuits, a new complex quantity, the complex power $S(j\omega)$ is defined (4.3.14). For simplicity of description argument ω is omitted at the right side of the equation.

$$S(j\omega) = P + jQ = S\exp(j\varphi) \tag{4.3.14}$$

The real part of the complex power is the average power, the imaginary part is the reactive power, its modulus is the apparent power. Graphical interpretation in the complex plane is presented in Fig. 4.3.3, an inductive load is assumed. The complex power satisfies the power balance principle, as its both terms satisfy this principle.

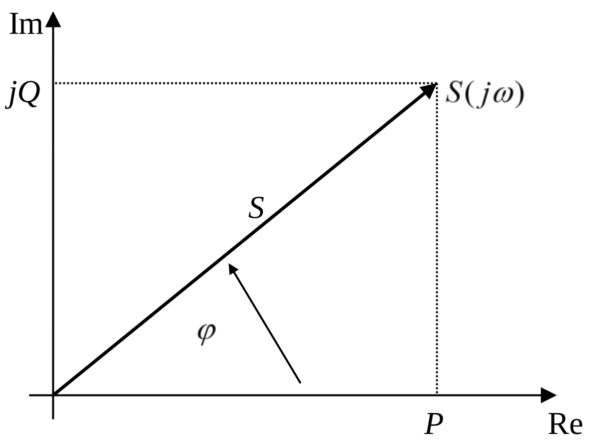


Fig. 4.3.3 Graphical interpretation of complex power and its components

MAXIMUM POWER TRANSFER

Same as in dc circuits, when designing ac circuit, it is frequently desirable to arrange for the maximum real power transfer to the load from the rest of the circuit. The whole circuit is divided into two parts (Fig. 4.3.1):

1. source, active subcircuit modeled by its Thevenin equivalent:

$$E_o(j\omega),\ Z_t(j\omega) = R_t(\omega) + jX_t(\omega) \tag{4.3.15a}$$

2. load, passive subcircuit modeled by its impedance

$$Z_l(j\omega) = R_l(\omega) + jX_l(\omega) \tag{4.3.15b}$$

Specifying $Z_l(j\omega)$, so that the average power absorbed by this impedance from the given active subcircuit is a maximum, is the task. Power absorbed by the load is a function of two arguments:

$$P = I^2 R_l = \left(\frac{E_o}{\sqrt{(R_t + R_l)^2 + (X_t + X_l)^2}} \right)^2 R_l = P(R_l, X_l) \qquad (4.3.16)$$

The values of R_l, X_l that maximize *P* are calculated from the following equations:

$$\frac{\partial P}{\partial R_l} = 0, \ \frac{\partial P}{\partial X_l} = 0 \qquad (4.3.17)$$

These values are:

$$R_l = R_t, \ X_l = -X_t \qquad (4.3.18)$$

and the maximum power transfer condition can be formulated.

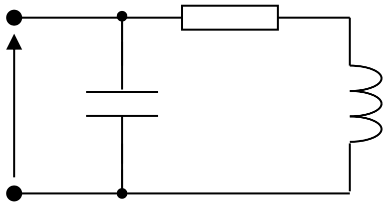
The maximum power is transferred to the load of $Z_l(j\omega)$ from the source with Thevenin equivalent impedance of $Z_t(j\omega)$, if these impedances are complex conjugates:

$$Z_l(j\omega) = Z_t^*(j\omega) \ \equiv \ Z_l(\omega) = Z_t(\omega), \ \varphi_l(\omega) = -\varphi_t(\omega) \qquad (4.3.18a)$$

Drill problems 4.3

1. A load has the inductive impedance $Z(j\omega) = 100 + j100\ \Omega$. Find the parallel impedance required to correct the power factor to 1.0 for $\omega = 500$ rad/s.
2. Find the real power absorbed by the subcircuit:
   $U(j\omega) = 20e^{j45^0}$ V, $|X_C| = X_L = R = 10\Omega$.

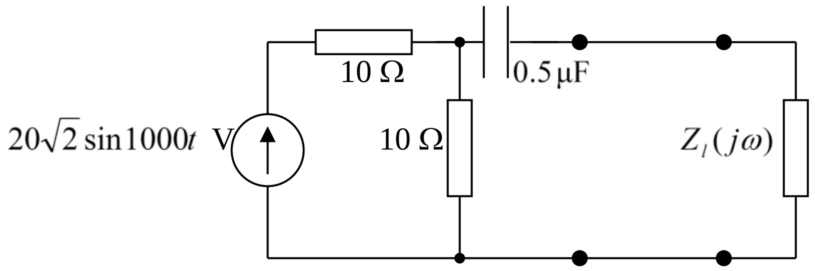
   Fig. P.4.3.2

3. The parallel *LC* circuit voltage is $u = U_m \sin\omega t$. Sketch (on one drawing) the instantaneous power of *L* and *C*, for $X_L = |X_C|$.
4. The series *LC* circuit current is $i = I_m \sin\omega t$. Sketch (on one drawing) the instantaneous power of *L* and *C*, for $X_L = |X_C|$.
5. The current $i = 10\sqrt{2}\sin(314t + 45^\circ)$ mA flows through $Z(j\omega) = 3 + j4$ kΩ load. What real, apparent and reactive power absorbs the load? What energy is dissipated in one period?

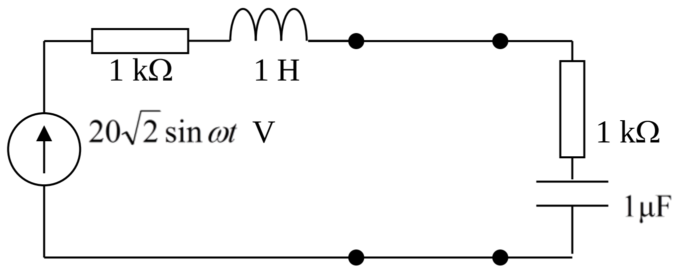
6. The voltage across $Z(j\omega)=3+j4\ \text{k}\Omega$ load is $u=10\sqrt{2}\sin(314t+45^{\circ})$ V. What real, apparent and reactive power absorbs the load? What energy is dissipated in one period?
7. The inductive impedance $Z(j\omega)=6+j8\Omega$ is connected to 220 V line. Find the energy dissipated in 1 hour.
8. Find the load impedance $Z_l(j\omega)=R_l+jX_l$ that will absorb the maximum power. What would be the value of that maximum ?

Fig. P.4.3.8



9. The inductive load absorbs 270 W of real power at a *pf* of 0.75 lagging and its voltage is 120 V rms. Find the real power absorbed by the transmission line resistance of $R_{\text{line}}=10\Omega$.
10. At what frequency would the capacitive load receive maximum power and what would be the value of that maximum.

Fig. P.4.3.10



11. A practical coil has inductance 4 mH and resistance 8 Ω. Find the power dissipated in the coil when it is connected to 20 V source at: a) dc, b) $\omega=1000$ rad/s, c) $\omega=1500$ rad/s.
12. The power delivered to a capacitor is $p=10\sin 2t$ mW, its voltage is $u=5\sqrt{2}\sin t$ V. Find the current entering the positive terminal, the charge between times 0 and a) $\pi/4$ s, b) π s, and energy delivered within this time.
13. A resistive load consumes 400 W at 100 V, 60 Hz supply. Find a capacitor that should be connected in series, if the combination is supplied from 200 V, 50 Hz line and load consumes the same energy.
14. The series *RLC* circuit is connected to $e=20\sin 10^3 t$ V source. Find the instantaneous energy stored W_L and W_C at the moment when the source voltage is zero. Find the instantaneous energy stored W_L at the moment when W_C is zero. Find the instantaneous energy stored W_C at the moment when W_L is zero.
15. A single-phase motor is supplied with 1200 W from 240 V, 50 Hz line. The motor operates at a *pf* of 0.8. What current flows to the motor ? What is the apparent power ?

4.4 FREQUENCY CHARACTERISTICS OF TWO-TERMINAL SUBCIRCUIT

In ac steady-state circuit, all circuit responses are functions of the generator frequency ω. Response variations with this frequency form the frequency response of the circuit.

Summary of ideal elements, resistor, capacitor and coil, is presented at first. Then, use of these elements to model a practical capacitor and a practical coil is discussed.

Next, frequency response of resonant circuits, circuits that contain both capacitors and coils is investigated. Simple circuits, series *RLC* circuit and parallel *GLC* circuit are considered at first. Then, complex two-terminal circuits are discussed.

IDEAL ELEMENTS – SUMMARY

Resistor

1. Time-domain description

$$u = Ri \Rightarrow U\sqrt{2}\sin(\omega t + \alpha_u) = RI\sqrt{2}\sin(\omega t + \alpha_i) \Rightarrow U = RI,\ \alpha_u = \alpha_i \tag{4.4.1}$$

There is no phase shift between the voltage and the current. Fig. 4.4.1 presents a single period of resistor waveforms - current is denoted by the bold line, $\alpha_i = 0^\circ$ is assumed.

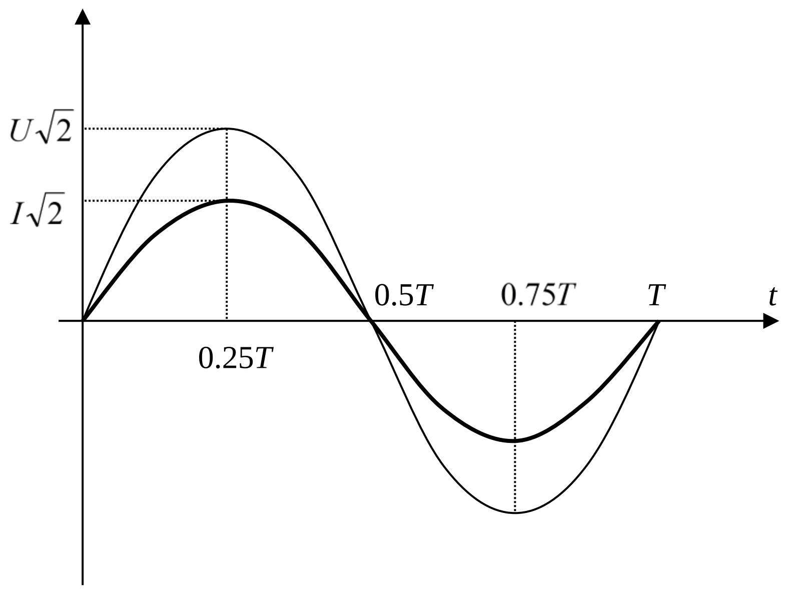


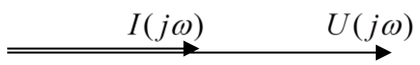
Fig. 4.4.1 Voltage and current waveforms for a resistor

2. Phasor-domain description

$$U(j\omega) = RI(j\omega) \tag{4.4.2}$$

Phasor diagram is presented in Fig. 4.4.2

Fig. 4.4.2 Phasor diagram for a resistor



3. Impedance

$$Z(j\omega) = R \Rightarrow Z(\omega) = R,\ \varphi = 0^{\circ} \tag{4.4.3}$$

Fig. 4.4.3 presents modulus (magnitude) and phase frequency characteristics.

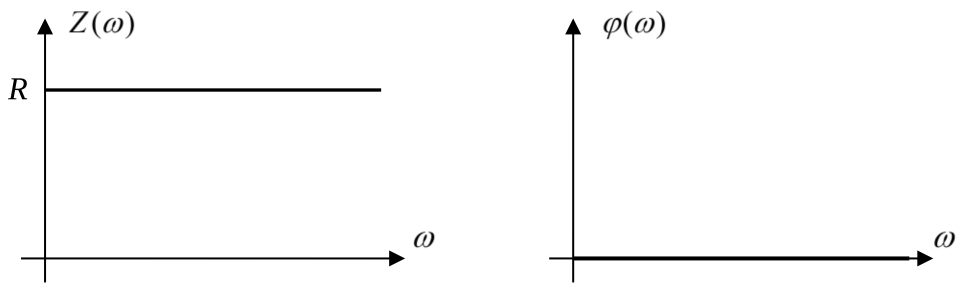


Fig. 4.4.3 Magnitude and phase frequency characteristics for a resistor

4. Power

Instantaneous power: $p = ui = 2UI\sin^2 \omega t = UI - UI\cos 2\omega t$ (4.4.4)

as presented in Fig. 4.4.4 - the dissipated energy is denoted by the shaded area.

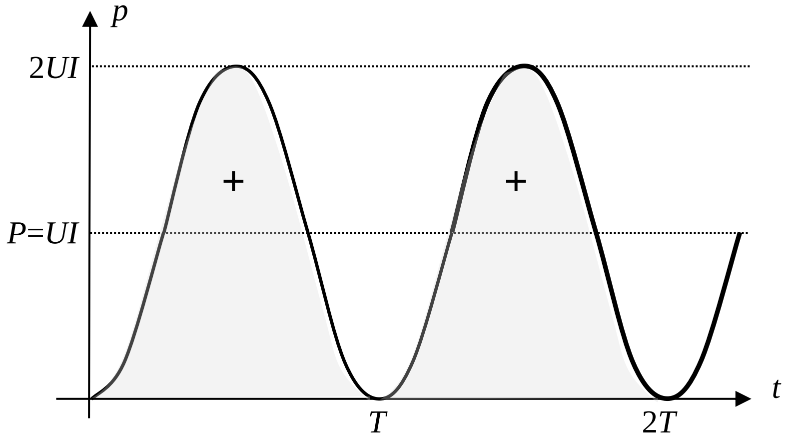


Fig. 4.4.4 Resistor instantaneous power

Average power, power factor: $P = I^2 R,\quad pf = 1$ (4.4.5)

Reactive power: $Q = 0$ (4.4.6)

Energy dissipated in one period: $W_T = I^2 RT$ (4.4.7)

Inductor

1. Time-domain description

$$u = L\frac{di}{dt} \Rightarrow U\sqrt{2}\sin(\omega t + \alpha_u) = \omega LI\sqrt{2}\sin(\omega t + \alpha_i + 90^{\circ}) \Rightarrow U = \omega LI,\ \alpha_u = \alpha_i + 90^{\circ}$$

(4.4.8)

The voltage leads the current by 90°. Fig. 4.4.5 presents a single period of inductor waveforms - current is denoted by the bold line, $\alpha_i = 0^{\circ}$ is assumed.

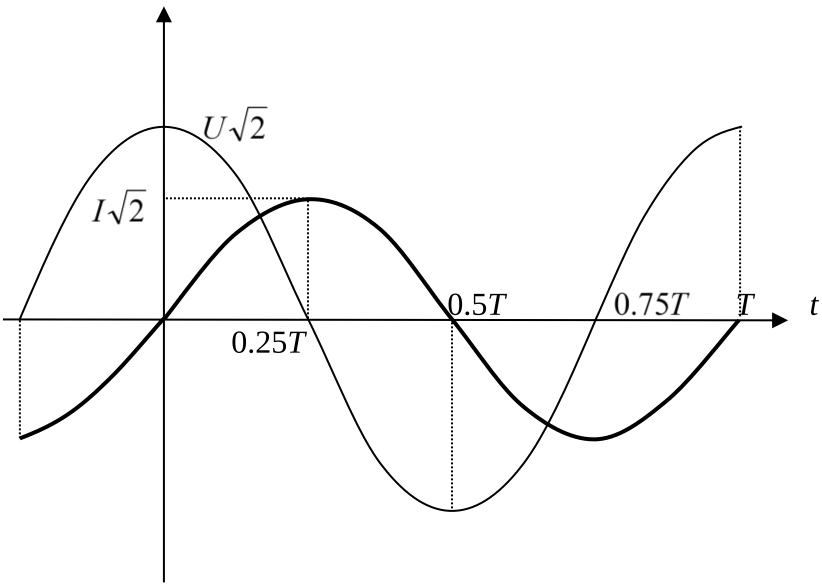


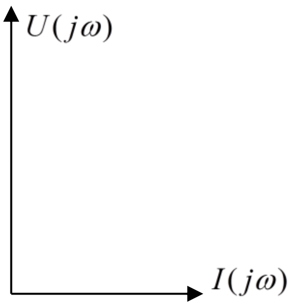
Fig. 4.4.5 Voltage and current waveforms for a coil

2. <u>Phasor-domain description</u>

$$U(j\omega) = j\omega L I(j\omega) \tag{4.4.9}$$

Phasor diagram is presented in Fig. 4.4.6

Fig. 4.4.6 Phasor diagram for a coil



3. <u>Impedance</u>

$$Z(j\omega) = j\omega L \Rightarrow Z(\omega) = \omega L,\ \varphi = 90^{\circ} \tag{4.4.10}$$

Fig. 4.4.7 presents modulus (magnitude) and phase frequency characteristics.

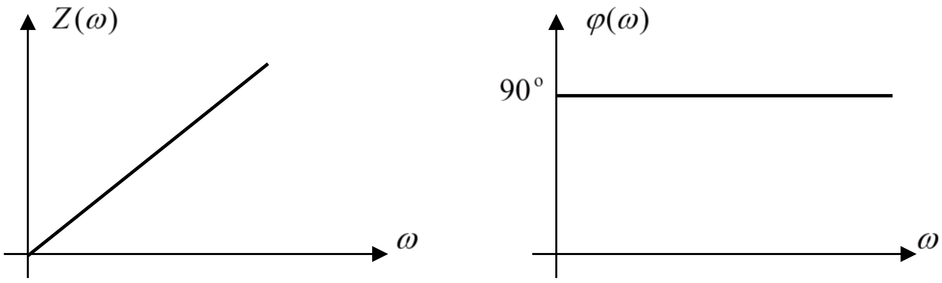


Fig. 4.4.7 Magnitude and phase frequency characteristics for a coil

In dc steady-state ($\omega = 0$), coil is a short-circuit. For $\omega \to \infty$, coil is an open-circuit.

4. Power

Instantaneous power: $p == UI\cos(2\omega t + 90^{o}) = UI\sin 2\omega t$ (4.4.11)

as presented in Fig. 4.4.8.

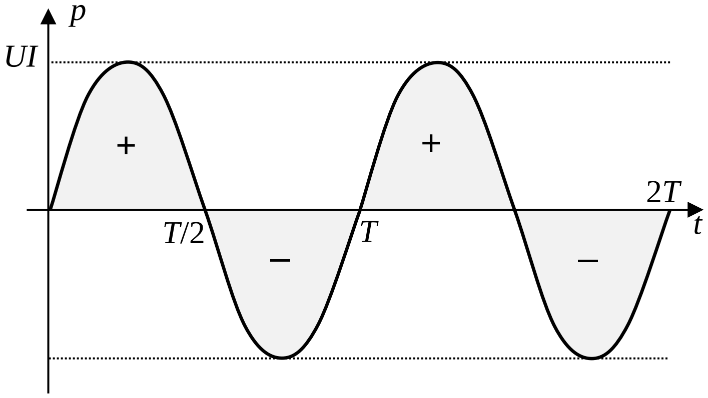


Fig. 4.4.8 Coil or capacitor instantaneous power

Average power, power factor: $P = 0, \quad pf = 0$ (4.4.12)

Reactive power: $Q = UI$ (4.4.13)

Energy absorbed in one period: $W_T = 0$ (4.4.14)

Inductor exchanges energy back and forth with the rest of the circuit - in Fig. 4.4.8 the exchanged energy is denoted by the shaded area.

$t = 0$ $\quad W_0 = 0$,

$t \in (0, T/4)$ energy is stored,

$t = T/4$ maximum energy stored: $W_{T/4} = \dfrac{L(i(T/4))^2}{2} = LI^2$, (4.4.14a)

$t \in (T/4, T/2)$ energy is given back to the rest of the circuit,

$t = T/2$ $\quad W_{T/2} = 0$ and the process of energy exchange repeats.

Capacitor

1. Time-domain description

$$i = C\frac{du}{dt} \Rightarrow I\sqrt{2}\sin(\omega t + \alpha_i) = \omega CU\sqrt{2}\sin(\omega t + \alpha_u + 90^{o}) \Rightarrow I = \omega CU,\ \alpha_i = \alpha_u + 90^{o}$$

(4.4.15)

The current leads the voltage by 90^{o}. Fig. 4.4.9 presents a single period of inductor waveforms - current is denoted by the bold line, $\alpha_u = 0^{o}$ is assumed.

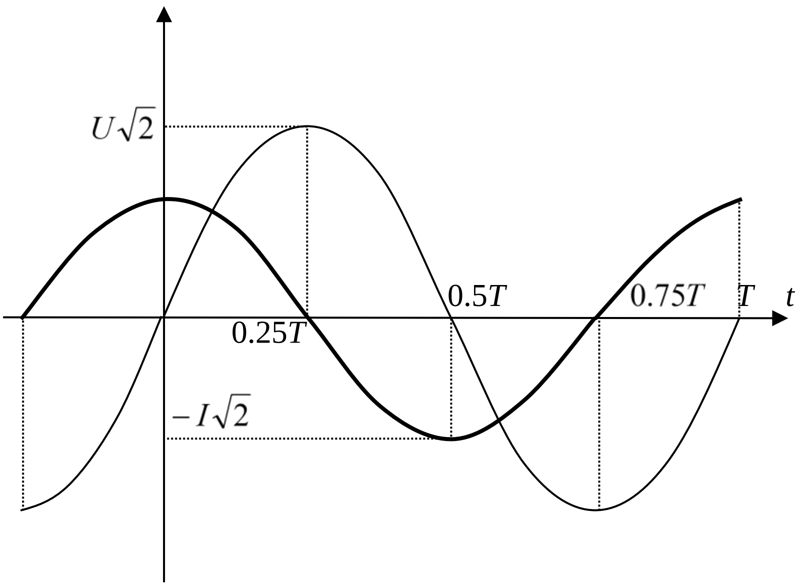


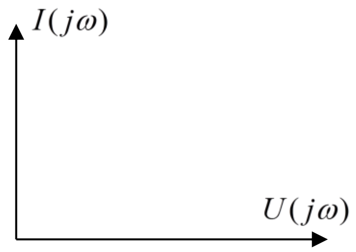
Fig. 4.4.9 Voltage and current waveforms for a capacitor

2. <u>Phasor-domain description</u>

$$I(j\omega) = j\omega C U(j\omega) \tag{4.4.16}$$

Phasor diagram is presented in Fig. 4.4.10

Fig. 4.4.10 Phasor diagram for a capacitor



3. <u>Impedance</u>

$$Z(j\omega) = \frac{1}{j\omega C} \Rightarrow Z(\omega) = \frac{1}{\omega C},\ \varphi = -90^{\circ} \tag{4.4.17}$$

Fig. 4.4.11 presents modulus (magnitude) and phase frequency characteristics.

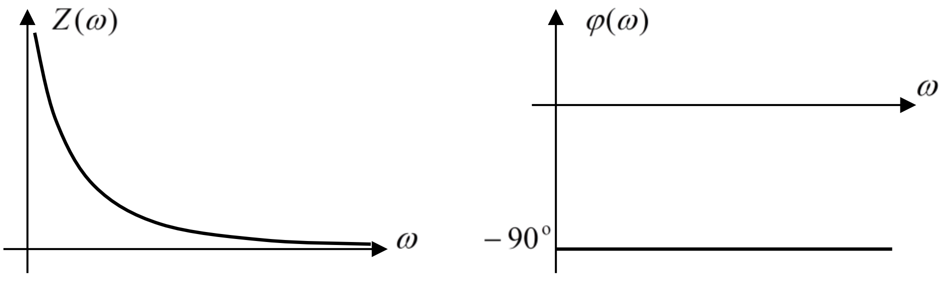


Fig. 4.4.11 Magnitude and phase frequency characteristics for a capacitor

In dc steady-state ($\omega = 0$), capacitor is an open-circuit. For $\omega \to \infty$, capacitor is a short-circuit.

4. Power

Instantaneous power: $p = UI\cos(2\omega t + 90^\circ) = UI\sin 2\omega t$ (4.4.18)

as presented in Fig. 4.4.8.

Average power, power factor: $P = 0,\quad pf = 0$ (4.4.19)

Reactive power: $Q = -UI$ (4.4.20)

Energy absorbed in one period: $W_T = 0$ (4.4.21)

Capacitor exchanges energy back and forth with the rest of the circuit - in Fig. 4.4.7 the exchanged energy is denoted by the hatched area.

$t = 0$ $W_0 = 0$,

$t \in (0, T/4)$ energy is stored (charging),

$t = T/4$ maximum energy stored: $W_{T/4} = \dfrac{C(u(T/4))^2}{2} = CU^2$, (4.4.21a)

$t \in (T/4, T/2)$ energy is given back to the rest of the circuit (discharging),

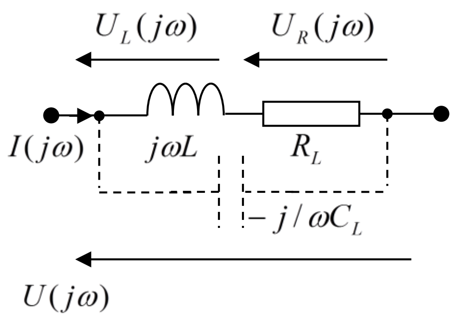
$t = T/2$ $W_{T/2} = 0$ and the process of energy exchange repeats.

PRACTICAL COIL and PRACTICAL CAPACITOR

A practical (nonideal) inductor or a practical capacitor is modeled by an ideal inductor or capacitor, together with some other parasitic elements to account for losses and coupling.

For a **practical inductor**, first of all, a winding resistance $R_L = R_{\mathrm{Cu}}$ (resistance in copper) has to be taken into account. This resistance may be modeled by inserting a series resistor into the circuit model for a practical inductor, as presented in Fig. 4.4.12.

Fig. 4.4.12 Circuit model for a practical coil



At first, model taking into account only this winding resistance is considered. From KVL:

$$U(j\omega) = U_L(j\omega) + U_R(j\omega) \tag{4.4.22}$$

The phasor diagram is presented in Fig. 4.4.13a ($\alpha_i = 0^\circ$ is assumed).
A practical coil equivalent impedance is

$$Z_L(j\omega) = R_L + j\omega L \Rightarrow Z_L(\omega) = \sqrt{R_L^2 + (\omega L)^2} \tag{4.4.23}$$

Frequency characteristic of the magnitude is presented in Fig. 4.4.13b (solid line).

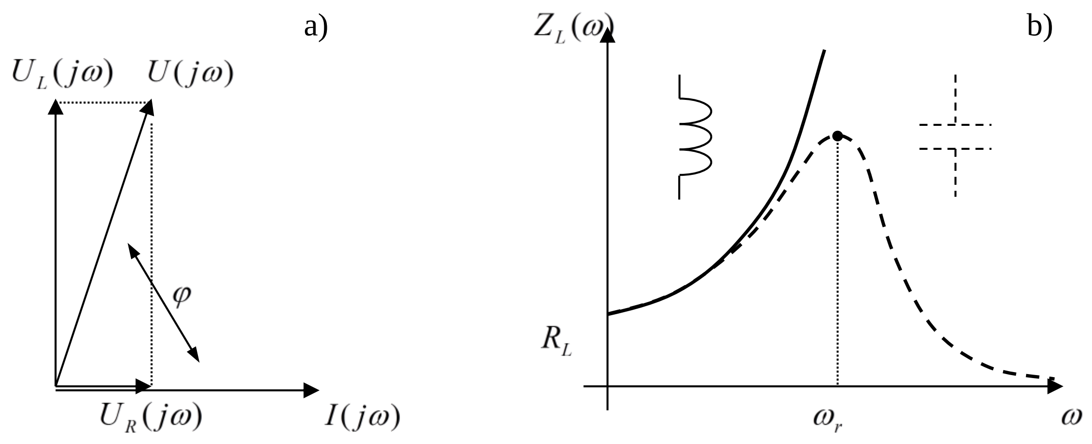


Fig. 4.4.13 Practical coil: a) phasor diagram, b) magnitude as a function of frequency

If a practical inductor or capacitor is described by the equivalent impedance, $Z(j\omega) = R(\omega) + jX(\omega)$, then the reactance $X(\omega)$ is the primary parameter of concern, and the resistance $R(\omega)$ represents the parasitic effect. The magnitude of $X(\omega)$ is usually much greater than the magnitude of $R(\omega)$. The ratio, called the **quality factor** of practical element

$$Q(\omega) = \frac{|X(\omega)|}{R(\omega)} \tag{4.4.24}$$

provides a measure of how close the practical element is to an ideal element. The inclusion of ω in equation (4.4.24) is to emphasize the fact that the quality factor depends on the frequency.

For an inductor, modeled as shown in Fig 4.4.12 – solid lines, the **quality factor** is designated by the following equation:

$$Q_L(\omega) = \frac{\omega L}{R_L} \tag{4.4.25}$$

For an exemplary coil characterized by the following parameters: $L = 0.1$ H, $R_L = 10\,\Omega$, at the audio frequency of $f = 1000$ Hz the quality factor is equal to 62.8.

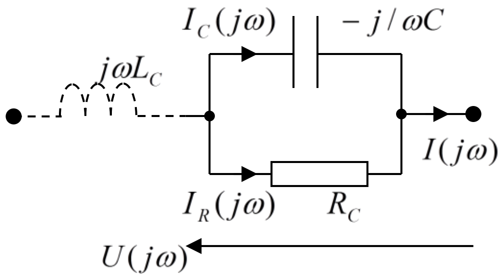
A coil model denoted in Fig. 4.4.12 by solid lines is valid for low and medium frequencies of up to few megahertz. For high frequencies the parasitic capacitance between a coil terminals C_L, the so called stray capacitance, has to be taken into account, as denoted in Fig. 4.4.12 by the dashed line. A practical coil equivalent circuit has inductive character up to some frequency, called the resonant frequency (meaning of this notion is explained in the next section of this Chapter):

$$\omega_r \cong \frac{1}{\sqrt{LC_L}} \tag{4.4.26}$$

Above the resonant frequency the equivalent circuit reveals the capacitive character! For the exemplary coil and stray capacitance of $C_L = 0.1$ pF, the resonant frequency is equal to $\omega_r = 10^7$ rad/s $\Rightarrow f_r = 1.6$ MHz.

For a **practical capacitor**, first of all, a leakage resistance $R_C = R_{\text{leak}}$ has to be taken into account. This resistance may be modeled by inserting a parallel resistor into the circuit model for a practical capacitor, as presented in Fig. 4.4.14.

Fig. 4.4.14 Circuit model for a practical capacitor



At first, model taking into account only this leakage resistance is considered. From KCL:

$$I(j\omega) = I_C(j\omega) + I_R(j\omega) \tag{4.4.27}$$

The phasor diagram is presented in Fig. 4.3.15a ($\alpha_u = 0^\circ$ is assumed).
A practical coil equivalent impedance is

$$Z_C(j\omega) = \frac{1}{Y_C(j\omega)} = \frac{1}{G_C + j\omega C} = \frac{G_C - j\omega C}{G_C^2 + (\omega C)^2} \Rightarrow Z_C(\omega) = \frac{1}{\sqrt{G_C^2 + (\omega C)^2}} \tag{4.4.28}$$

Frequency characteristic of the magnitude is presented in Fig. 4.3.15b (solid line).

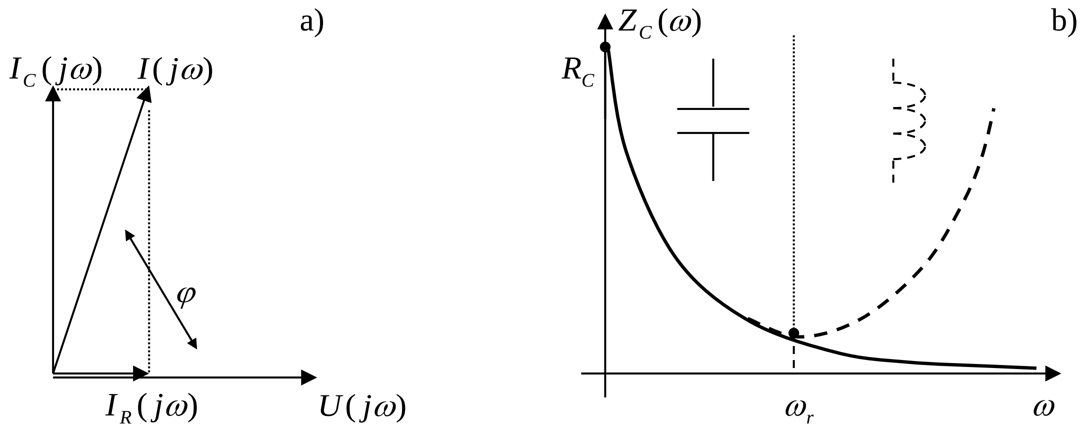


Fig. 4.4.15 Practical coil: a) phasor diagram, b) magnitude as a function of frequency

For a capacitor, modeled as shown in Fig 4.4.14 – solid lines, the **quality factor** is designated by the following equation:

$$Q_C(\omega) = \frac{|B(\omega)|}{G(\omega)} = \frac{\omega C}{G_C} \tag{4.4.29}$$

A capacitor quality factor is usually much greater than a coil quality factor, it is often assumed to be infinite. The reciprocal of $Q_C(\omega)$ is called the **dissipation factor** $d_C(\omega)$:

$$d_C(\omega) = 1/Q_C(\omega) \tag{4.3.30}$$

For an exemplary capacitor characterized by the following parameters: C=1μF, $R_C = 5\,\text{M}\Omega$, at the audio frequency of $f = 1000$ Hz, the quality factor is equal to 31400.

A capacitor model denoted in Fig. 4.4.14 by solid lines is valid for low and medium frequencies. For high frequencies the parasitic inductance of connecting wires L_C has to be taken into account, as denoted in Fig. 4.4.14 by dashed connections. A practical capacitor equivalent circuit has capacitive character up to some frequency, called the resonant frequency (meaning of this notion is explained in the following section):

$$\omega_r \cong \frac{1}{\sqrt{L_C C}} \tag{4.4.31}$$

Above the resonant frequency the equivalent circuit reveals the inductive character! For the exemplary capacitor and parasitic inductance of $L_C = 10$ nH, the resonant frequency is equal to $\omega_r = 10^7$ rad/s$\Rightarrow f_r = 1.6$ MHz.

A circuit design should take this phenomenon into account and resonant frequencies of all used reactive elements, capacitors and coils, should be greater than the maximum operating frequency.

RESONANT CIRCUITS

Consider the two-terminal passive circuit that contains at least one inductor and at least one capacitor, connected to a sinusoidal generator of variable frequency, as presented in Fig. 4.4.16.

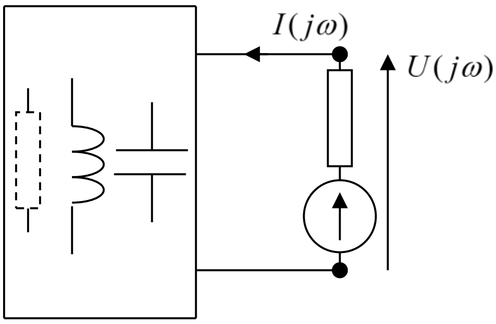


Fig. 4.4.16 Two-terminal RLC circuit

Frequency characteristics will be investigated, first for simple two-terminal *RLC* circuits built of three elements in series or parallel configuration, then for complex circuits. Phenomenon of resonance will be discussed.

The term **resonance** reflects the condition that the source voltage $U(j\omega)$ and current $I(j\omega)$ are in phase, $\varphi(\omega) = \alpha_u(\omega) - \alpha_i(\omega) = 0^{\circ}$. The frequency at which this phenomenon occurs is called the **resonant frequency** ω_r. If ω_r exists, then the circuit is a **resonant circuit**.

Series-resonant circuit *RLC*

The series *RLC* circuit is presented in Fig. 4.4.17. The circuit impedance

$$Z(j\omega) = R + j(\omega L - \frac{1}{\omega C}) = R + jX(\omega) \tag{4.4.32}$$

consists of the fixed real part (resistance) and imaginary part (reactance) that varies with frequency. The frequency at which reactance vanishes and the impedance is at a minimum magnitude, the resonant frequency, is

$$\omega_r = \frac{1}{\sqrt{LC}} \tag{4.4.33}$$

It is worth to observe, that this frequency coincides with the undamped resonant frequency (3.2.26c) of natural response. The reactance frequency characteristic is presented in Fig. 4.4.18.

Fig. 4.4.17 Series RLC circuit

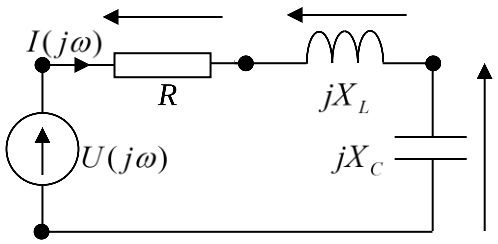
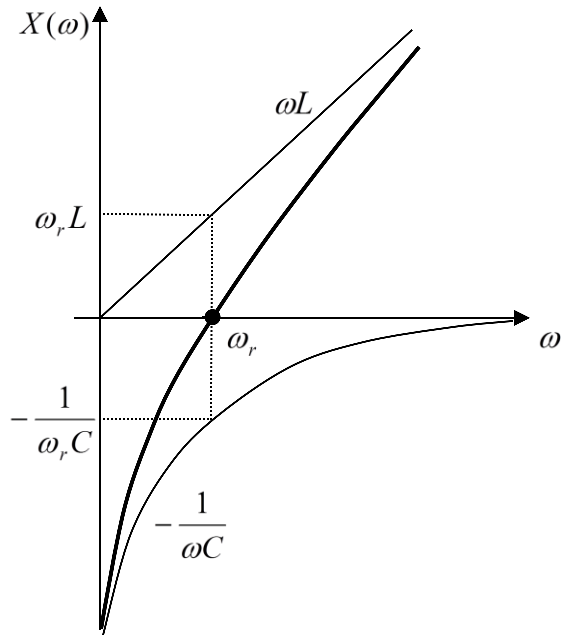


Fig. 4.4.18 Frequency characteristic of reactance in RLC series circuit



- for dc input voltage $\omega = 0$, reactance is an open-circuit,
- for low frequencies $\omega \in (0, \omega_r)$, reactance is negative (capacitive),

$$X(\omega) = -\frac{1}{\omega C_s(\omega)} = \omega L - \frac{1}{\omega C} \tag{4.4.34a}$$

where $C_s(\omega)$ is the equivalent series capacitance,

- for the resonant frequency ω_r,
  reactance is zero (short-circuit) – circuit has resistive character,

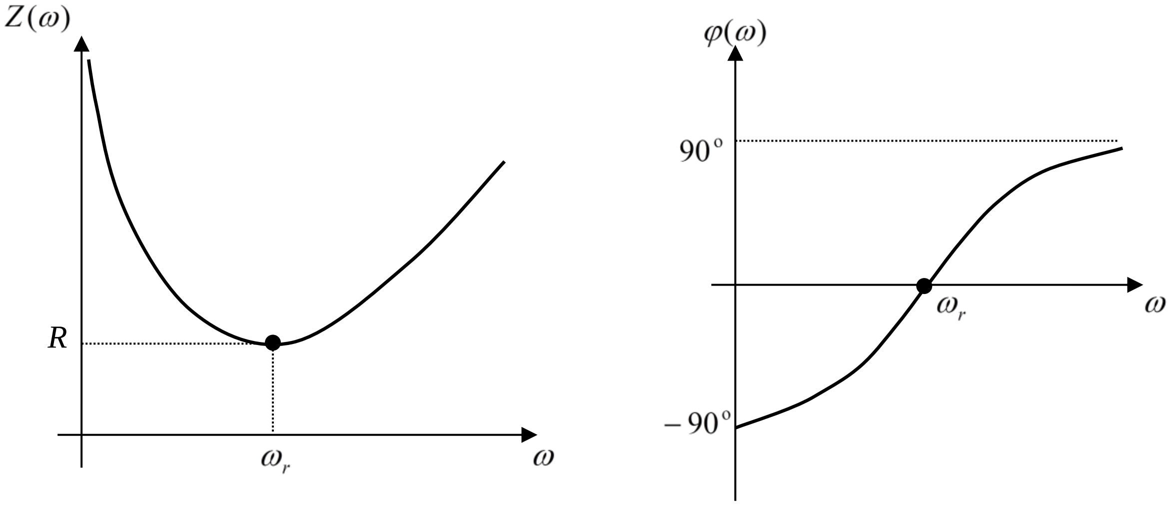
- for high frequencies, $\omega \in (\omega_r, \infty)$, reactance is positive (inductive),

$$X(\omega) = \omega L_s(\omega) = \omega L - \frac{1}{\omega C} \tag{4.4.34b}$$

where $L_s(\omega)$ is the equivalent series inductance,

- for $\omega = \infty$, reactance is an open-circuit.

The magnitude and phase characteristics of impedance are presented in Fig. 4.4.19.



Fig, 4.4.19 Magnitude and phase characteristics of impedance of the series RLC circuit

Consider the input to this circuit to be its voltage $U(j\omega) = U = const,\ \omega \in <0, \infty>$ and its output to be its current $I(j\omega)$. Magnitude of the current,

$$I(\omega) = \frac{U}{Z(\omega)} = \frac{U}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \tag{4.4.35}$$

is normally plotted in relation to the resonant current

$$I_r = I(\omega_r) = \frac{U}{R} \tag{4.4.36}$$

The obtained gain curve:

$$\frac{I(\omega)}{I_r} = \frac{1}{\sqrt{1 + Q^2(\eta - 1/\eta)^2}} \tag{4.4.37}$$

is called the **resonant curve** and it is depicted in Fig. 4.4.20, for two values of *Q*, where η is the **normalized frequency** and *Q* is the **quality factor** of the series *RLC* circuit:

$$\eta = \omega / \omega_r \tag{4.3.38}$$

$$Q = \frac{1}{R}\sqrt{\frac{L}{C}} \tag{4.4.39}$$

This curve can be also considered as the ratio of admittances.

$$\frac{Y(\omega)}{Y_r} = \frac{UI(\omega)}{UI_r} = \frac{1}{\sqrt{1+Q^2(\eta - 1/\eta)^2}} \tag{4.4.37a}$$

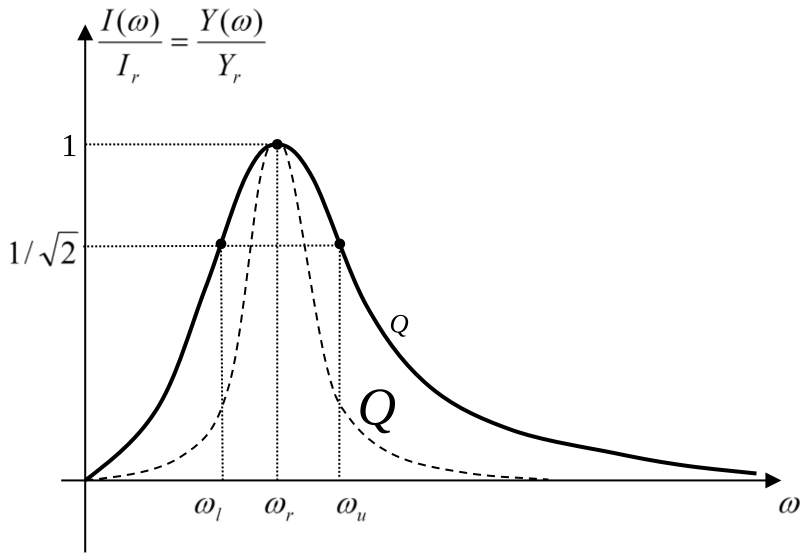


Fig. 4.4.20 Resonant curve of series RLC circuit for two values of Q

As can be observed, a phenomenon of resonance makes a circuit frequency selective. To have a good measure of this selectivity, to measure sharpness of the resonant curve, term of quality factor (4.4.39) is used. Circuits with high quality factors are very frequency selective, and this implies low values of resistance *R*. Since practical inductors include significant winding resistance R_{Cu}, it is difficult and expensive to design high-*Q* resonant circuits passively, that is, solely with *RLC* elements (active resonant circuits are not discussed).

In essence, *Q* is a measure of the energy storage property of a circuit in relation to its energy dissipation property. It can be easily proved that

$$Q = 2\pi \frac{\text{maximum energy stored}}{\text{total energy dissipated per cycle}} \tag{4.4.39a}$$

To measure the width of the frequency band within which the circuit is behaving in near-resonant fashion, the term of bandwidth $\Delta\omega$ is defined.

The circuit **bandwidth** is the range of frequencies that lie between two frequencies ω_l, ω_u where the magnitude of the gain is $1/\sqrt{2}$.

The gain of $1/\sqrt{2}$ corresponds to the power ratio of 1/2

$$\frac{P(\omega)}{P_r} = \frac{I(\omega)^2 R}{I_r R} = \frac{1}{2} \Rightarrow \frac{I(\omega)}{I_r} = \frac{1}{\sqrt{2}} \tag{4.4.40}$$

Boundary frequencies, the **upper half-power frequency** ω_u and the **lower half-power frequency** ω_l are calculated from the following equation:

$$\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{1+Q^2(\eta - 1/\eta)^2}} \tag{4.4.41}$$

Leaving the algebra, after subtraction of the two frequencies, the bandwidth for the series *RLC* is simply

$$\Delta\omega = \omega_u - \omega_l = \frac{R}{L} = \frac{\omega_r}{Q} \tag{4.4.42}$$

It is worth to emphasize, that at the resonant frequency magnitude of the coil and the capacitor voltages is the same

$$U_{Lr} = \omega_r L I_r = U_{Cr} = \frac{1}{\omega_r C} I_r \tag{4.4.43}$$

Then, another expression for the quality factor can be formulated

$$Q = \frac{U_{Lr}}{U} = \frac{U_{Cr}}{U} = \frac{\omega_r L}{R} = \frac{1}{\omega_r CR} \tag{4.3.44}$$

The frequency characteristics of magnitudes of coil and capacitor voltages are presented in Fig. 4.4.21.

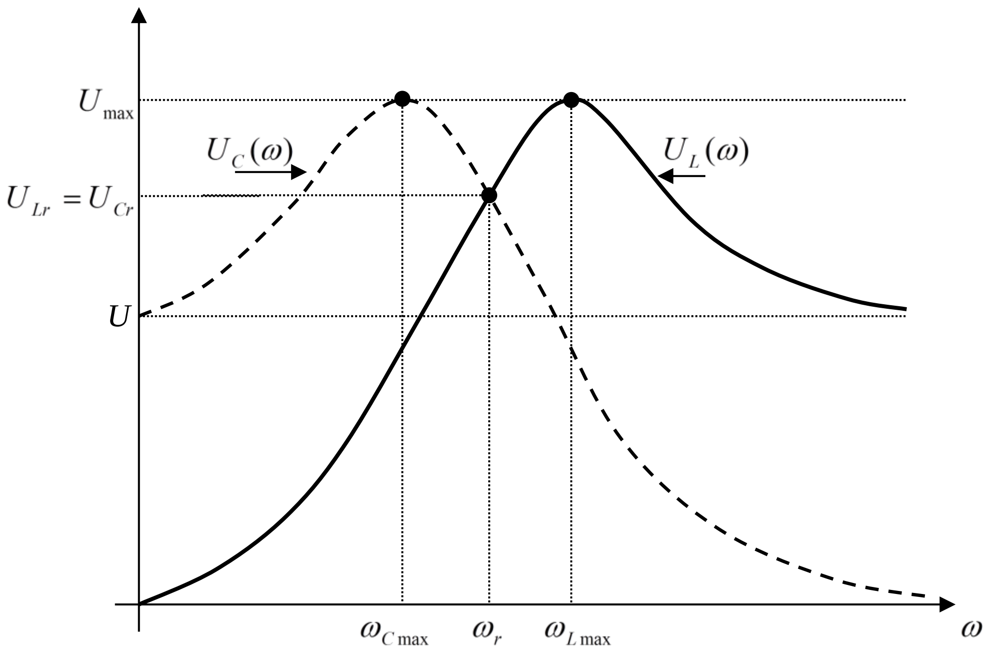


Fig. 4.4.21 Frequency characteristics of coil (bold) and capacitor voltage

For low values of Q, the voltages do not exceed the supply voltage. For larger values, the voltages exceed the supply voltage, as presented in Fig. 4.4.21. For even larger values of Q, practically for $Q > 10$, $\omega_{C\max} = \omega_{L\max} = \omega_r$ and $U_{\max} = U_{Lr} = U_{Cr} = UQ$! The coil or capacitor resonant voltage significantly exceeds the supply voltage and this effect is called the **resonance overvoltage**.

Phasor diagrams for three frequencies: $\omega_1 = 0.5\omega_r$, $\omega_2 = \omega_r$, $\omega_3 = 2\omega_r$ and $Q = 2/3$ are presented in Fig. 4.4.22.

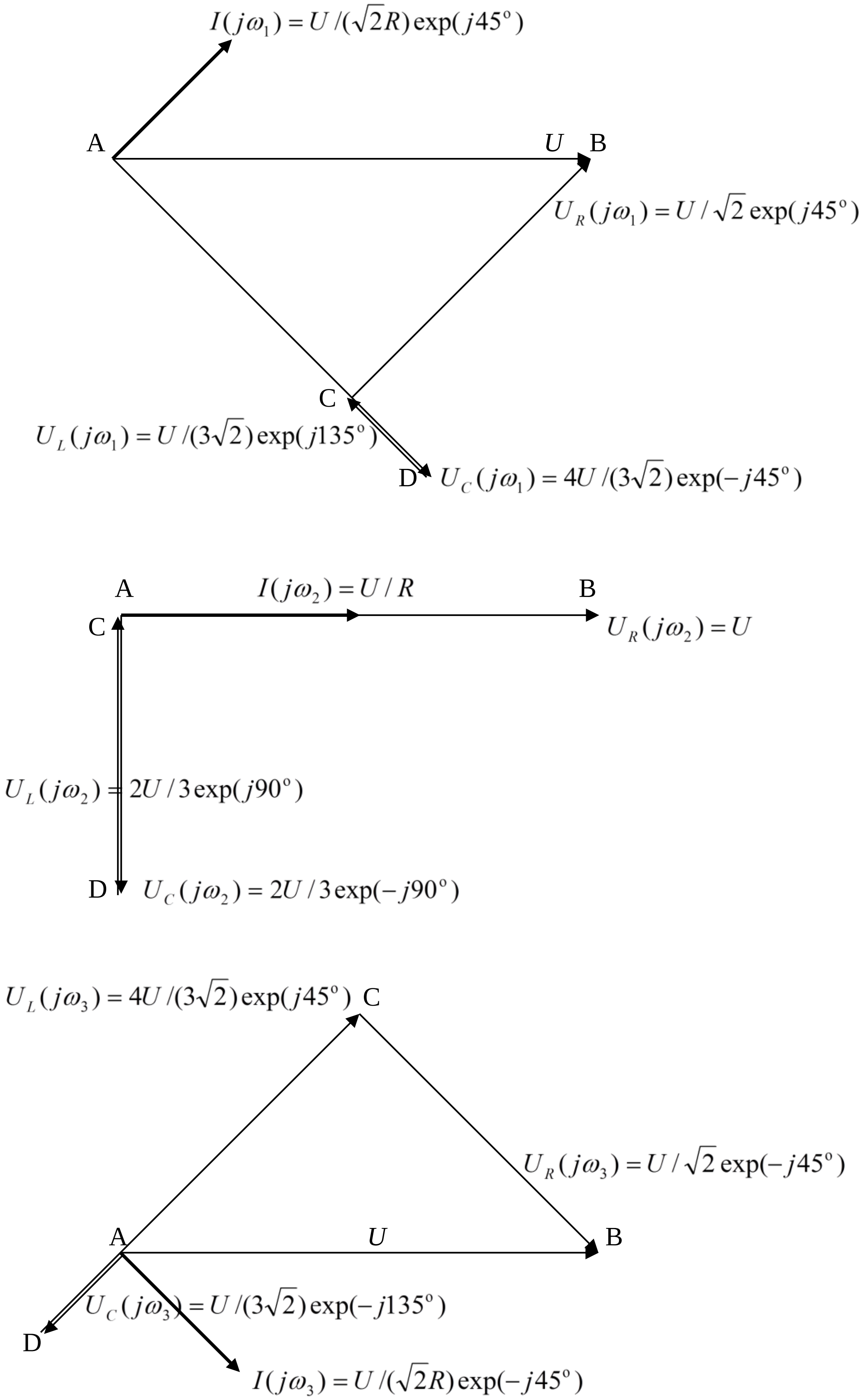


Fig. 4.4.22 Phasor diagrams for three frequencies: ω_1=0.5ω_r, ω_2=ω_r, ω_3=2ω_r, Q=2/3

Example 4.4.1

For the given parameters of *RLC* series circuit: $R = 10\Omega$, $L = 1\text{H}$, $C = 1\mu\text{F}$ and the input voltage of $U = 1$ V, find magnitude of the coil/capacitor voltage at the resonant frequency.

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The resonant frequency is $\omega_r = 1000$ rad/sec, the quality factor is $Q = 100$. For these values, the resonant current is $I_r = 0.1\text{A}$ and the maximum rms coil/capacitor voltage is $U_{\max} \cong 100\,\text{V}$!!!

♦

Parallel-resonant circuit *RLC*

The parallel *RLC* circuit is presented in Fig. 4.4.23. The circuit admittance

$$Y(j\omega) = G + j(\omega C - \frac{1}{\omega L}) = G + jB(\omega) \tag{4.4.45}$$

consists of the fixed real part (conductance) and imaginary part (susceptance) that varies with frequency. The frequency at which susceptance vanishes and the admittance is at a minimum magnitude, the resonant frequency is the same as in the series circuit (4.4.33). The susceptance frequency characteristic is presented in Fig. 4.4.24.

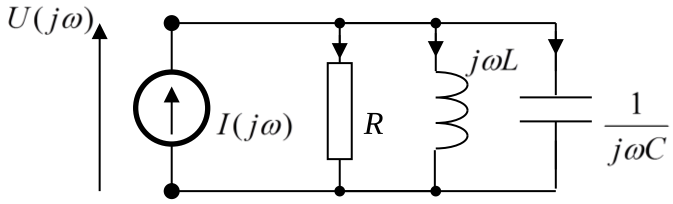


Fig. 4.4.23 Parallel RLC circuit

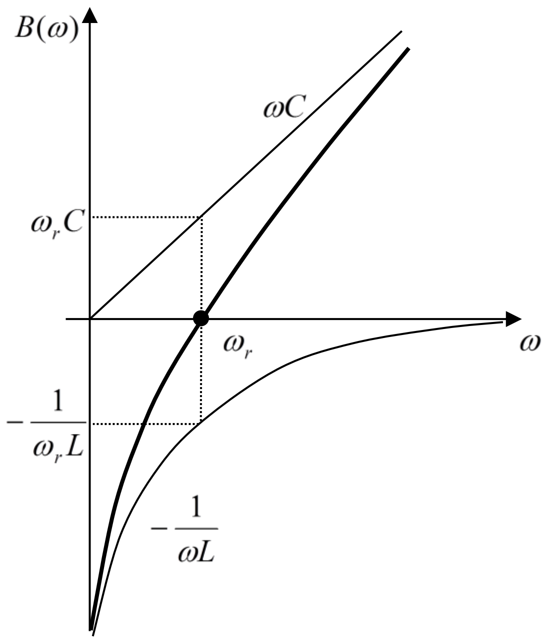


Fig. 4.4.24 Frequency characteristic of susceptance in RLC series circuit

- for dc voltage $\omega = 0$, susceptance is a short-circuit,
- for low frequencies $\omega \in (0, \omega_r)$, susceptance is negative (inductive),

$$B(\omega) = -\frac{1}{\omega L_p(\omega)} = \omega C - \frac{1}{\omega L} \qquad (4.4.46a)$$

where $L_p(\omega)$ is the equivalent parallel inductance,

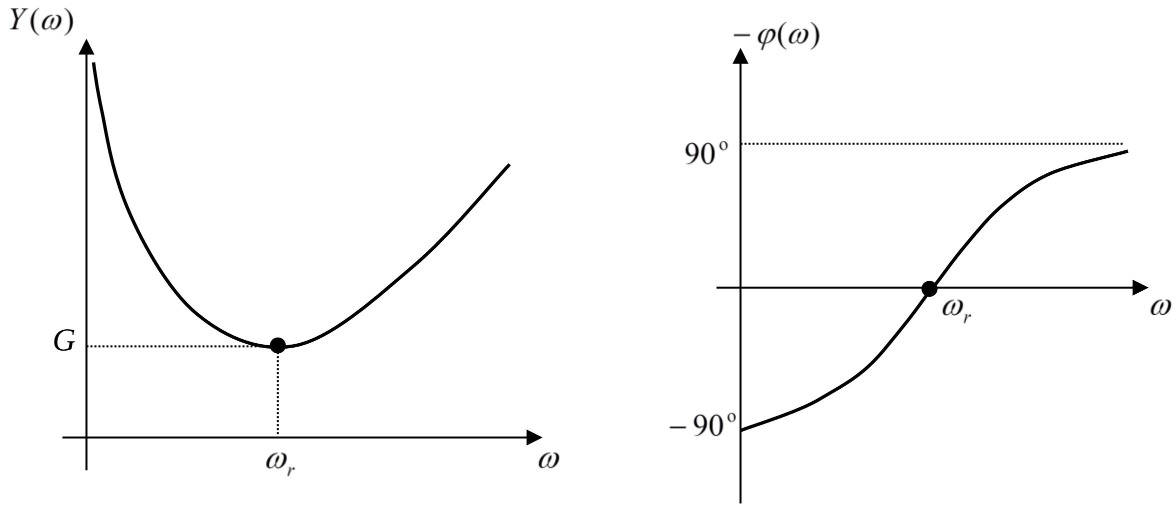
- for the resonant frequency ω_r,
  susceptance is zero (open-circuit) – circuit has resistive character,
- for high frequencies, $\omega \in (\omega_r, \infty)$, reactance is positive (capacitive),

$$B(\omega) = \omega C_p(\omega) = \omega C - \frac{1}{\omega L} \qquad (4.4.46b)$$

where $C_p(\omega)$ is the equivalent parallel capacitance,

- for $\omega = \infty$, susceptance is a short-circuit.

Magnitude and phase characteristics of admittance are presented in Fig. 4.4.25.



Fig, 4.4.25 Magnitude and phase characteristics of admittance of the parallel RLC circuit

Magnitude of the voltage is normally plotted in relation to the resonant voltage

$$U_r = U(\omega_r) = \frac{I}{G} \qquad (4.4.47)$$

The obtained gain curve,

$$\frac{U(\omega)}{U_r} = \frac{1}{\sqrt{1 + Q^2(\eta - 1/\eta)^2}} \qquad (4.4.48)$$

is called the resonant curve and it is depicted in Fig. 4.4.26, for two values of Q, where: η is the normalized frequency (4.4.38) and Q is the circuit quality factor

$$Q = \frac{1}{G}\sqrt{\frac{C}{L}} \qquad (4.4.49)$$

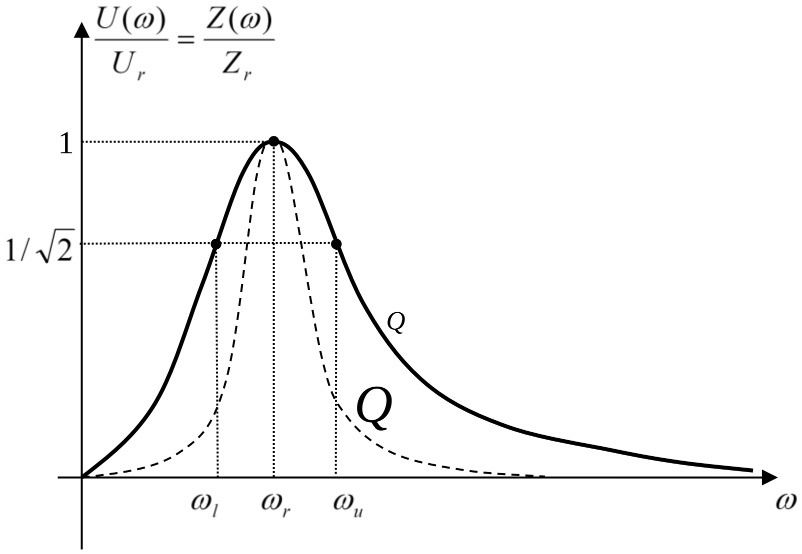


Fig. 4.4.26 Resonant curve of parallel RLC circuit for two values of Q

As can be observed, the parallel resonant circuit can be studied by repeating the results noted for the series *RLC* case while making substitution of $I(j\omega)$ for $U(j\omega)$ and vice-versa, *L* for *C* and vice-versa, and *G* for *R*. For large values of *Q*, the effect of the **resonance overcurrent** can be observed.

Example 4.4.2

For the given parameters of *RLC* parallel circuit: $R = 10\,\mathrm{k\Omega}$, $L = 1\mathrm{H}$, $C = 1\mathrm{\mu F}$ and the input current of $I = 10$ mA, find magnitude of the coil/capacitor current at the resonant frequency.

÷

The resonant frequency is $\omega_r = 1000$ rad/s, the quality factor is $Q = 100$. For these values, the resonant voltage is $U_r = 10\,\mathrm{V}$ and the maximum rms coil/capacitor current is $I_{\max} \cong IQ = 1$ A !

♦

Complex-resonant circuit

In complex-resonant circuits more than one resonance may occur. The analysis of such circuits is generally laborious and not especially illuminating. At the resonant frequency, the terminal voltage $U(j\omega)$ and current $I(j\omega)$ are in phase, and this is achieved when reactance is equal to zero (susceptance is equal to infinity) or reactance is equal to infinity (susceptance is equal to zero). Then, in a complex circuit resonant frequencies may be calculated from the following equations:

$$X(\omega) = 0 \text{ or } B(\omega) = \infty \qquad (4.4.50a)$$

and

$$X(\omega) = \infty \text{ or } B(\omega) = 0 \qquad (4.4.50b)$$

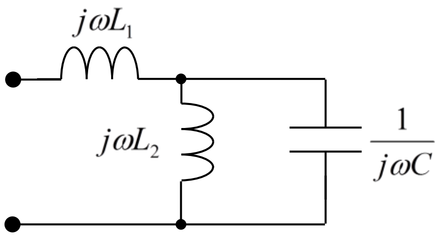
Resonant frequencies designated from (4.4.50a) are alternating with frequencies designated from (4.4.50b) and

$$dX(\omega)/d\omega \geq 0\,. \qquad (4.4.51)$$

Example 4.4.3

Plot the frequency characteristic $X(\omega)$ and calculate the resonant frequencies.

Fig. 4.4.27 Circuit for Example 4.4.3



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Two resonant frequencies are expected.

- For $\omega = 0$ the circuit is a short-circuit.
- For low frequencies the reactance is positive (circuit has inductive character), up to the frequency of the first resonance designated from (4.4.53b). At this frequency parallel connection of L_2 and C gives an open-circuit.
- For medium frequencies the reactance is negative (circuit has capacitive character), up to the frequency of the second resonance designated from (4.4.53a). At this frequency all three elements give a short-circuit.
- For high frequencies the reactance is positive (circuit has inductive character) again.
- For $\omega = \infty$ the reactance is an open-circuit.

The frequency characteristic is plotted in Fig. 4.4.28.

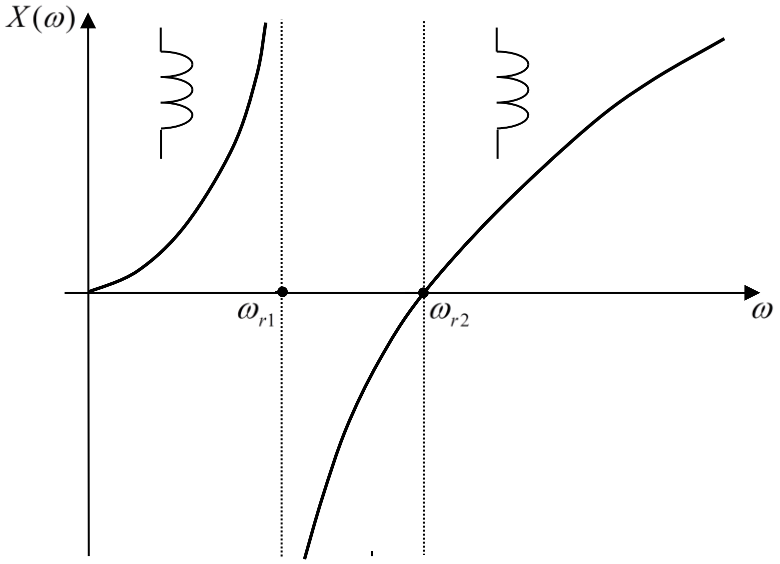


Fig. 4.4.28 Frequency characteristic for Example 4.4.3

The circuit impedance is

$$Z(j\omega) = j\omega L_1 + \frac{j\omega L_2 \frac{1}{j\omega C}}{j\omega L_2 + \frac{1}{j\omega C}} = j\left(\omega L_1 - \frac{\omega L_2}{\omega^2 L_2 C - 1}\right) = jX(\omega) \tag{4.4.52}$$

Then, the resonant frequencies calculated from (4.4.50) are:

$$\omega L_1 - \frac{\omega L_2}{\omega^2 L_2 C - 1} = 0 \Rightarrow \omega_{r2} = \sqrt{\frac{L_1 + L_2}{L_1 L_2 C}} \tag{4.4.53a}$$

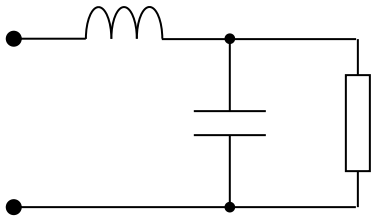
$$\omega L_1 - \frac{\omega L_2}{\omega^2 L_2 C - 1} = \infty \Rightarrow \omega_{r1} = \frac{1}{\sqrt{L_2 C}} \tag{4.4.53b}$$

♦

Drill problems 4.4

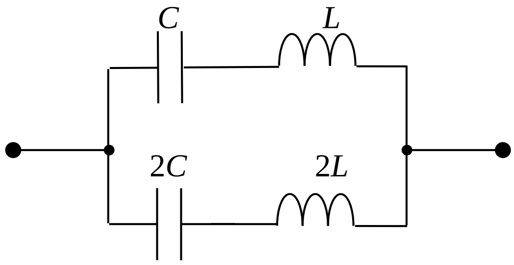
1. Draw the frequency characteristic $Z(\omega)$. Assume: $R = 100\Omega$, $L = 1\text{H}$, $C = 1\mu\text{F}$.

Fig. P.4.4.1



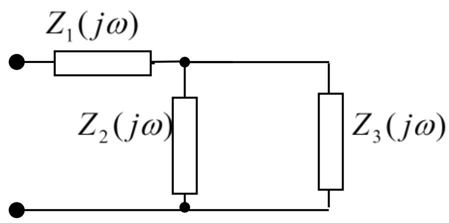
2 Draw the frequency characteristic $Z(\omega)$. Assume: $L = 1\text{H}$, $C = 1\mu\text{F}$.

Fig. P.4.4.2



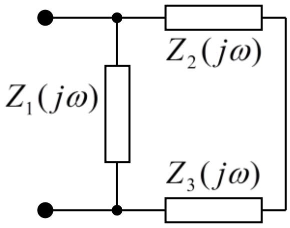
3. The series resonant circuit has L=1 mH and C=10 μF. Find the required Q and R when it is desired that the bandwidth be 16 Hz.
4. Make sketches of $Z(\omega)$, $R(\omega)$, $X(\omega)$ if the series element $Z_1(j\omega)$ is R, L or C and the parallel combination $Z_2(j\omega) \| Z_3(j\omega)$ is RL, RC or LC.

Fig. P.4.4.4



5. Rework Problem 4.4.4 if the parallel element is R, L or C and the series combination is RL, RC or LC.

Fig. P.4.4.5

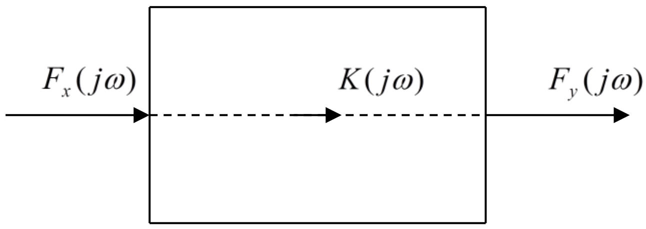


6. The series *RLC* circuit, $R = 10\,\Omega$; $L = 1$ H; $C = 1\,\mu\text{F}$, is connected to $U = 1$ V source. Calculate element voltages at the resonant frequency. Draw frequency characteristics of all voltages.
7. Draw the frequency characteristic $Z(\omega)$ for Fig. 4.2.11 circuit.
8. The parallel *RLC* circuit, $R = 100\,\text{k}\Omega$; $L = 1$ H, $C = 1\,\mu\text{F}$ is connected to $I = 1$ mA source. Calculate element currents at the resonant frequency. Draw frequency characteristics of all currents.

4.5 TRANSFER FUNCTION IN FREQUENCY DOMAIN-FREQUENCY RESPONSE

In most cases, ac steady-state analysis is much narrowly defined than that of finding all responses (amplitudes and phases) at single frequency excitation. A convenient way to test a linear circuit is to inject a sinusoid as the input and observe the sinusoidal steady-state output (amplitude and/or phase) at different frequencies. In many practical circuits, observation of response variations with frequency, the so called **frequency response**, is the fundamental part of ac analysis. In such case, the analysis is limited to the SISO analysis and the transfer function approach is utilized. The SISO circuit described in the frequency domain is presented in Fig. 4.5.1 – to avoid collision of notations, frequency-domain signals are denoted $F_x(j\omega)$ and $F_y(j\omega)$,while the *s*-domain signals have been denoted $X(s)$ and $Y(s)$.

Fig. 4.5.1 SISO linear circuit described in frequency domain



The **frequency response function** $K(j\omega)$, the transfer function $K(s)$ with *s* replaced by $j\omega$, scales the input phasor to yield the output phasor.

$$K(j\omega) = K(s)\Big|_{s=j\omega} = \frac{F_y(j\omega)}{F_x(j\omega)} = K(\omega)\exp[j\varphi(\omega)] \qquad (4.5.1)$$

$$K(\omega) = \frac{F_y(\omega)}{F_x(\omega)} \qquad (4.5.1a)$$

$$\varphi(\omega) = \alpha_y(\omega) - \alpha_x(\omega) \qquad (4.5.1b)$$

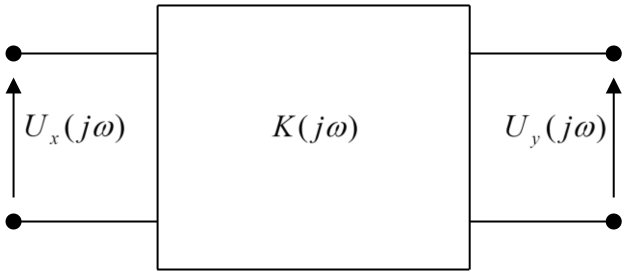
The curves for gain $K(\omega)$ versus ω and phase shift $\varphi(\omega)$ versus ω are called the **magnitude** or **amplitude** (frequency) **response** and **phase response**, respectively.

The SISO circuit gain $K(\omega)$ and phase shift $\varphi(\omega)$ completely describe of how the circuit responds to inputs at any frequency.

For the two-terminal circuit, if $F_x(j\omega)$ is its current $I(j\omega)$ and $F_y(j\omega)$ is its voltage $U(j\omega)$, or vice-versa, then $K(j\omega) = Z(j\omega)$ or $K(j\omega) = Y(j\omega)$, respectively, and this case has been already discussed in the preceding Chapter. In electronics, in most practical applications two-port is considered, as shown in Fig. 4.5.2. Then, the frequency response function (magnitude response) is the ratio of two voltages:

$$K(j\omega) = \frac{U_y(j\omega)}{U_x(j\omega)}, \quad K(\omega) = \frac{U_y(\omega)}{U_x(\omega)} \qquad (4.5.2)$$

Fig. 4.5.2 Two-port linear circuit described by frequency response function

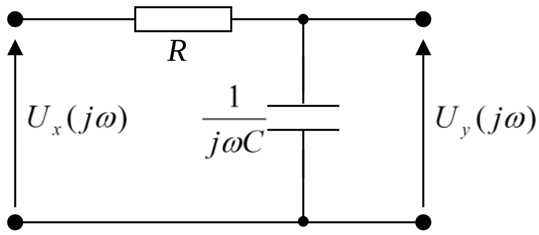


The circuit frequency response can be expressed graphically. The locus of the frequency response function can be plotted in the complex plan, or separate curves for phase shift and magnitude versus ω can be graphed. The latter one is normally graphed in logarithmic scale, as described in the next section of this Chapter.

<u>Example 4.5.1</u>

Plot the locus of the frequency response function, the phase response and the magnitude response of the two-port *RC* circuit shown in Fig. 4.5.3.

Fig. 4.5.3 Circuit for Example 4.5.1



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The circuit is the practical integrator considered already in Chapter 3.3. From its transfer function (3.3.7a) the frequency response function is obtained, with *s* replaced by $j\omega$.

$$K(s) = \frac{1}{1+sT} \Rightarrow K(j\omega) = \frac{1}{1+j\omega T} \tag{4.5.3}$$

Then, the magnitude and phase responses are

$$K(\omega) = \frac{1}{\sqrt{1+(\omega T)^2}},\ \varphi(\omega) = -\arctan(\omega T) \tag{4.5.3a}$$

The frequency response function, gain and phase shift, are collected in Table 4.5.1, for three characteristic frequencies.

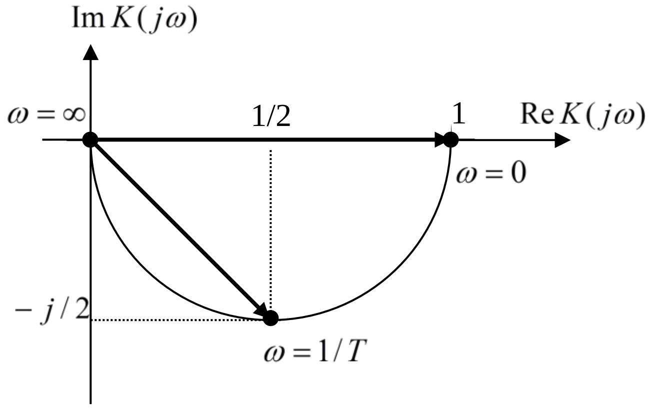
Table 5.5.1

Example 4.5.1 frequency response at selected frequencies

ω	0	$\omega_c = 1/T$	∞
$K(j\omega)$	1	$1/2 - j1/2$	0
$K(\omega)$	1	$1/\sqrt{2}$	0
$\varphi(\omega)$	0^o	-45^o	-90^o

It can be proved that locus described by the function (4.5.3) is a semicircle, as shown in Fig. 4.5.4. The amplitude response (**gain curve**) and phase response (**phase shift curve**) are presented in Fig. 4.5.5.

Fig. 4.5.4 Locus of $K(j\omega)$ for Example 4.5.1



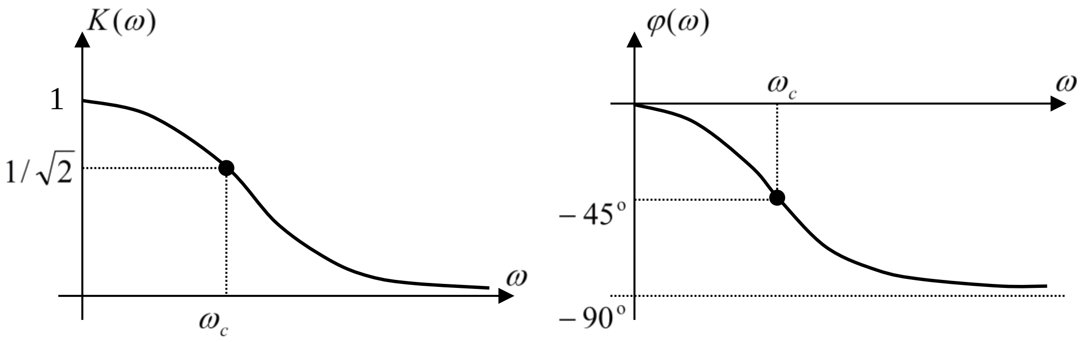


Fig. 4.5.5 Amplitude response and phase response for Example 4.5.1

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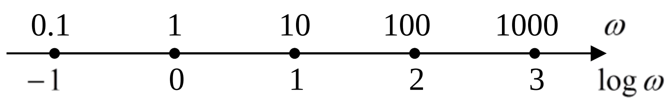
BODE (LOGARITHMIC) PLOT

The use of linear scale to measure gain has its limitations. Small dynamic range it makes available for graphing is the most important drawback of such scale. It is desirable to have equal ratios mapped into equal displacements and this can be achieved by using a logarithmic scale.
On a logarithmic scale, equal intervals represent a certain multiple, an increase of unity in the common logarithm, $\log_{10}(x)=\log(x)$, represents multiplication by ten:

$$\omega_2=10\omega_1 \Rightarrow \log\frac{\omega_2}{\omega_1}=\log 10=1 \tag{4.5.4}$$

Fig. 4.5.6 Frequency axis in logarithmic scale



In case of frequency such an interval is called a decade. In case of magnitude such an interval is called bell. For some practical reasons $10\cdot \text{bel} = \textbf{decibel}$ [dB] was adopted, first for the power ratio

$$10\log\frac{P_y(\omega)}{P_x(\omega)} \tag{4.5.5}$$

From this ratio, the voltage gain in logarithmic scale is obtained

$$\frac{P_y(\omega)}{P_x(\omega)} = \frac{U_y^2(\omega)G}{U_x^2(\omega)G} \Rightarrow K_{\mathrm{dB}}(\omega) = 20\log K(\omega) \tag{4.5.5a}$$

Then, for voltages (or currents) the gain is measured in decibels by twenty times its common logarithm. Table 4.5.1 presents the common decibel conversion table.

Table 4.5.1

Decibel conversion table

$K(\omega)$	0.1	$1/\sqrt{2} \cong 0.707$	1	$\sqrt{2}$	2	3	4	5	10	100
$K_{\mathrm{dB}}(\omega)$	–20	$-3.01 \cong -3$	0	≈3	≈6	≈10	≈12	≈14	20	40

An exact plot of gain versus frequency (logarithmic plot) is somewhat tedious to produce. In the 1930s the German-born engineer Hendrick Bode devised a simple method for graphing the logarithmic plot, it bears his name **Bode gain plot**.

Consider a transfer function (3.3.2) with s replaced by $j\omega$

$$K(j\omega) = K\frac{L(j\omega)}{M(j\omega)} = K\frac{\prod_{j=1}^{l}(j\omega - q_j)}{\prod_{k=1}^{m}(j\omega - s_k)} = C\frac{\prod_{j=1}^{l}(1 + j\omega T_{Lj})}{\prod_{k=1}^{m}(1 + j\omega T_{Mk})} \tag{4.5.6}$$

where,

q_j, s_k are roots of polynomials $L(s), M(s)$, zeroes and poles of $K(s)$,

$K = const$ is gain,

$T_{Lj} = -\dfrac{1}{q_j}, T_{Mk} = -\dfrac{1}{s_k}$ are time constants.

Then, the logarithmic plot is given by the following equation:

$$K_{\mathrm{dB}}(\omega) = 20\log C + \sum_{j=1}^{l} 20\log\sqrt{1+(\omega T_{Lj})^2} - \sum_{k=1}^{m} 20\log\sqrt{1+(\omega T_{Mk})^2} \tag{4.5.7}$$

The strategy for plotting the gain in decibels will be to plot each term of (4.5.7) separately and then add these component plots graphically.

The graph of the first term clearly is a flat straight line at the level of $20\log C$.

The graph of a term $f(\omega) = 20\log\sqrt{1+(\omega T)^2}$ can be approximated by two linear segments:

I. $\omega T << 1 \equiv \omega << \omega_c = 1/T \Rightarrow f(\omega) = 0$ (4.5.8a)

II. $\omega T >> 1 \equiv \omega >> \omega_c = 1/T \Rightarrow f(\omega) = 20\log(\omega T)$ (4.5.8b)

as presented in Fig. 4.5.7.

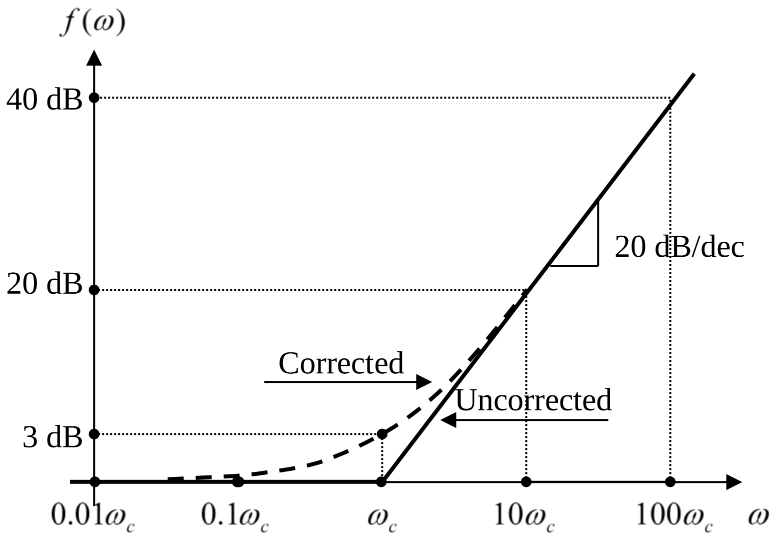


Fig. 4.5.7 Bode plot for $f(\omega) = 20\log\sqrt{1+(\omega T)^2}$

Such plot of PWL approximation, a pair of lines meeting at the **break frequency** or **cut-off frequency** ω_c, is called the **uncorrected Bode plot**. It is worth to observe that slope of the second segment (4.5.8b) is equal to 20 dB/decade. The true or **corrected Bode plot** is denoted by the dashed curve. Note that the maximum error occurs at the break frequency and it is equal to 3 dB. Far from this frequency the uncorrected and corrected plots merge smoothly. The technique for using component graphs to generate Bode plot will be illustrated in Example 4.5.2.

Example 4.5.2

Find Bode plot of the following transfer function

$$K(s) = 10\frac{1+s}{1+s10} \tag{4.5.9}$$

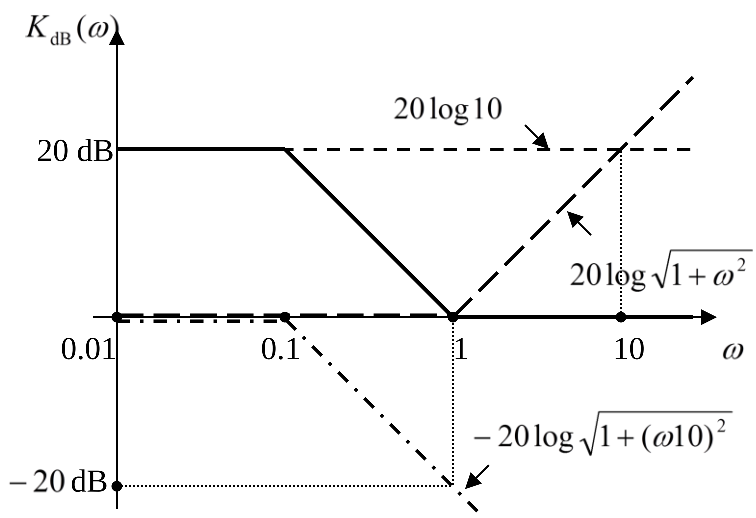


Fig. 4.5.8 Bode plot (uncorrected) for Example 4.5.2

The corresponding Bode plot

$$K_{\mathrm{dB}}(\omega) = 20\log 10 + 20\log\sqrt{1+\omega^2} - 20\log\sqrt{1+(\omega 10)^2} \tag{4.5.10}$$

has three terms, denoted in Fig. 4.5.8. by the dashed lines. These terms generate Bode plot, denoted by the solid line.

♦

Example 4.5.1 – cont.

Draw Bode gain plot for $T = 1\mathrm{s}$.

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For the gain expressed in linear scale by equation (4.5.3a), the corresponding Bode plot is

$$K_{\mathrm{dB}}(\omega) = -20\log\sqrt{1+(\omega T)^2} \tag{4.5.11}$$

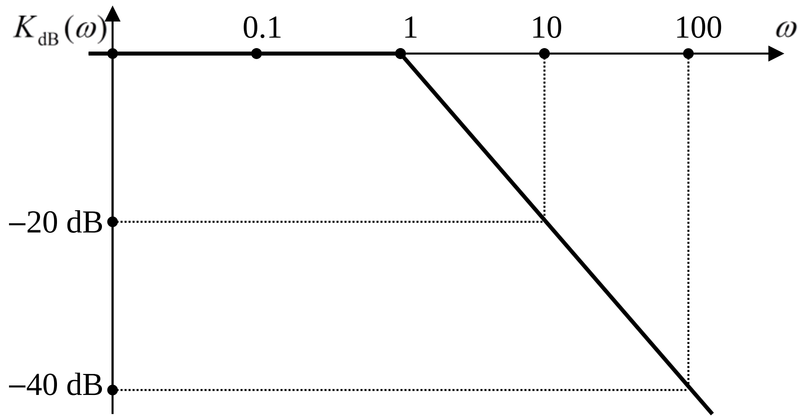
Values of the gain for $\omega = k\omega_c;\ k = 0,\ 0.1,\ 1,\ 10,\ 100$, in linear and decibel scale, are collected in Table 4.5.2. Bode plot is presented in Fig. 4.5.9.

Table 4.5.2

Gain in linear and decibel scale for Example 4.5.1

ω	0	0.1	1	10	100
$K(\omega)$	1	0.995	0.707	0.099	0.01
$K_{\mathrm{dB}}(\omega)$	0	–0.04	–3.01	–20.04	–40.00

Fig. 4.5.9 Bode plot for Example 4.5.1



♦

FILTERS

Filters are among the most common two-ports found in general circuit design. Every practical electronic circuit of any complexity contains at least one filter.

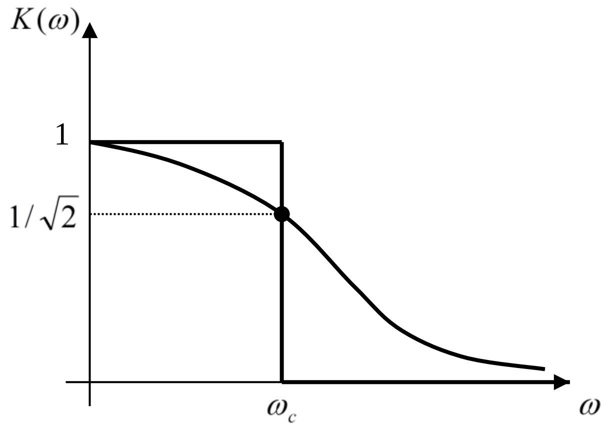
An **electrical filter** is a (two-port) circuit, as presented in Fig. 4.5.2, that is designated to introduce amplitude gain or loss over a predefined range of frequencies, impedes the passage of signals whose frequencies fall within a band called the **stopband**, while permitting those in another band, called the **passband**, to pass relatively unchanged.

Ideal filters block stopband signals completely while passing passband signals without any change. For nonideal filters the band-limiting frequency(ies) are defined as the half-power frequency(ies). In other words, the band-limiting frequency is the frequency at which the gain is 3 dB below its maximum value. The location of the pass(stop)band designates character of a filter, and four types of filters can be distinguished.

Low-Pass Filter - LPF

The magnitude response of a low-pass filter with band-limiting frequency, so called cutoff frequency ω_c, is presented in Fig. 4.5.10, for both ideal (solid) and nonideal (dashed) case.

Fig. 4.5.10 Low-pass filter gain curve

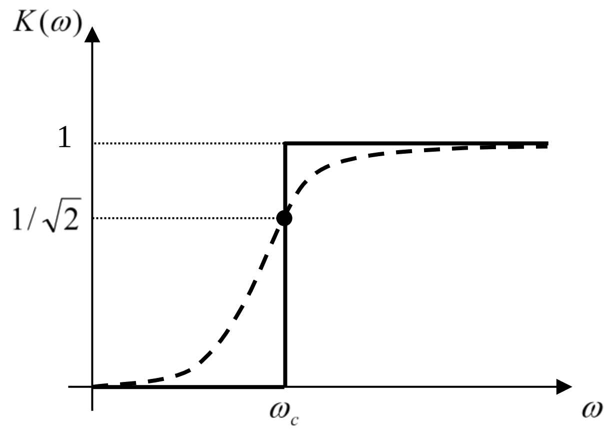


The two-port *RC* circuit of Example. 4.5.1 can be considered as the simplest low-pass filter. Its gain curve is the dashed curve of Fig. 4.5.10 and its Bode plot is presented in Fig. 4.5.9 ($\omega_c = 1$ rad/s).

High-Pass Filter - HPF

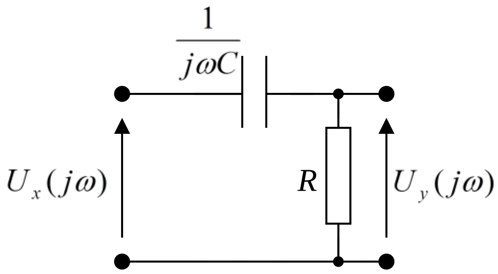
The magnitude response of a high-pass filter with cutoff frequency ω_c is presented in Fig. 4.5.11, for both ideal (solid) and nonideal (dashed) case.

Fig. 4.5.11 High-pass filter gain curve



The simplest, *RC* circuit realization is presented in Fig. 4.5.12. This circuit is the simplest differentiator considered in Chapter 3.3. Its transfer function is described by equation (3.3.11a).

Fig. 4.5.12 RC high-pass filter

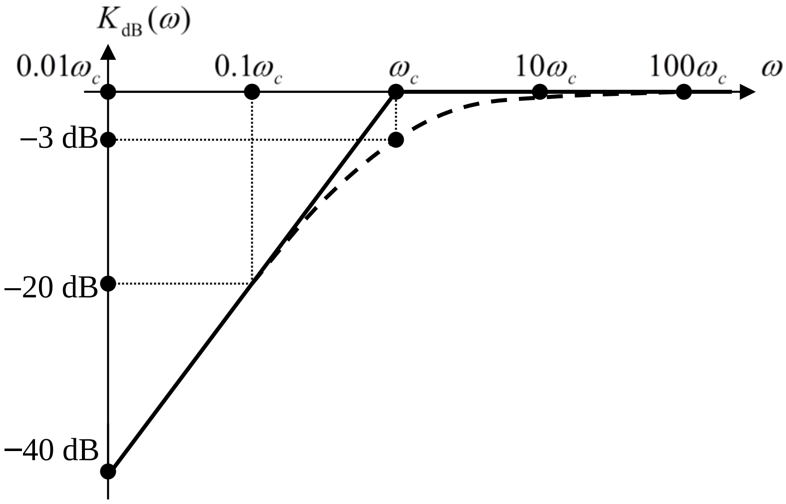


Then, the frequency response function is

$$K(\omega) = \frac{\omega T}{\sqrt{1+(\omega T)^2}}, \; K_{\mathrm{dB}}(\omega) = 20\log(\omega T) - 20\log\sqrt{1+(\omega T)^2} \tag{4.5.12}$$

The linear-scale gain is denoted in Fig. 4.5.11 by the dashed curve. The corresponding Bode plot is presented in Fig. 4.5.13.

Fig. 4.5.13 Bode plot of RC high-pass filter



Band-Pass Filter - BPF

The magnitude response of a band-pass filter with lower and upper boundary frequencies ω_l, ω_u is presented in Fig. 4.5.14, for both ideal case (solid) and nonideal case (dashed). The simplest, *RLC* circuit realizations are presented in Fig. 4.5.15.

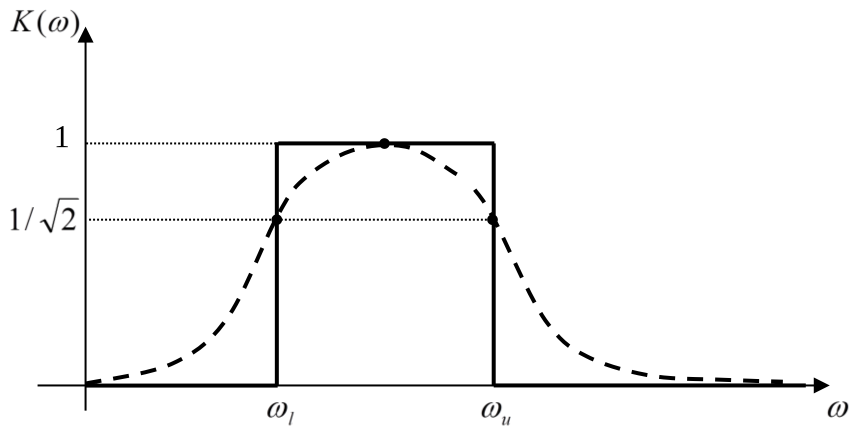


Fig. 4.5.14 Magnitude response of a band-pass filter

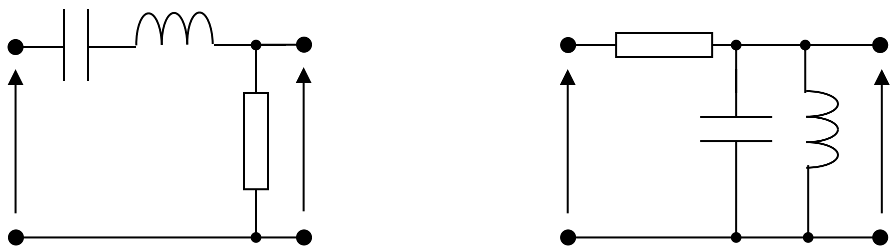


Fig. 4.5.15 RLC realizations of band-pass filter

Band-Stop Filter - BSF

The magnitude response of a band-stop filter with lower and upper boundary frequencies ω_l, ω_u is presented in Fig. 4.5.16, for both ideal case (solid) and nonideal case (dashed). The simplest, *RLC* circuit realizations are presented in Fig. 4.5.17.

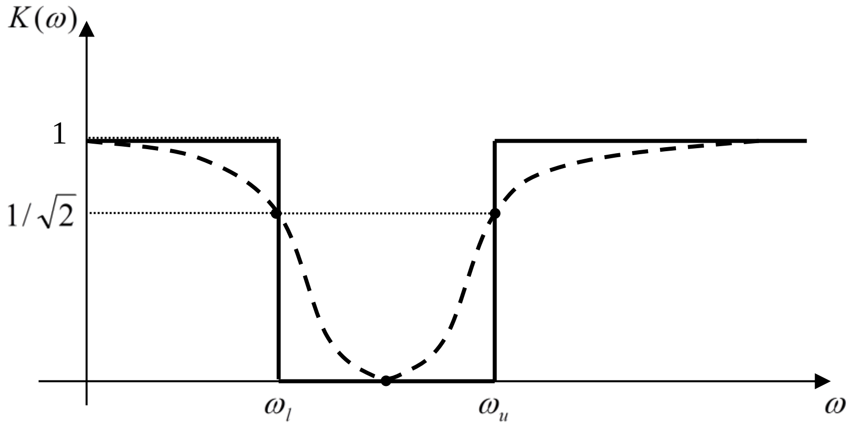


Fig. 4.5.16 Band-stop filter gain curve

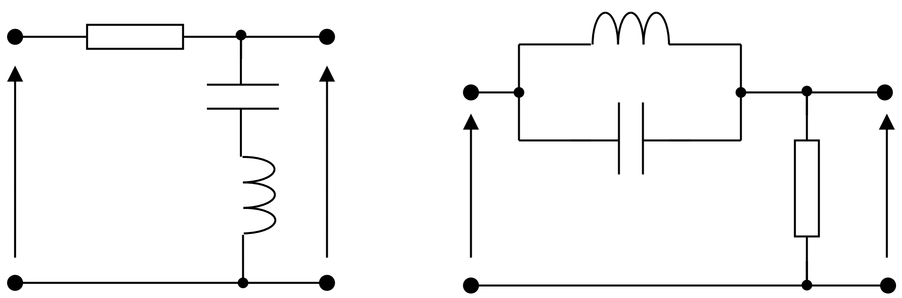


Fig. 4.5.17 RLC realizations of band-stop filter

Drill problems 4.5

1. Draw the amplitude response in logarithmic scale (Bode plot) for the transfer function $K(s) = (A + Bs)/(C + Ds)$ and the following combinations of its coefficients:
   a) *A*=0.1, *B*=0, *C*=1, *D*=10; b) *A*=1, *B*=0.1, *C*=0, *D*=1; c) *A*=0, *B*=10, *C*=1, *D*=10.
2. Draw the logarithmic plot of the *RC* high(low)-pass filter, $R = 10\,\mathrm{k\Omega}$, $C = 1\,\mu\mathrm{F}$.
3. Draw the logarithmic plot of the ideal integrator (differentiator) characterized by the integration (differentiation) constant *T*=10 s.

4. What is the simplest structure of a filter giving the following amplitude response?

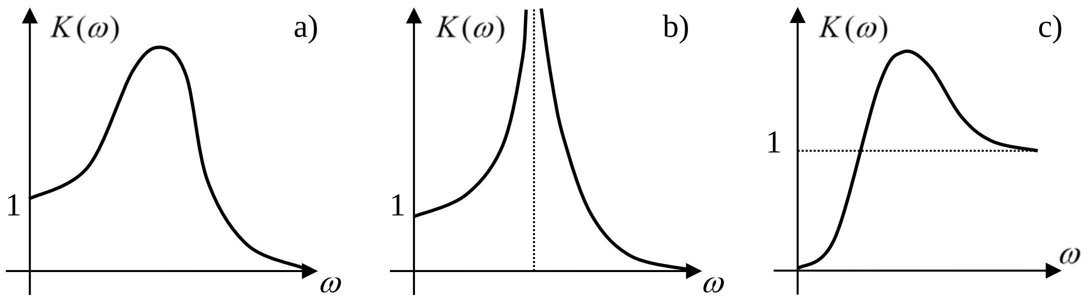
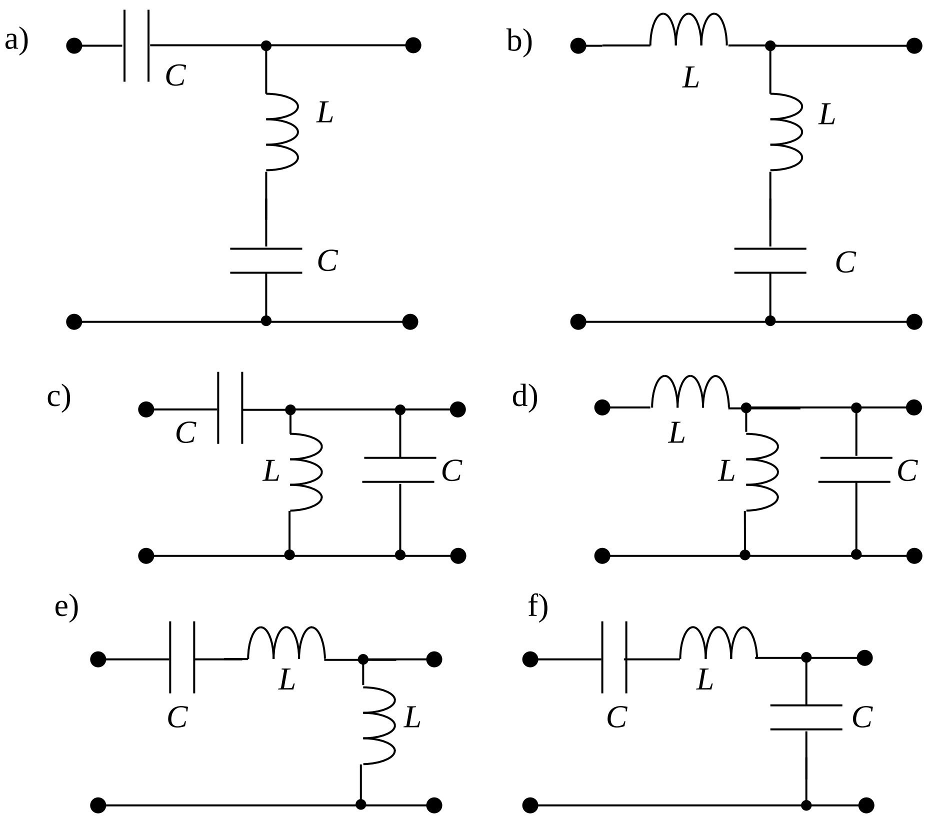


Fig. P.4.5.4

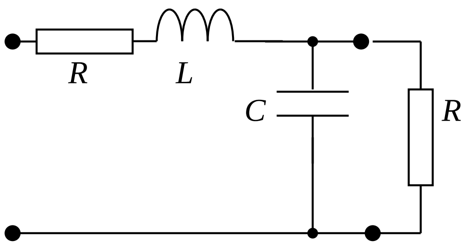
5. Sketch amplitude response $K(\omega)$ of the given filters.

Fig. P.4.5.5



6. Sketch amplitude response $K(\omega)$ of the given loaded filter.

Fig. P.4.5.6



7. Calculate the frequency response function $K(j\omega)$ of the RC filter. Sketch Bode plot of this filter.

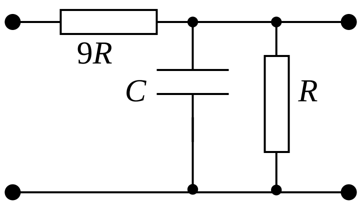


Fig. P.4.5.7

4.6 ANALYSIS OF CIRCUIT RESPONSE WHEN ONE CIRCUIT CONSTANT VARIES

In some practical applications study of a circuit behavior when its one constant (parameter) *q*, such as: resistance *R*, inductance *L* or capacitance *C*, varies from $Q_{\min}$ to $Q_{\max}$ is necessary at the design stage. A locus of circuit phasor response when *q* varies is plotted in the complex plane and its two shapes will be discussed:

a) (half)line,

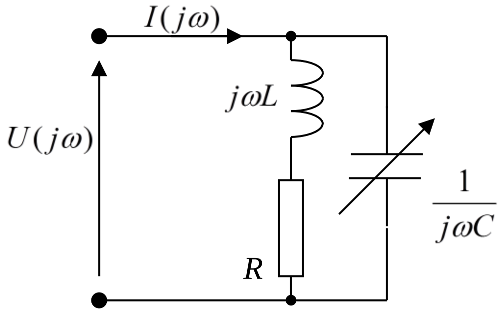
b) (semi)circle.

The technique for plotting such loci will be illustrated in Example 4.6.1.

<u>Example 4.6.1a – straight line example</u>

For the input voltage $U(j\omega) = U = 10/\sqrt{2}$ V plot locus of the input current phasor, if *C* varies from 0 to infinity. Other circuit constants are: $\omega L = R = 10\,\Omega$, $\omega = \sqrt{2}$ rad/s.

Fig. 4.6.1 Circuit for Example 4.6.1



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The circuit current is

$$I(jq) = \frac{U}{R + j\omega L} + jU\omega C \tag{4.6.1}$$

and its locus for $q = C \in <0, \infty>$ is a half-line.

In general, a straight line locus is described by the following equation

$$F(jq) = \mathbf{A} + \mathbf{B}q \tag{4.6.2}$$

where, **A** and **B** are complex numbers. For exemplary vectors: $\mathbf{A} = 1 - j$, $\mathbf{B} = 1 + 2j$ and $q \in <0, \infty>$ a half-line locus shown in Fig. 4.6.2 is obtained.

For the phasor described by (4.6.1) and $q = C \in \langle 0, \omega \rangle$:

$$\mathbf{A} = \frac{U}{R + j\omega L} = 5 - j5 \tag{4.6.2a}$$

$$\mathbf{B} = jU\omega = j10 \tag{4.6.2b}$$

and a half-line locus is obtained, as shown in Fig. 4.6.3.

Fig. 4.6.2 Exemplary line-shape locus

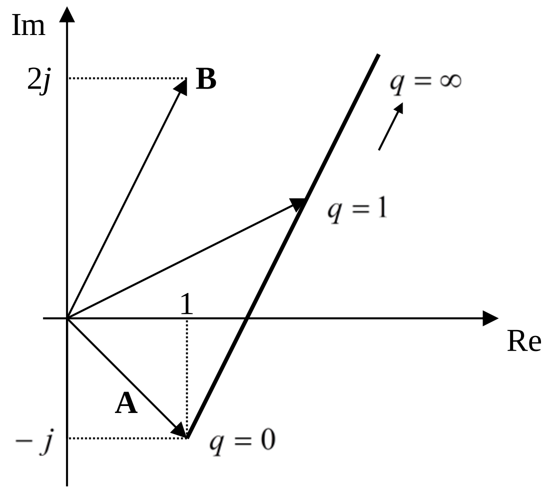
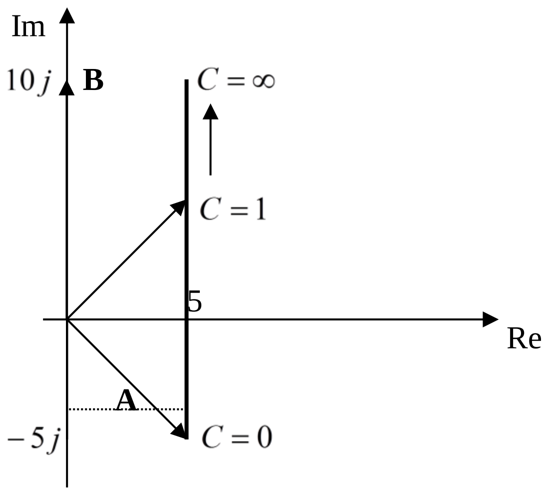


Fig. 4.6.3 Locus of Example 4.6.1a



Example 4.6.1b – semicircle example

For the circuit of Fig. 4.6.1 and the input voltage $U(j\omega) = U = 10/\sqrt{2}$, plot locus of the input current phasor, if L varies from 0 to infinity. Other circuit constants are:

$$\frac{1}{\omega C} = 4R, R = 10\Omega,\ \omega = \sqrt{2} \text{ rad/s.}$$

÷

Locus of the coil current described by the following equation

$$I_L(jL) = \frac{U}{R + j\omega L} \qquad (4.6.3a)$$

for $L \in \langle 0, \infty \rangle$ is a semicircle with the center coordinates of $[\,\mathrm{Im} = 0,\ \mathrm{Re} = U/2R\,]$ and the radius of $U/2R$. Following equation (4.6.1), this locus is added to the fixed capacitor current

$$I_C(jL) = j\omega CU = jU/4R \qquad (4.6.3b)$$

and the total locus is obtained, as shown in Fig. 4.6.4

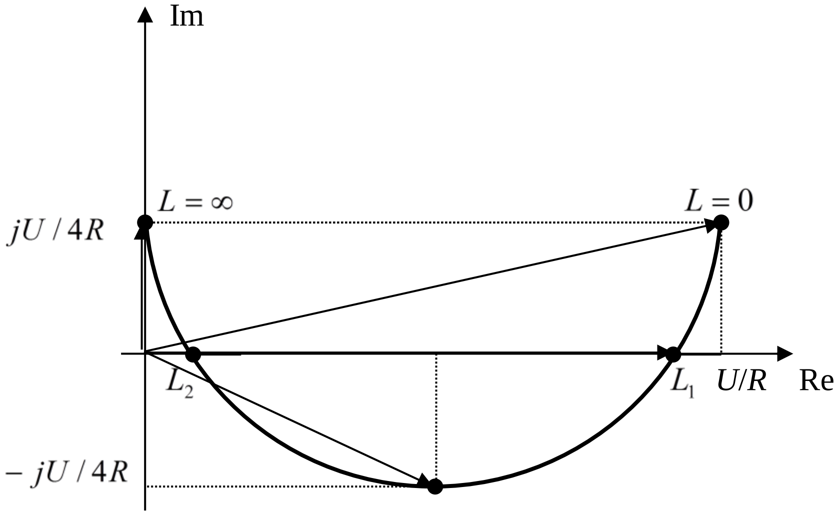


Fig. 4.6.4 Locus of Example 4.6.1b

The circuit behavior as L varies from 0 to ∞ can be studied:

- for $L \in \langle 0, L_1)$ inductive character, $I(jL)|_{L=0} = U/R + jU/4R$
- for $L = L_1$ resistive character (1st resonance)
- for $L \in (L_1, L_2)$ capacitive character, $I(jL)|_{L=R/\omega} = U/2R - jU/4R$
- for $L = L_2$ resistive character (2nd resonance)
- for $L \in (L_2, \infty\rangle$ inductive character, $I(jL)|_{L=\infty} = I_C(jL) = jU/4R$

♦

<u>Drill problems 4.6</u>

1. Plot in the complex plane a locus of current $I(j\omega)$ that flows through: a) *RC* parallel circuit *R*=10 Ω, $C \in \langle 0, \infty \rangle$, b) *RL* series circuit *R*=10 Ω, $L \in \langle 0, \infty \rangle$, supplied from $U(j\omega) = U = 10$ V source.
2. Plot in the complex plane a locus of voltage $U(j\omega)$ at terminals of: a) *RC* series circuit *R*=10 Ω, $C \in \langle 0, \infty \rangle$, b) *RL* series circuit *R*=10 Ω, $L \in \langle 0, \infty \rangle$, supplied from $I(j\omega) = I = 10$ A source.
3. For the *RC* high-pass filter (Fig. 4.5.12) find the gain $K = U_y / U_x$ in terms of *R* and *C*.

4.7 MUTUAL INDUCTANCE AND TRANSFORMERS

The previous analysis of a coil assumed that the only flux linking a coil was that due to its own current, and consequently, the only voltage induced was that due to this current. In this Chapter coupled coils and phenomenon of mutual inductance will be discussed. A two-port equations expressing voltages by currents, both in time-domain and frequency-domain, will be considered. Then, the most practical use of this phenomenon in transformers will be studied. Basic transformer built of practical coils will be considered at first, next an ideal transformer. Finally the ideal transformer based model of practical transformer will be described.

MUTUAL INDUCTANCE – BASIC TRANSFORMER

Consider two **coupled coils**, as shown in Fig. 4.7.1.

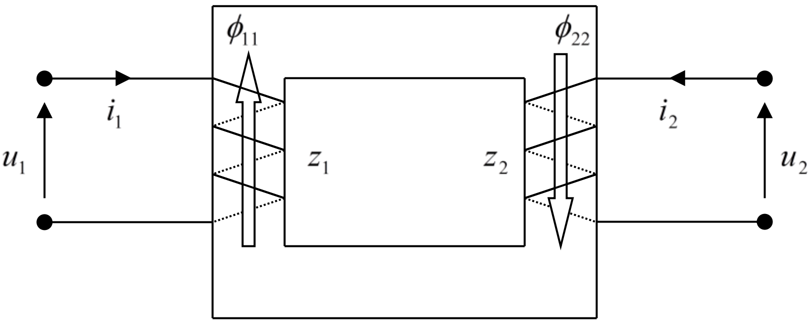


Fig. 4.7.1 Pair of coupled coils

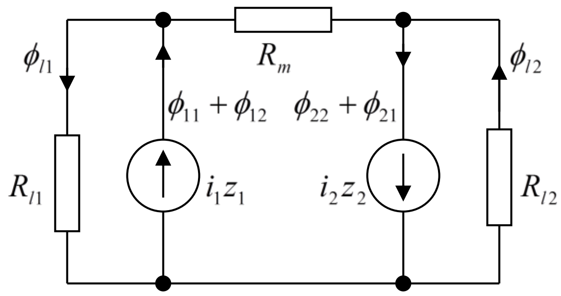
The current i_1 produces in coil 1 flux ϕ_{11}. Part of this flux threads coil 2, the remainder is coil 1 leakage flux. They are denoted ϕ_{21} and ϕ_{l1}, respectively. Similarly, i_2 produces in coil 2 flux ϕ_{22} that is split into two fluxes, ϕ_{12} and ϕ_{l2}.

$$\phi_{11} = \phi_{21} + \phi_{l1} \tag{4.7.1a}$$

$$\phi_{22} = \phi_{12} + \phi_{l2} \tag{4.7.1b}$$

Electric analog of this magnetic circuit is presented in Fig. 4.7.2

Fig. 4.7.2 Electric analog of magnetic circuit



where R_{l1}, R_{l2} are magnetic resistances of leakages, R_m is a core magnetic resistance. Superposition principle can be applied and a circuit of Fig. 4.7.2 can be split into two subcircuits, as presented in Fig. 4.7.2a.

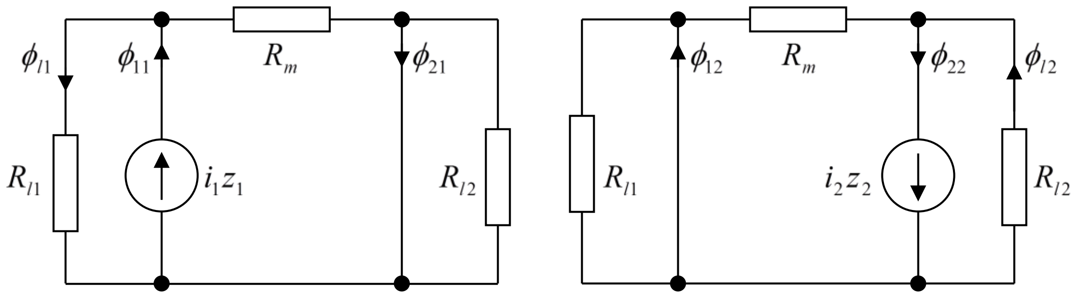


Fig. 4.7.2a Electric analog of magnetic circuit split into two subcircuits

The total flux threading coil 1 is the sum of two components:

$$\phi_{t1} = z_1\phi_1 = z_1\phi_{11} \pm z_1\phi_{12} = z_1\left(\frac{i_1 z_1}{R_{l1}} + \frac{i_1 z_1}{R_m}\right) \pm z_1 \frac{i_2 z_2}{R_m} = L_1 i_1 \pm M i_2 \tag{4.7.2a}$$

Similarly, the total flux threading coil 2

$$\phi_{t2} = z_2\phi_2 = z_2\phi_{22} \pm z_2\phi_{21} = z_2\left(\frac{i_2 z_2}{R_{l2}} + \frac{i_2 z_2}{R_m}\right) \pm z_2 \frac{i_1 z_1}{R_m} = L_2 i_2 \pm M i_1 \tag{4.7.2b}$$

Constant of proportionality between one coil current and a flux that is produces in the coupled coil is called the **mutual inductance** *M*. This constant can be expressed by self inductances of individual coils

$$M = k\sqrt{L_1 L_2} \tag{4.7.3}$$

where, $k \in <0,1>$ is the **coefficient of coupling**, a measure of the degree to which the flux produced by one coil threads another. If there is no coupling then, $k = 0 \Rightarrow M = 0$. For tightly coupled coils, which is the most desirable situation, $k \cong 1 \Rightarrow M \cong \sqrt{L_1 L_2}$. From (4.7.2), two-port equations can be obtained

$$u_1 = \frac{d\phi_{t1}}{dt} = L_1 \frac{di_1}{dt} \pm M \frac{di_2}{dt} \tag{4.7.4a}$$

$$u_2 = \frac{d\phi_{t2}}{dt} = L_2 \frac{di_2}{dt} \pm M \frac{di_1}{dt}$$

The reason for $\pm$ sign in the coil equation is that the flux produced by the coupled coil may be in the same or opposite direction as the produced by the coil itself. For Fig. 4.7.1 coupling, sign + should be used. For unique denotation of the coupling sign, the so called dot convention is used.

Dot convention

Currents entering the dotted ends are creating additive fluxes. Dotted ends have a positive voltage at the same time.

A circuit symbol of coupled coils is presented in Fig. 4.7.3.

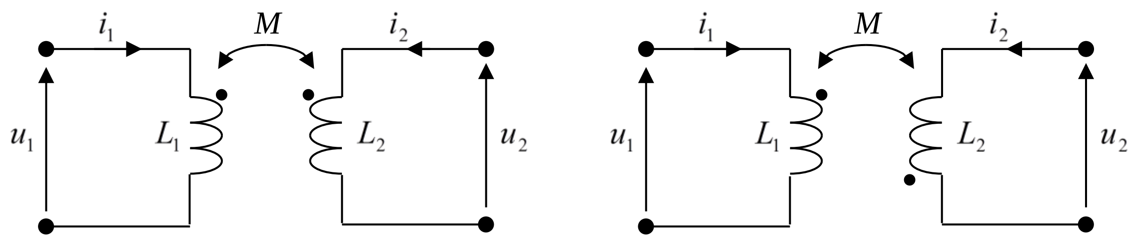


Fig. 4.7.3 Circuit symbol for coupled coils for positive and negative coupling

Using the phasor notation, coupled coils equations are

$$U_1(j\omega) = j\omega L_1 I_1(j\omega) \pm j\omega M I_2(j\omega) \tag{4.7.4b}$$

$$U_2(j\omega) = j\omega L_2 I_2(j\omega) \pm j\omega M I_1(j\omega)$$

<u>Example 4.7.1</u>
Consider two practical coupled coils connected in series, as shown in Fig. 4.7.4. Find the circuit equivalent.

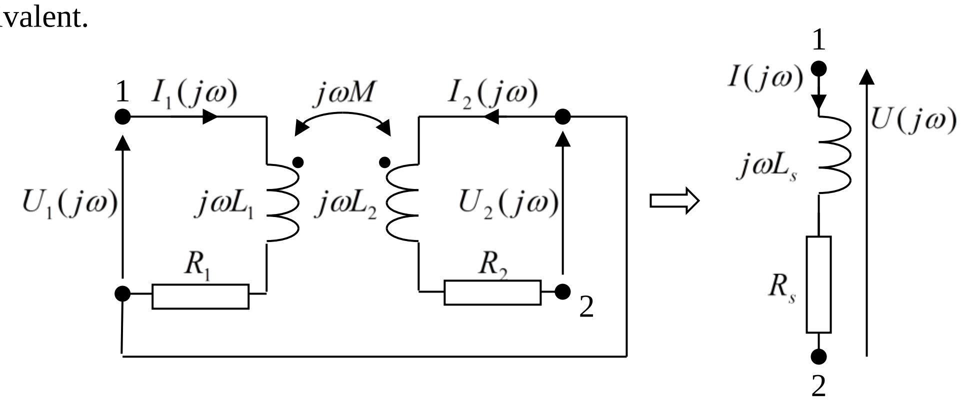


Fig. 4.7.4 Two coupled coils connected in series

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The total voltage expressed by the currents is

$$U(j\omega) = U_1(j\omega) + U_2(j\omega) = I(j\omega)[R_1 + j\omega(L_1 + M) + R_2 + j\omega(L_2 + M)] \tag{4.7.5a}$$

$$U(j\omega) = I(j\omega)[R_s + j\omega L_s] \tag{4.7.5b}$$

From (4.7.5), the series equivalent resistance and inductance are

$$R_s = R_1 + R_2 \tag{4.7.6a}$$

$$L_s = L_1 + L_2 + 2M \tag{4.7.6b}$$

The equivalent series inductance of two coupled coils is sum of self inductances plus the doubled mutual inductance. For negative coupling the doubled mutual inductance should be subtracted.

♦

<u>Example 4.7.2</u>

A coil of $z = 100$ turns connected to $U = 100$ V, $\omega = 1000$ rad/s supply has the following parameters: $L = 1$ H, $R = 10\,\Omega$. Consider a short-circuit of one turn. Study the effect of such failure.

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A coil with one turn shorted can be considered as pair of coupled coils, as presented in Fig. 4.7.5.

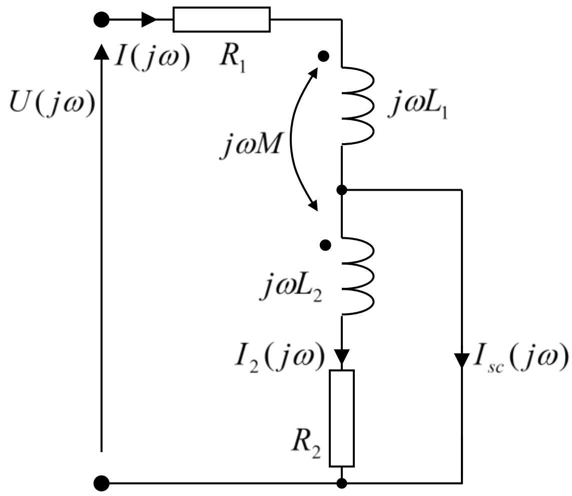


Fig. 4.7.5 Coil with one turn shorted

It can be assumed that $L_1 \cong L = 1$ H, $R_1 \cong R = 10\,\Omega$ and $L_2 = 0.1$ mH, $R_2 = 0.1\,\Omega$, $M = 0.01$ H. Two cases are studied.

I. Before shorting

$$Z(j\omega) = R + j\omega L \cong 10 + j1000 \Rightarrow Z \cong 1000\,\Omega \tag{4.7.7}$$

$I = U/Z \cong 0.1$ A

and the dissipated power is

$$P = I^2 R \cong 0.1\,\text{W}$$

II. After shorting

The circuit is described by the following KVL equations

$$R_1 I(j\omega) + j\omega L_1 I(j\omega) + j\omega M I_2(j\omega) = U$$

$$R_2 I_2(j\omega) + j\omega L_2 I_2(j\omega) + j\omega M I(j\omega) = 0$$

From these equations, the circuit impedance is

$$Z(j\omega) = \frac{U}{I(j\omega)} = R_1 + \frac{(\omega M)^2 R_2}{R_2^2 + (\omega L_2)^2} + j\omega\left[L_1 - \frac{(\omega M)^2 L_2}{R_2^2 + (\omega L_2)^2}\right] \cong 1010 + j2 \Rightarrow Z \cong 1000\,\Omega \tag{4.7.8b}$$

$I \cong 0.1$ A, and the dissipated power is $P = I^2 R(\omega) \cong 10\,\mathrm{W}$.

The impedance magnitude practically has not changed, and consequently rms current remains unchanged, however the resistance and then dissipated power has increased 100 times !!! This causes rapid increase of temperature, an isolation melts and next turns are shorted, what completely destroys (burns) the coil.

♦

Basic transformer, an air-core (liear) transformer built of two coupled coils is considered next.

A transformer is an electronic device that uses magnetically coupled coils to transfer energy from one circuit to another. This device has two ports available for connection to external circuitry. One of this ports is called the **primary port** and usually it is connected to an external source. The other port is called the **secondary port** and usually it is connected to a load. Then, terms primary/secondary circuit, winding, voltage or current are used. The primary circuit has been simplified to its Thevenin equivalent, the load has been reduced to its equivalent impedance. It should be emphasized, that the primary and the secondary circuit are electrically isolated. Basic transformer circuit described in phasor domain is presented in Fig. 4.7.6. For simplicity of further derivations, it has been assumed that coils are resistanceless.

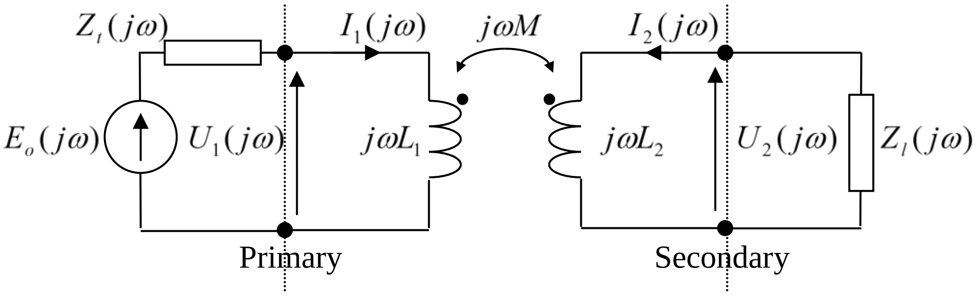


Fig. 4.7.6 Basic transformer circuit

Transformer itself is described in phasor domain by equations (4.7.4b). The secondary circuit (load) is described by the following equation

$$U_2(j\omega) = -Z_l(j\omega) I_2(j\omega) \tag{4.7.9}$$

The load current and voltage do not satisfy the passive sign convention, and for that reason sign minus appears at the rights side of equation (4.7.9). It is interesting to study the impedance looking into the primary port

$$Z_1(j\omega) = U_1(j\omega) / I_1(j\omega) \tag{4.7.10}$$

Combining equations (4.7.4b), (4.7.9) and (4.7.10), the following impedance is obtained

$$Z_1(j\omega) = j\omega L_1 + \frac{\omega^2 M^2}{Z_l(j\omega) + j\omega L_2} \tag{4.7.10a}$$

The first term depends on the primary coil while the second term is due to the coupling, so called the **reflected impedance**

$$Z_r(j\omega) = \frac{\omega^2 M^2}{R_l + j(X_l + \omega L_2)} = R_r + jX_r \tag{4.7.10b}$$

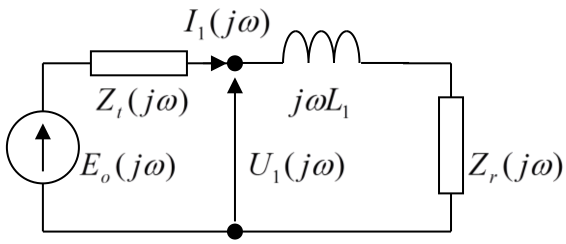


Fig. 4.7.7 Secondary winding inductance and load reflected into primary

Then, the transformer together with its load can be replaced by series connection of two impedances. The obtained one-loop circuit, presented in Fig. 4.7.7, is often used to simplify analysis of the basic transformer circuit of Fig. 4.7.6.

IDEAL TRANSFORMER

A transformer scales, or transforms, the voltage, current and impedance levels of the circuit. The secondary-to-primary current and voltage ratios will be discussed for an ideal transformer that may be thought of as the first-order model of an iron-core transformer (complex model will be discussed in the next section of this Chapter). An **ideal transformer** satisfies the following three assumptions.

1. Windings are resistanceless: $R_1 = R_2 = 0$.
2. Leakage fluxes are zero: $\phi_{l1} = \phi_{l2} = 0 \Rightarrow k = 1$.
3. The core magnetic material has unboundedly large permeability: $\mu_r = \infty \Rightarrow R_m = 0 \Rightarrow L_1, L_2$ are unboundedly large, however their ratio is finite, $L_2 / L_1 = (z_2 / z_1)^2 = n^2$ (see equations (4.7.2)).

An ideal transformer electric analog is presented in Fig. 4.7.8.

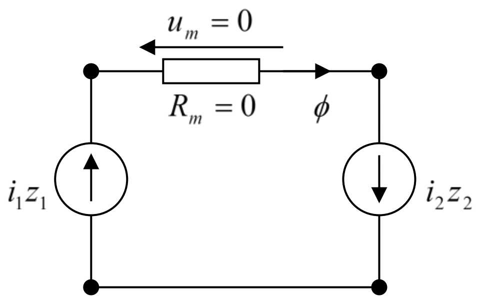


Fig. 4.7.8 Electric analog of ideal transformer

As windings are resistanceless (assumption 1) and same flux flows through both coils (assumption 2), then their voltages are

$$u_1 = z_1 \frac{d\phi}{dt},\; u_2 = z_2 \frac{d\phi}{dt} \tag{4.7.11}$$

As the core magnetic resistance (voltage) is zero (assumption 3), then the currents are related by the following KVL equation

$$i_1 z_1 + i_2 z_2 = R_m \phi = 0 \tag{4.7.12}$$

From these equations, secondary-to-primary current and voltage ratios are

$$\frac{u_2}{u_1} = n, \quad \frac{i_2}{i_1} = -\frac{1}{n} \tag{4.7.13}$$

where

$$n = \frac{z_2}{z_1} \tag{4.7.14}$$

is the only parameter that characterizes an ideal transformer, so called **turns ratio**. Its value defines the transformer character:

- $n > 1$: step-up transformer,
- $n < 1$: step-down transformer,
- $n = 1$: isolating transformer.

It is worth to observe, that an ideal transformer secondary-to-primary current and voltage ratios are fixed, independent of load and frequency. An ideal transformer phasor equations are

$$U_2(j\omega) = nU_1(j\omega) \tag{4.7.15}$$
$$I_2(j\omega) = -\frac{1}{n} I_1(j\omega)$$

The circuit symbol for an ideal transformer is similar to that for nonideal transformer (Fig. 4.6.6) except that the turns ratio is specified rather than inductances (all three are infinitely large) and pair of parallel lines is drawn. An ideal transformer with primary source and secondary load is presented in Fig. 4.7.9.

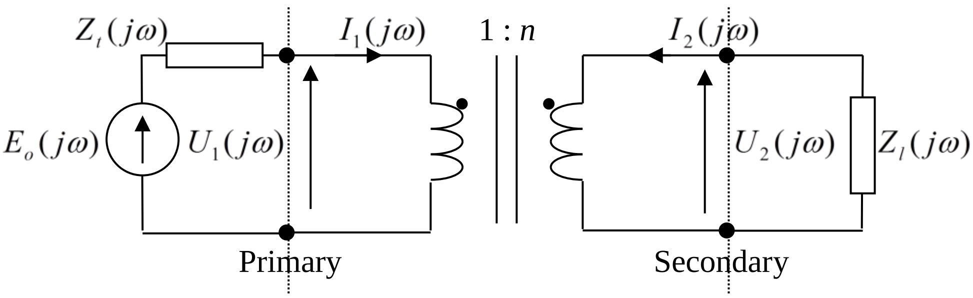


Fig. 4.7.9 Ideal transformer with primary source and secondary load

The impedance looking into the primary port is

$$Z_1(j\omega) = \frac{U_1(j\omega)}{I_1(j\omega)} = -\frac{1}{n^2}\frac{U_2(j\omega)}{I_2(j\omega)} = \frac{1}{n^2} Z_l(j\omega) \tag{4.7.16}$$

Thus, an ideal transformer together with its secondary load $Z_l(j\omega)$ is equivalent to an impedance of value $Z_l(j\omega)/n^2$ reflected into the primary circuit. Then, apart of previously discussed uses to isolate two circuits and step-up or step down a voltage level, an ideal transformer can be used to impedance scale a load such that maximum power transfer condition is achieved. The latter use will be illustrated in the next example.

Example 4.7.3

An audio amplifier produces the Thevenin equivalent voltage phasor $E_o(j\omega)$ through $Z_t(j\omega) = R_t = 72\,\Omega$ impedance (resistance). The produced power is to be delivered to a load, $Z_l(j\omega) = R_l = 8\,\Omega$ speaker, through an ideal transformer. Determination of the turns ratio n to maximize the power supplied by the source (absorbed by the speaker) is the task. Amplifier-transformer-speaker circuit is presented in Fig. 4.7.9.

÷

For the maximum power transfer, the load (speaker) impedance $Z_l(j\omega)$ seen by the source should be equal to the complex conjugate of the source impedance, which is purely real $Z_t(j\omega) = R_t = 72\,\Omega$. Thus, for the maximum power supplied to the speaker from the amplifier

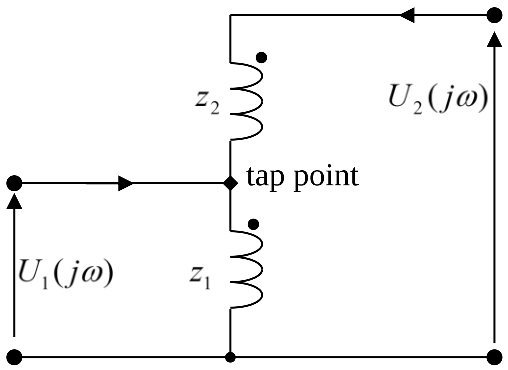
$$Z_1(j\omega) = \frac{1}{n^2} R_l = R_t \Rightarrow n = \sqrt{\frac{R_l}{R_t}} = \frac{1}{3} \tag{4.7.17}$$

what means that 1:3 step-down transformer should be used to impedance match to the given load.

♦

In some practical applications, transformer is replaced by an **autotransformer**. Autotransformer is built of a single winding with the **tap point**. Then, it can be considered as two coupled coils connected in series. Primary circuit is connected between the tap point and one terminal of the winding, secondary circuit is connected to winding terminals, as presented in Fig. 4.7.10a. Same effect can be obtained by series connection of two-winding transformer coils, as presented in Fig. 4.7.10b.

Fig. 4.7.10a Autotransformer circuit



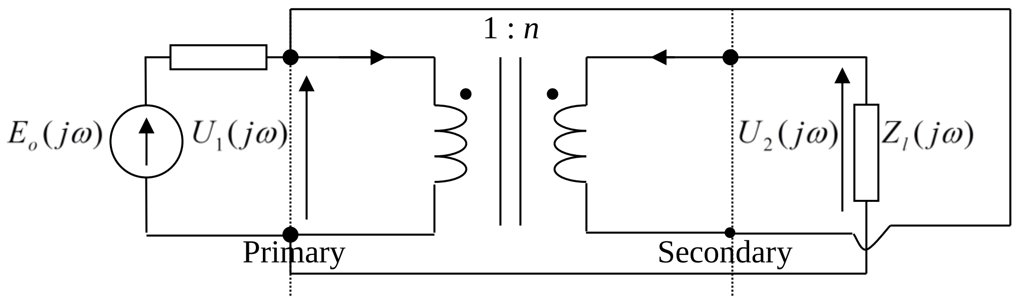


Fig. 4.7.10b Autotransformer obtained from two-winding transformer

An autotransformer KVL equation is

$$U_2(j\omega) = U_1(j\omega) + nU_1(j\omega) = (n+1)U_1(j\omega) \tag{4.7.18}$$

Then, the turns ratio is

$$n_a = \frac{z_1 + z_2}{z_1} = 1 + n \tag{4.7.19}$$

It should be emphasized, that autotransformer couples primary and secondary circuit both electrically and magnetically (no electrical isolation between circuits).

Example 4.6.4

A 2300/230 V two-winding transformer is connected as an autotransformer. Determine the voltage rating.

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$$U_2 = 2300\left(1 + \frac{230}{2300}\right) = 2530\,\mathrm{V}, \;\; n_a = \frac{2530}{2300} = 1 + 0.1 = 1.1 \tag{4.7.20}$$

♦

Example 4.7.4 – cont.

Determine the new secondary voltage and turns ratio after reversing the low-voltage winding.

÷

$$U_2 = 2300\left(1 - \frac{230}{2300}\right) = 2070\,\mathrm{V}, \;\; n_a = \frac{2070}{2300} = 1 - 0.1 = 0.9 \tag{4.7.20a}$$

♦

PRACTICAL IRON-CORE TRANSFORMER

In a practical iron-core transformer all three nonidealities: leakage fluxes, winding losses and nonideality of magnetic material have to be taken into account. One model of a practical transformer has been already discussed. The other, commonly used model consists of an ideal transformer supplemented by elements representing nonidealities. Leakage fluxes and winding losses are taken into account by series connection of the leakage inductance L_{l1} (L_{l2}) and winding resistance R_1 (R_2) in series with the primary (secondary) winding of an ideal transformer. Nonideality of a magnetic material (non-infinite inductances L_1, L_2, M) is taken into account by parallel connection of inductance L_m (magnetization inductance of a core) with the primary winding. The obtained practical transformer circuit is presented in Fig. 4.7.11.

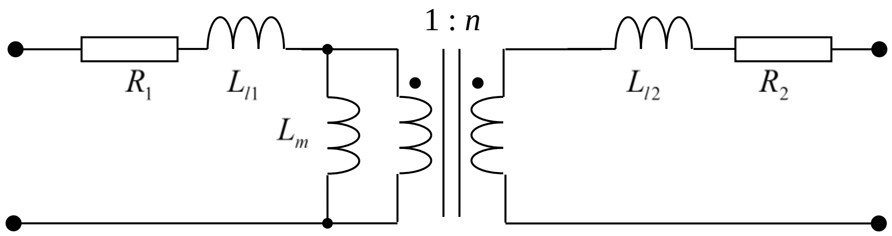


Fig. 4.7.11 Practical transformer circuit for medium frequencies

This circuit is used for medium frequencies. For low frequencies leakage inductances may be disregarded, nonideality of magnetic core is prevailing. Then, magnetization inductance shorts primary winding what makes transformation difficult, even impossible - the circuit presented in Fig. 4.7.11a is obtained.

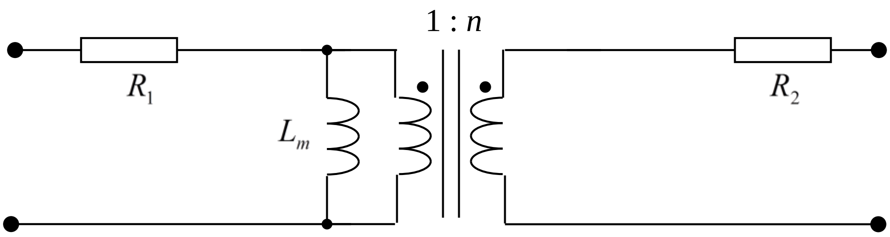


Fig. 4.7.11a Practical transformer circuit for low frequencies

For high frequencies leakage inductances prevail, winding resistances and magnetization may be disregarded. Moreover, shunting capacitances have to be taken into account, as presented in Fig. 4.7.11b.

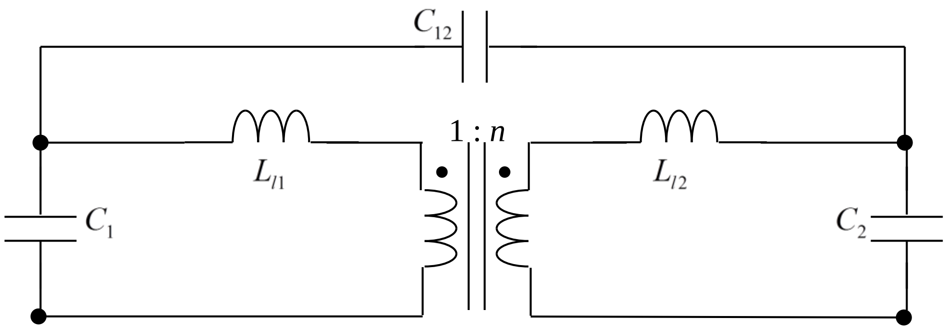


Fig. 4.7.11b Practical transformer circuit for high frequencies

Drill problems 4.7

1. Find equivalent inductance of two ideal coupled coils connected in parallel.
2. A low-frequency amplifier has an output impedance of 5 kΩ. It is to supply a maximum amount of power to an 8 Ω load (speaker). What should be the turns ratio of the matching transformer.

3. For an ideal transformer of $z_1 = 20$, $z_2 = 100$ and $j1000\ \Omega$ load impedance, find u_1 if $i_1 = 50\sqrt{2}\sin 314t$ mA.

4. An ideal transformer of $n = 1$ is loaded by a) 1H inductance, b) $1\ \mu\mathrm{F}$ capacitance. For the measured input voltage $u_1 = 100\sqrt{2}\sin 1000t$ V, find i_1.

5. An ideal transformer of $n = 1$ is loaded by $Z_l(j\omega) = 10 + j10\ \Omega$. For the measured input current: $i_1 = 2\sin 314t$ A, find the real power supplied to the load.

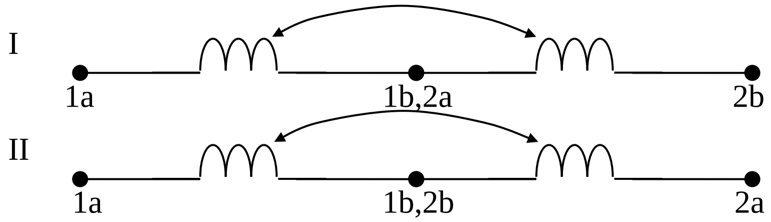
6. The impedance $Z_l(j\omega) = 10 - j10\ \Omega$ loads an ideal isolating transformer (n=1). Find the real power supplied by $U_1 = 6$ V ac primary source.

7. An ideal isolating transformer primary voltage is $10\exp(j0^{\circ})$ V, the secondary load impedance is $Z_l(j\omega) = 12 - j16\ \Omega$. Find the rms primary current.

8. The primary terminals of a basic transformer of $L_1 = L_2 = 1$ H; k=0.5, are connected to a voltage source $u_1 = 10\sqrt{2}\sin 10t$ V. An ideal rms a) ammeter, b) voltmeter is connected to the secondary terminals. Calculate its indication and the reflected inductance.

9. Two tightly coupled (k=1) coils have been connected as shown and the following total inductances have been measured: $L_{\mathrm{I}} = 40$ mH, $L_{\mathrm{II}} = 60$ mH. If 1a is the dotted terminal, which terminal of the second coil is the dotted one ? Calculate the mutual inductance.

Fig. P.4.7.9



4.8 THREE-PHASE CIRCUITS

One very important use of ac steady-state analysis is its application to power systems. Alternating voltage can be stepped up for transmission and stepped down for distribution with transformers, and this subject is not discussed. For reason of economics and performance, almost all electric power systems are three-phase systems. In such system, the source is the three-phase **balanced generator**. Such generator produces a balanced set of voltages, the voltages having the same amplitude and frequency but displaced in phase by 120^{o}. For the conventionally assumed zero initial phase angle of the 1st phase and the **phase sequence**, that is the sequence in which phase voltages reach a positive peak, 123 or ABC, these voltages are

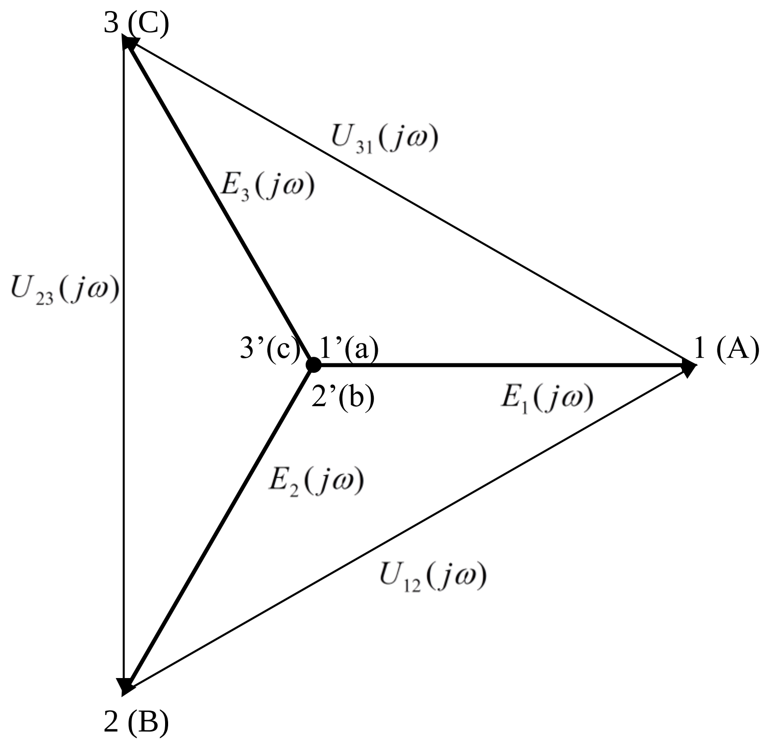
$E_1(j\omega) = E\exp(j0^{\mathrm{o}})$ between terminals 1 and 1' or A and a, (4.8.1)

$E_2(j\omega) = E\exp(j-120^{\mathrm{o}})$ between terminals 2 and 2' or B and b,

$E_3(j\omega) = E\exp(j120^{\mathrm{o}})$ between terminals 3 and 3' or C and c,

as presented in Fig. 4.8.1 phasor diagram.

Fig. 4.8.1 Phasor diagram of three-phase voltages



For the assumed sequence, instead of specifying three balanced sources, it is sufficient to specify one, e.g. $E_1(j\omega)$. Then

$$E_2(j\omega) = E_1(j\omega)\exp(-j120^{\mathrm{o}}),\ E_3(j\omega) = E_1(j\omega)\exp(j120^{\mathrm{o}}) \qquad (4.8.2)$$

The sum of any balanced set of three-phase voltages is always zero

$$\sum_{i=1}^{3} E_i(j\omega) = 0 \qquad (4.8.3)$$

The three-phase generator is equivalent to three single-phase generators and individual generators may be connected to individual loads, to produce three single-phase circuits. However, for reason of economics, individual generators and individual loads are connected into one three-phase circuit, and three methods of connecting three-phase circuit are possible:

1. wye-wye or Y–Y or star-star connection,
2. delta-delta or $\Delta - \Delta$ or mesh-mesh connection,
3. mixed, $\mathrm{Y} - \Delta$ or $\Delta - \mathrm{Y}$ connection.

WYE-WYE SYSTEMS

Consider the three-phase source that has line terminals 1, 2 and 3 and a neutral terminal 0, in which terminals 1’, 2’ and 3’ are connected. In this case the source is said to be wye(Y)-connected or star-connected, as shown in Fig. 4.8.2 for two representations: wye-shape representation and somewhat easier to draw equivalent representation. Line terminals are normally denoted by letters A, B, C and neutral by N. However, for compliance with description of dc multi-terminal circuits, these terminals are denoted by numbers 1, 2, 3 and 0.

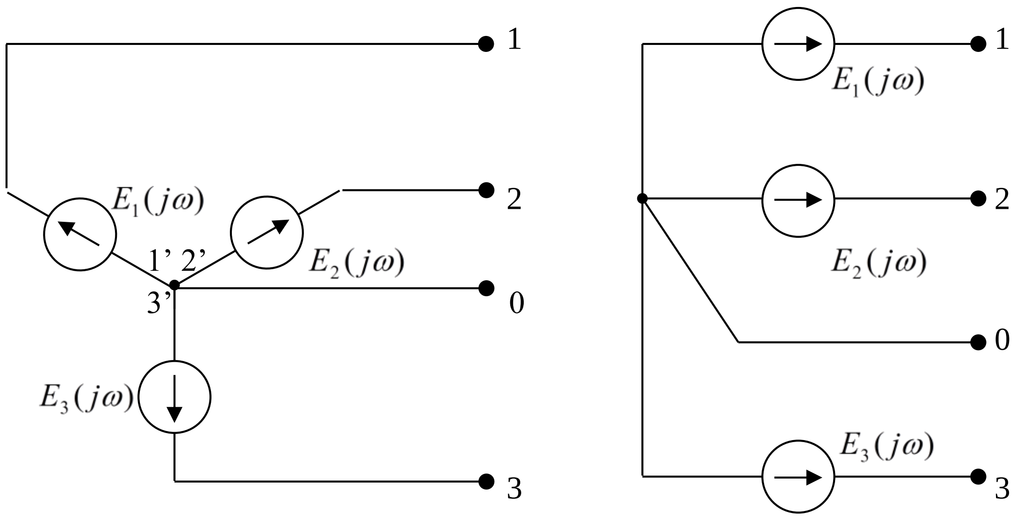


Fig. 4.8.2 Two representations of wye connected three-phase source

Same connection may be applied to a load, and then, three single-phase circuits are connected in a wye-wye three-phase four wire system, as shown in Fig. 4.8.3. For the assumed resistanceless lines, voltages between line terminals and the neutral terminal of three-phase load are source voltages

$$V = E,\ V_i(j\omega) = E_i(j\omega);\ i = 1,2,3 \tag{4.8.4}$$

They are called **phase voltages** and are the same as load voltages of three single-phase circuits. These voltages are denoted in Fig. 4.8.1 phasor diagram (bold), together with line-to-line voltages or simply **line voltages**

$$U_{31}(j\omega) = V_3(j\omega) - V_1(j\omega) = U\exp(j150^\circ) \tag{4.8.5}$$

$$U_{23}(j\omega) = V_2(j\omega) - V_3(j\omega) = U\exp(j270^\circ)$$

$$U_{12}(j\omega) = V_1(j\omega) - V_2(j\omega) = U\exp(j30^\circ)$$

where magnitude of line voltage is:

$$U = \sqrt{3}V = \sqrt{3}E \tag{4.8.6}$$

For a phase rms voltage of 230 V, line rms voltage is equal to ≈ 400 V.

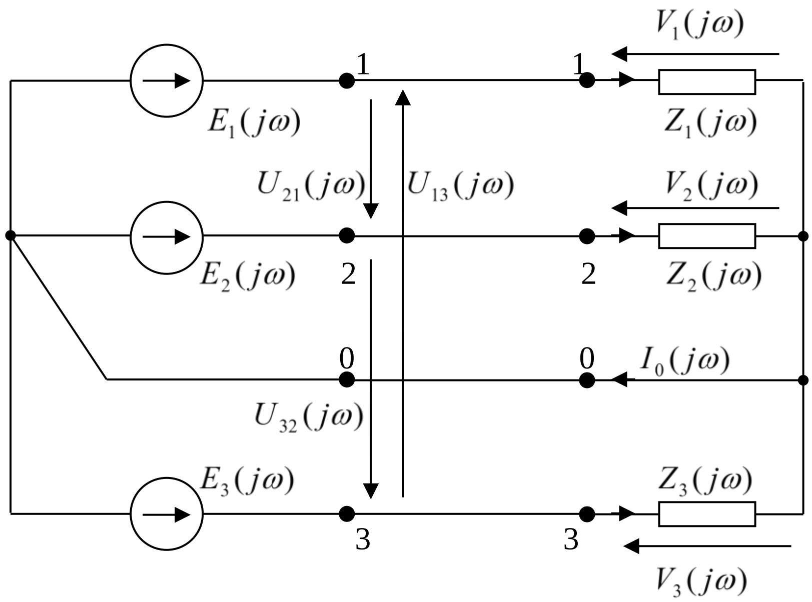


Fig. 4.8.3 Wye-wye three-phase four wire system

Phase currents $I_i(j\omega)$, line currents $J_i(j\omega)$ at the same time, are designated by load impedances of individual phases

$$I_i(j\omega) = J_i(j\omega) = \frac{V_i(j\omega)}{Z_i(j\omega)};\ i = 1,2,3 \tag{4.8.7}$$

The **neutral line current** is

$$I_0(j\omega) = \sum_{i=1}^{3} I_i(j\omega) \tag{4.8.8}$$

For a balanced load:

$$Z_1(j\omega) = Z_2(j\omega) = Z_3(j\omega) = Z(j\omega) \tag{4.8.9}$$

the neutral line current is zero

$$I_0(j\omega) = \frac{1}{Z(j\omega)}\sum_{i=1}^{3} V_i(j\omega) = 0 \tag{4.8.10}$$

and this line can be omitted to form three-phase three-wire system, as shown in Fig. 4.8.4.

In a four-wire unbalanced system, the load neutral point is fixed in potential by connection to the source neutral. If the neutral wire is removed, the load neutral, denoted by 0^*, is no longer fixed but is free to float, its potential is determined by the values of load impedances. For a balanced load, still $V_0(j\omega) = 0$. If for some reasons, such as short-circuit or open-circuit in one phase, three-wire system becomes unbalanced, then significant deviations of phase voltages should be expected, in both magnitude and phase. These cases will be illustrated by the next two examples.

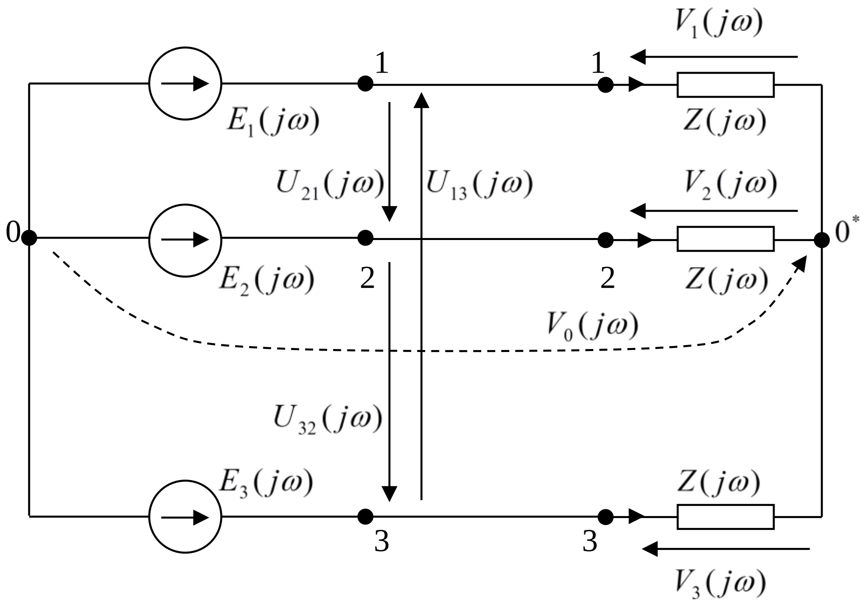


Fig. 4.8.4 Wye-wye three-phase three wire system

Example 4.8.1

Consider a three-wire wye-wye balanced system with an open-circuit in phase 3, as shown in Fig. 4.8.5. Draw the phasor diagram and calculate phase voltages.

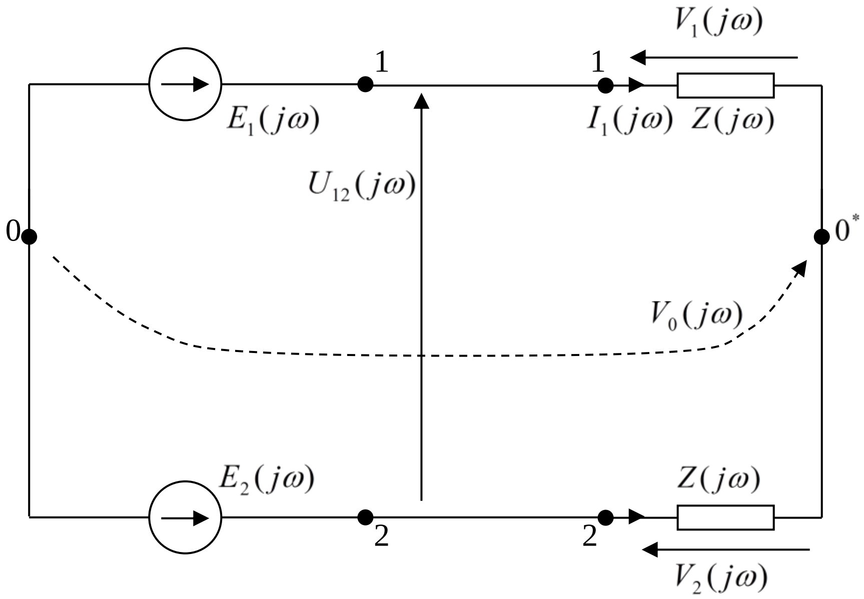


Fig. 4.8.5 Three-wire wye-wye system with an open-circuit in phase 2

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A three-phase circuit degenerates to two-phase (one-loop) circuit. Its mesh current is

$$I_1(j\omega) = -I_2(j\omega) = \frac{U_{12}(j\omega)}{2Z(j\omega)} \tag{4.8.11}$$

The phase voltages are

$$V_1(j\omega) = \frac{U_{12}(j\omega)}{2} = \frac{E\sqrt{3}}{2}\exp(j30^\circ) \tag{4.8.12}$$
$$V_2(j\omega) = -\frac{U_{12}(j\omega)}{2} = \frac{E\sqrt{3}}{2}\exp(j210^\circ)$$

The phasor diagram is shown in Fig. 4.8.6. The effect of an open-circuit in one phase is voltage drop in two other phases, from E to $\approx 0.86E$. For 230 V generator, phase voltage drops to ≈ 199 V.

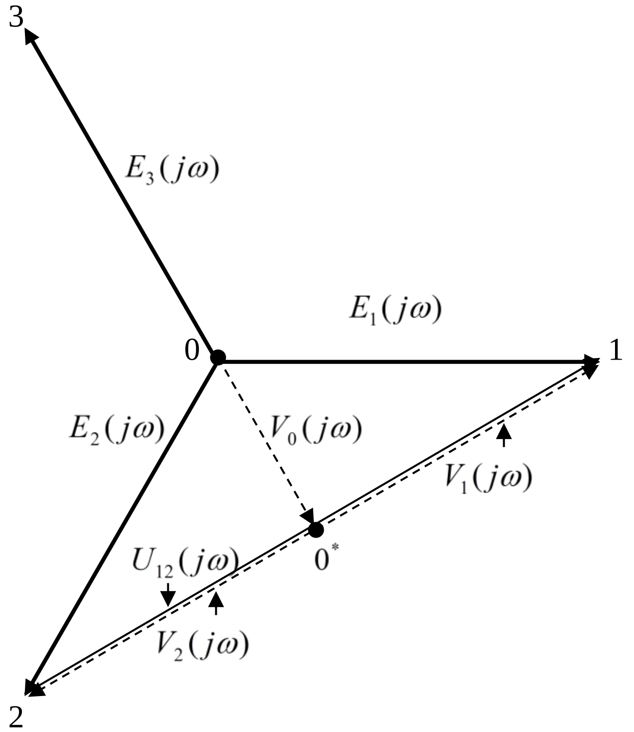


Fig. 4.8.6 Phasor diagram of a three-wire wye-wye system with an open-circuit in phase 2

♦

Example 4.8.2

Consider a three-wire wye-wye balanced system with a short-circuit in phase 3, as shown in Fig. 4.8.7. Draw the phasor diagram and calculate the phase voltages.

÷

A system remains three-phase system, however now voltages of "healthy" phases are line voltages

$$V_1(j\omega) = U_{13}(j\omega) \tag{4.8.13}$$
$$V_2(j\omega) = U_{23}(j\omega)$$

The load neutral 0^* has floated from source neutral 0 to line terminal 2, as presented in Fig. 4.8.8 phasor diagram. The effect of a short-circuit in one phase is voltage jump in other phases by $\sqrt{3}$. For 230 V generator, phase voltage jumps to ≈ 400 V and this may evidently cause damage of load device.

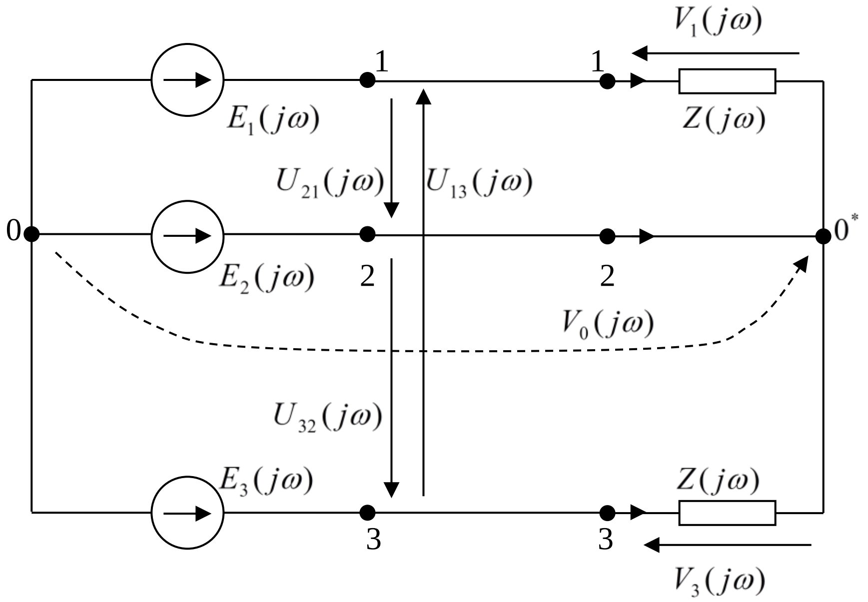


Fig. 4.8.7 Three-wire wye-wye system with a short-circuit in phase 2

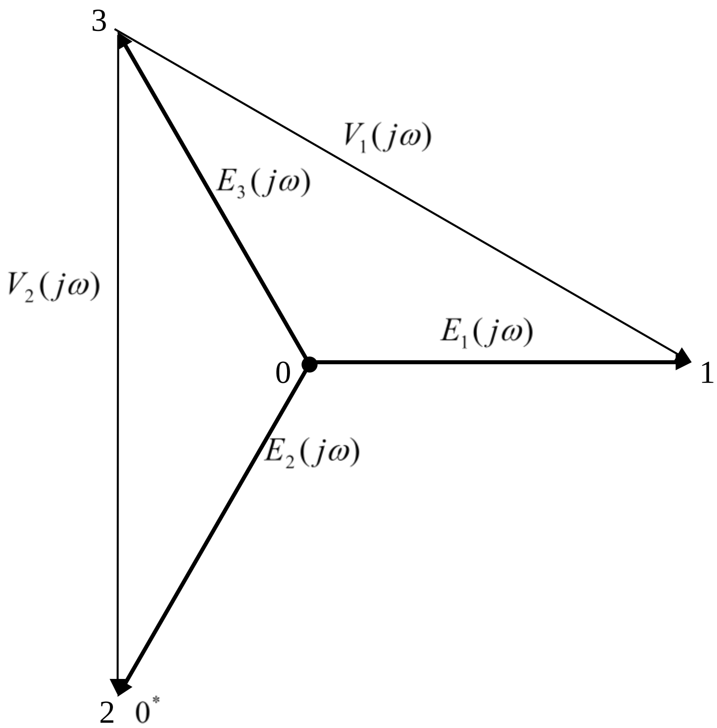


Fig. 4.8.8 Phasor diagram of a three-wire wye-wye system with a short-circuit in phase 2

♦

DELTA-DELTA and WYE-DELTA SYSTEMS

Consider the three-phase source connected as shown in Fig. 4.8.9 for two representations: delta-shape representation and somewhat easier to draw equivalent representation. In this case the source is said to be delta(Δ)-connected or mesh-connected. Same connection may be applied to load impedances, and then, delta-delta three-phase system is obtained, as presented in Fig. 4.8.10.

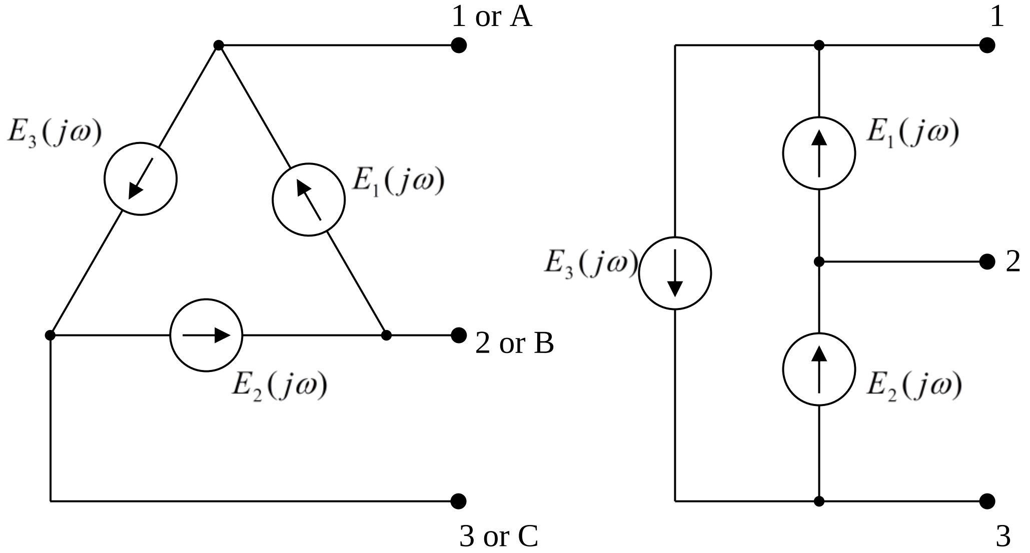


Fig. 4.8.9 Two representations of delta connected three-phase source

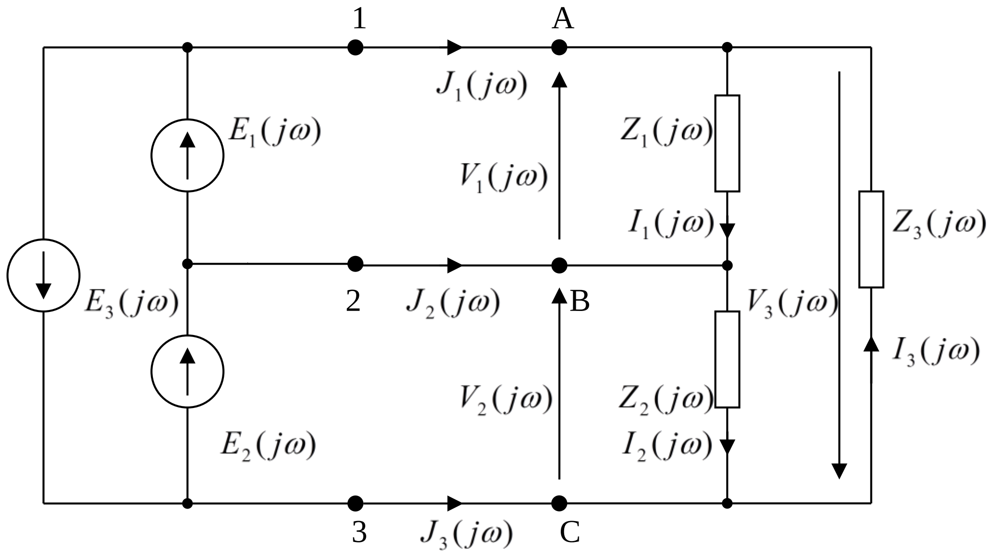


Fig. 4.8.10 Delta-delta three-phase system

Obviously, systems with delta-connected loads are three-wire system, since there is no neutral connection. Phase voltages are at the same time line voltages. Each line current is the difference of two phase currents (4.8.14) and clearly, for the balanced load magnitude of line current is related to that of phase current by equation (4.8.15).

$$J_1(j\omega) = I_1(j\omega) - I_3(j\omega) \tag{4.8.14}$$
$$J_2(j\omega) = I_2(j\omega) - I_1(j\omega)$$
$$J_3(j\omega) = I_3(j\omega) - I_2(j\omega)$$

$$J = \sqrt{3}I \tag{4.8.15}$$

Sources are rarely delta-connected, however wye-delta connections are frequently used. In case of such connection, phase voltages are voltages between three-phase source terminals (4.8.5), they are $\sqrt{3}$ times higher than those of the wye connection of a load.

COMBINATIONAL SYSTEMS

In case many loads are connected to three-phase lines, some of them may be wye-connected, some others delta-connected. Then, the combinational system is created. An exemplary combinational system is presented in Fig. 4.8.11.

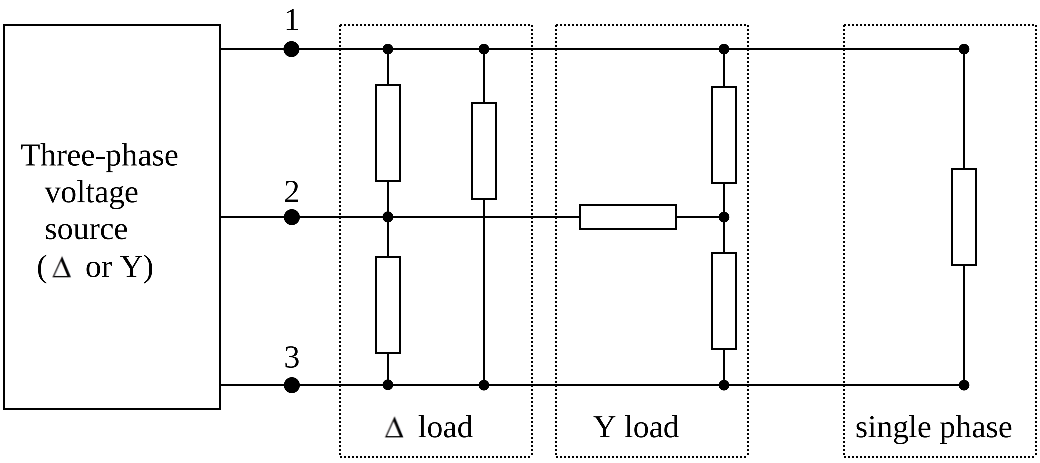


Fig. 4.8.11 Exemplary combinational three-phase system

POWER IN THREE-PHASE SYSTEMS

The total real power P transferred in the three-phase n-wire system, $n = 3$ or 4, can be measured by $n-1$ wattmeters or calculated, through calculation of $n-1$ pairs $J_i(j\omega)$, $U_i(j\omega)$, where $J_i(j\omega)$ are line currents and $U_i(j\omega)$ are line voltages. For $n = 4$ and neutral as the reference, line voltages are phase voltages: $U_i(j\omega) = V_i(j\omega);\ i = 1,2,3$. For $n = 3$, one line, say $C = 3$, is taken as the reference, and then line voltages are: $U_i(j\omega) = U_{i3}(j\omega);\ i = 1,2$. From the power balance, the total power is equal to the sum of powers supplied to individual phases

$$P = P_1 + P_2 + P_3 \tag{4.8.16}$$
$$P_i = V_i I_i \cos\varphi_i;\ i = 1,2,3$$

For a balanced system, both three-wire and four-wire, $P_1 = P_2 = P_3$ and the total power is:

$$P = 3VI\cos\varphi$$

Then, in 4-wire balanced system the total power may be measured by one wattmeter. Same strategy may be applied in 3-wire balanced system after creation of the **artificial neutral point**, as presented in Fig. 4.8.12, where $R >> Z(\omega)$.

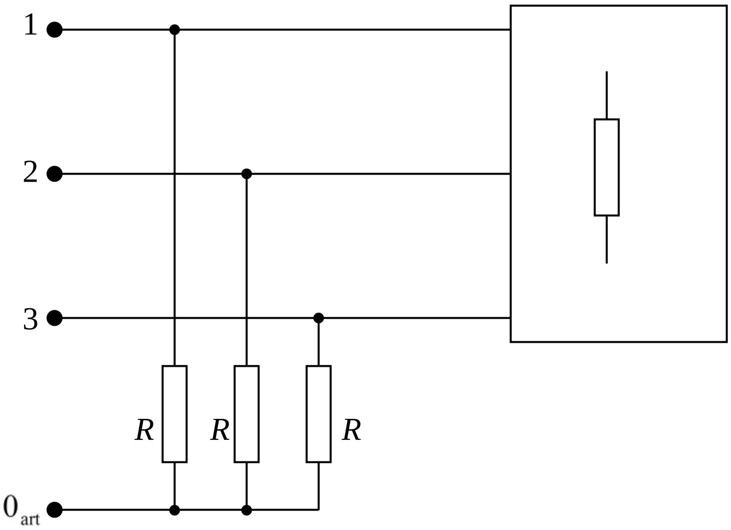


Fig. 4.8.12 Three-wire system with artificial neutral point

The following benefits of three-phase power systems can be enlisted.

1. Savings in copper, four or three wires instead of six wires.
2. Availability of two different voltages, line and phase voltage, in case of four-wire (wye-wye) system.
3. Availability of the rotating field, which can be used to energize electric motors.

Drill problems 4.8

1. Sketch the phasor diagram for a three-phase 3-wire unbalanced $\mathrm{Y}-\mathrm{Y}$ system: $Z_1(j\omega) = \infty$, $Z_2(j\omega) = Z_3(j\omega) = 100 + j100\ \Omega$.
2. The total power of a three-phase balanced wye load is 6 kW. What is the power factor of each phase load, if the line voltage is 400 V and the line current is 10 A?
3. A balanced delta load with phase impedance of 10exp(j30) is connected to 230 V lines. Determine the total real power supplied.
4. A balanced wye load with phase impedance of 100 Ω is connected to 230 V lines. Determine the total real power delivered to the load.
5. For the balanced wye-delta system: $E = 100\,\mathrm{V}$, $Z(j\omega) = 100\,\Omega$, find rms value of a line current.

6. A 230 V/100 W heater is connected to line terminals of 3x230 V generator. What energy is supplied in 1 hour?

7. For the three-phase 4-wire balanced system: $E = 100$ V, $Z(j\omega) = 100\,\Omega$, find the total power supplied after opening phase 1, draw the phasor diagram.

8. For the three-phase 3-wire balanced system in $\mathrm{Y}-\mathrm{Y}$ configuration: $E = 100$ V, $Z(j\omega) = 100\,\Omega$, find the total power supplied after shorting phase 1, draw the phasor diagram.

9. Find the total power supplied: $E = 100\,\mathrm{V}$, $R = 100\,\Omega$.

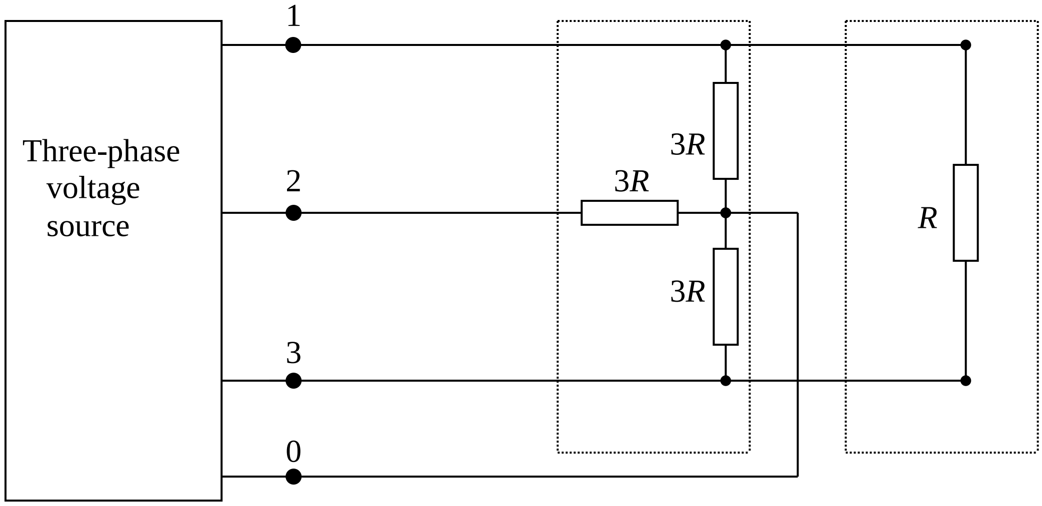


Fig. P.4.8.9

10. Three adjacent houses take electricity from the three successive phases of a 230 V supply. Find the resultant neutral current when they consume, in order of phase sequence, 1 A at unity power factor, 2 A at 0.75 lagging and 2A at leading. What is the total power absorbed ?

5 TRANSMISSION LINE

5.1 INTRODUCTION

Transmission line commonly used to carry electrical energy and/or information over a distance is considered. It is assumed that a transmission line is any arrangement of two continuous conductors having the required length *l* and a uniform cross-section. In general, transmission line connects two circuits built of lumped components, each of which is usually designated to have one parameter dominant: inductance, capacitance, resistance or conductance which are **lumped constants** (parameters). Transmission system built of input circuit, the transmitter or source, transmission line and output circuit, the receiver or load, is shown in Fig. 5.1.1.

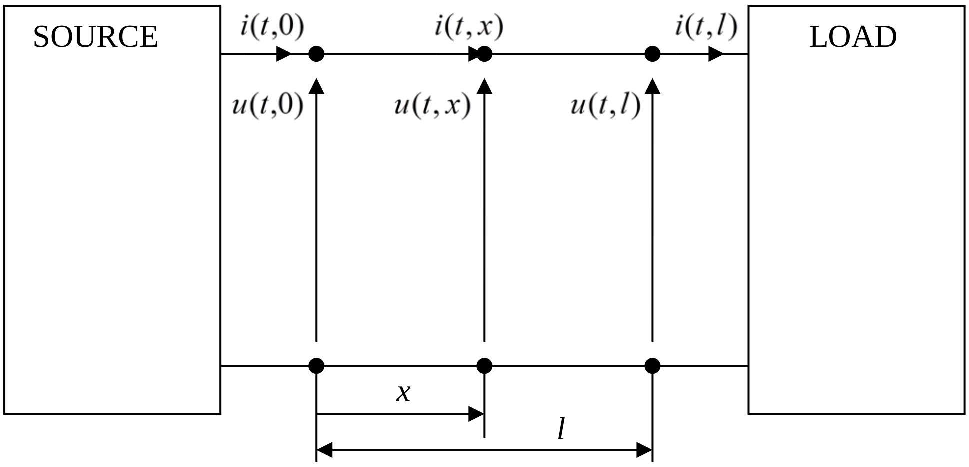


Fig. 5.1.1 Transmission system

Like any device that carries current and sustains a voltage, the transmission line must inevitably have some inductance, capacitance, resistance and conductance. The values of all four parameters (constants) increase with the line length. Thus, an axially uniform transmission line is characterized by its length *l* and **distributed constants** (parameters). They are called **per unit length parameters** or **primary parameters**:

- inductance per unit length $L_{ul} = L$ [H/m], (5.1.1)
- capacitance per unit length $C_{ul} = C$ [F/m],
- resistance per unit length $R_{ul} = R$ [Ω/m],
- conductance per unit length $G_{ul} = G$ [S/m].

Inductance and resistance must act like series elements, since they each are additive when in series, while capacitance and conductance, which are each additive when in parallel, are shunt elements. Transmission line of any length can therefore be represented by dividing the line into small elements of length $\Delta x \to 0$, each of which is a two-port section having infinitesimally small components, as shown in Fig. 5.1.2.

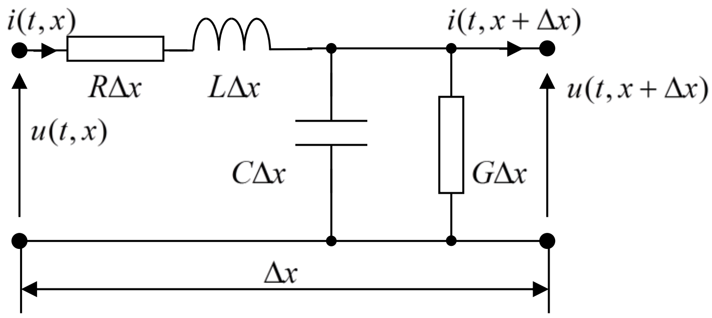


Fig. 5.1.2 An elementary section of transmission line

Then, it is obvious that current or voltage at any distance x is function of two arguments, time and distance:

$$\begin{aligned} i(t,x) &= i_x(t) \\ u(t,x) &= u_x(t) \end{aligned} \tag{5.1.2}$$

Applying KVL to Fig. 5.1.2 two-port

$$u(t,x) - R\Delta x i(t,x) - L\Delta x \frac{\partial i(t,x)}{\partial t} - u(t,x+\Delta x) = 0 \tag{5.1.3a}$$

which leads to

$$-\frac{u(t,x+\Delta x) - u(t,x)}{\Delta x} = -\frac{\Delta u(t,x)}{\Delta x} = Ri(t,x) + L\frac{\partial i(t,x)}{\partial t}\,. \tag{5.1.3b}$$

On the limit as $\Delta x \to 0$, equation (5.1.3b) becomes

$$-\frac{\partial u(t,x)}{\partial x} = Ri(t,x) + L\frac{\partial i(t,x)}{\partial t} \tag{5.1.3}$$

Similarly, applying KCL, equation (5.1.4) is obtained

$$-\frac{\partial i(t,x)}{\partial x} = Gu(t,x) + C\frac{\partial u(t,x)}{\partial t} \tag{5.1.4}$$

Differential equations (5.1.3) and (5.1.4) are the general **transmission line equations**, so called **telegraphist's equations**. Solution of these equations, designation of the current and/or voltage (5.1.2), for the given source and load, is the task of transmission line analysis. Two types of analyses are discussed: transient analysis for aperiodic input signal and ac-steady state analysis.

5.2 TRANSIENT ANALYSIS

Differential equations (5.1.3) and (5.1.4) can be solved using Laplace transformation. Zero initial conditions are assumed

$$i(0,x) = i_x(0) = 0 \qquad (5.2.1)$$

$$u(0,x) = u_x(0) = 0$$

Transmission system described in the *s*-domain is presented in Fig. 5.2.1.

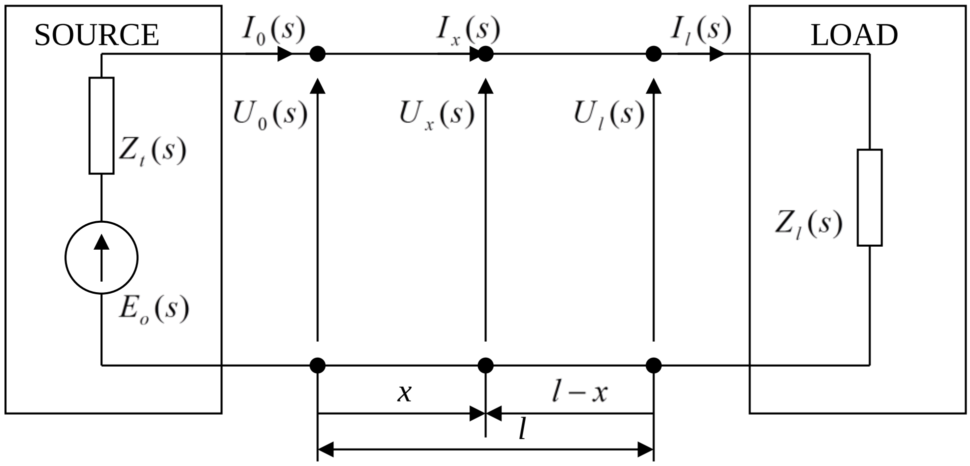


Fig. 5.2.1 Transmission system described in s-domain

Skipping the mathematics, the following solution in *s*-domain is obtained:

$$U_x(s) = U_0(s)\cosh[\gamma(s)x] - I_0(s)Z(s)\sinh[\gamma(s)x] \qquad (5.2.2a)$$

$$I_x(s) = I_0(s)\cosh[\gamma(s)x] - \frac{U_0(s)}{Z(s)}\sinh[\gamma(s)x]$$

$$U_x(s) = U_l(s)\cosh[\gamma(s)(l-x)] + I_l(s)Z(s)\sinh[\gamma(s)(l-x)] \qquad (5.2.2b)$$

$$I_x(s) = I_l(s)\cosh[\gamma(s)(l-x)] + \frac{U_l(s)}{Z(s)}\sinh[\gamma(s)(l-x)]$$

where

$$Z(s) = \sqrt{\frac{R+sL}{G+sC}} \qquad (5.2.3)$$

$$\gamma(s) = \sqrt{(R+sL)(G+sC)} \qquad (5.2.4)$$

are the line **secondary parameters**, so called the **characteristic impedance** and the **propagation constant**.

That way, the voltage and the current at a distance *x* are expressed in terms of the input voltage and current (5.2.2a) or the output voltage and current (5.2.2b).

The line input and output equations are

$$U_0(s) = E_o(s) - I_0(s)Z_t(s) \tag{5.2.5}$$
$$U_l(s) = I_l(s)Z_l(s)$$

From equations (5.2.2) and (5.2.5), equations that express the voltage and the current at a distance *x* by line secondary parameters and parameters of input and output circuits are

$$U_x(s) = E_o(s)\frac{Z(s)}{Z(s)+Z_t(s)}\frac{\exp[-\gamma(s)x] - N(s)\exp[-\gamma(s)(2l-x)]}{1-M(s)N(s)\exp[-2\gamma(s)l]} \tag{5.2.6}$$

$$I_x(s) = E_o(s)\frac{1}{Z(s)+Z_t(s)}\frac{\exp[-\gamma(s)x] + N(s)\exp[-\gamma(s)(2l-x)]}{1-M(s)N(s)\exp[-2\gamma(s)l]}$$

where

$$M(s) = \frac{Z(s)-Z_t(s)}{Z(s)+Z_t(s)} \tag{5.2.7}$$

$$N(s) = \frac{Z(s)-Z_l(s)}{Z(s)+Z_l(s)}$$

are **reflection coefficients**, for the line input and output, respectively.

Applying the following series expansion

$$\frac{1}{1-A\exp(B)} = \sum_{k=0}^{\infty} A^k \exp(kB) \tag{5.2.8}$$

the final voltage and current waveforms in the *s*-domain are sum of traveling waves

$$U_x(s) = U_x^f(s) + U_x^b(s) \tag{5.2.9}$$

$$I_x(s) = I_x^f(s) + I_x^b(s)$$

where

$$U_x^f(s) = E(s)\sum_{k=0}^{\infty}\big(M(s)N(s)\big)^k \exp[-\gamma(s)(2kl+x)] \tag{5.2.9a}$$

$$I_x^f(s) = J(s)\sum_{k=0}^{\infty}\big(M(s)N(s)\big)^k \exp[-\gamma(s)(2kl+x)]$$

are **forward traveling waves**, for $k=0$ the first incident wave, for $k=1,2,...$ waves reflected from the line input, and

$$U_x^b(s) = -E(s)N(s)\sum_{k=0}^{\infty}\big(M(s)N(s)\big)^k \exp\big(-\gamma(s)[2(k+1)-x]\big) \tag{5.2.9b}$$

$$I_x^b(s) = J(s)N(s)\sum_{k=0}^{\infty}\big(M(s)N(s)\big)^k \exp\big(-\gamma(s)[2(k+1)-x]\big)$$

are **backward traveling waves**, waves reflected from the line output.

The introduced equivalent voltage $E(s)$ and current $J(s)$ are

$$E(s) = E_o(s)\frac{Z(s)}{Z(s)+Z_t(s)} \tag{5.2.9c}$$

$$J(s) = \frac{E(s)}{Z(s)} = \frac{E_o(s)}{Z(s)+Z_t(s)}$$

The inverse transformation of waveforms expressed by equations (5.2.9) is the next step. The general expressions are relatively complicated. The following limiting cases have special significance, are of practical meaning.

1. Distortionless line

A distortionless line satisfies the following condition:

$$\frac{R}{L} = \frac{G}{C} \tag{5.2.10}$$

If the condition is satisfied, then expressions for both $\gamma(s)$ and $Z(s)$ simplify.

Characteristic impedance

$$Z(s) = \sqrt{\frac{L}{C}} = \rho = const \tag{5.2.11}$$

Propagation constant

$$\gamma(s) = \alpha + s/v \tag{5.2.12}$$

where

$$\alpha = \sqrt{RG} \tag{5.2.12a}$$

is the **attenuation constant** in [1/m]

$$v = \frac{1}{\sqrt{LC}} \tag{5.2.12b}$$

is the **propagation velocity**, in [m/s].

2. Lossless line

A lossless line is the special case of a distortionless line and it satisfies the following condition:

$$R = 0,\ G = 0 \tag{5.2.13}$$

Thus, exept for a vanishing attenuation constant, $\alpha = 0$, the characteristics of a lossless line are the same as those of a distortionless line.

For a distortionless line, equations (5.2.9) simplify.

$$U_x(s) = E(s)\exp(-\alpha x)\exp(-s\frac{x}{v}) - E(s)N(s)\exp\big(-\alpha(2l-x)\big)\exp\left(-s\frac{2l-x}{v}\right) +$$
$$+E(s)M(s)N(s)\exp\big(-\alpha(2l+x)\big) - E(s)M(s)N(s)^2\exp\big(-\alpha(4l-x)\big)\exp\left(-s\frac{4l-x}{v}\right) + \cdots \tag{5.2.14}$$

$$I_x(s) = \frac{1}{\rho}E(s)\exp(-\alpha x)\exp(-s\frac{x}{v}) + \frac{1}{\rho}E(s)N(s)\exp\big(-\alpha(2l-x)\big)\exp\left(-s\frac{2l-x}{v}\right) +$$
$$+\frac{1}{\rho}E(s)M(s)N(s)\exp\big(-\alpha(2l+x)\big) + \frac{1}{\rho}E(s)M(s)N(s)^2\exp\big(-\alpha(4l-x)\big)\exp\left(-s\frac{4l-x}{v}\right) + \cdots$$

For lines with resistive source and termination: $Z_t(s) = R_t$, $Z_l(S) = R_l$, reflection coefficients are real numbers $M(s) = M,\ N(s) = N$, and then, after inverse transformation, the following waveforms in the time-domain are obtained

$$u_x(t) = e\left(t-\frac{x}{v}\right)\exp(-\alpha x) - e\left(t-\frac{2l-x}{v}\right)N\exp\big(-\alpha(2l-x)\big) +$$
$$+e\left(t-\frac{2l+x}{v}\right)MN\exp\big(-\alpha(2l+x)\big) - e\left(t-\frac{4l-x}{v}\right)MN^2\exp\big(-\alpha(4l-x)\big) + \cdots \tag{5.2.14a}$$

$$i_x(t) = \frac{1}{\rho}e\left(t-\frac{x}{v}\right)\exp(-\alpha x) + \frac{1}{\rho}e\left(t-\frac{2l-x}{v}\right)N\exp\big(-\alpha(2l-x)\big) +$$
$$+\frac{1}{\rho}e\left(t-\frac{2l+x}{v}\right)MN\exp\big(-\alpha(2l+x)\big) + \frac{1}{\rho}e\left(t-\frac{4l-x}{v}\right)MN^2\exp\big(-\alpha(4l-x)\big) + \cdots$$

where: $e(t) = \boldsymbol{L}^{-1}\{E(s)\}\mathbf{1}(t) = u_0(0)$.

In case of step excitation $e_o(t) = E_o\mathbf{1}(t)$ and $e(t) = E\,\mathbf{1}(t)$; $E = E_o\dfrac{\rho}{\rho+R_t}$

Then, the waveforms become

$$u_x(t) = E\,\mathbf{1}\left(t-\frac{x}{v}\right)\exp(-\alpha x) - E\,\mathbf{1}\left(t-\frac{2l-x}{v}\right)N\exp\big(-\alpha(2l-x)\big) + \tag{5.2.14b}$$

$$+E\,\mathbf{1}\left(t-\frac{2l+x}{v}\right)MN\exp\big(-\alpha(2l+x)\big) - E\,\mathbf{1}\left(t-\frac{4l-x}{v}\right)MN^2\exp\big(-\alpha(4l-x)\big) + \cdots$$

$$i_x(t) = \frac{1}{\rho}E\,\mathbf{1}\left(t-\frac{x}{v}\right)\exp(-\alpha x) + \frac{1}{\rho}E\,\mathbf{1}\left(t-\frac{2l-x}{v}\right)N\exp\big(-\alpha(2l-x)\big) +$$
$$+\frac{1}{\rho}E\,\mathbf{1}\left(t-\frac{2l+x}{v}\right)MN\exp\big(-\alpha(2l+x)\big) + \frac{1}{\rho}E\,\mathbf{1}\left(t-\frac{4l-x}{v}\right)MN^2\exp\big(-\alpha(4l-x)\big) + \cdots$$

For a lossless line, attenuation constant vanishes and all exponentials are equal to 1. Then, the steady-state values are

$$u_x(\infty) = U_\infty = \frac{R_l}{R_l + R_t} E_o \,,\; i_x(\infty) = I_\infty = \frac{E_o}{R_l + R_t} \tag{5.2.15}$$

Graphical description of traveling waves

For a lossless line, the algebraic description of waveforms (5.2.14) can be represented graphically, as shown in Fig. 5.2.2.

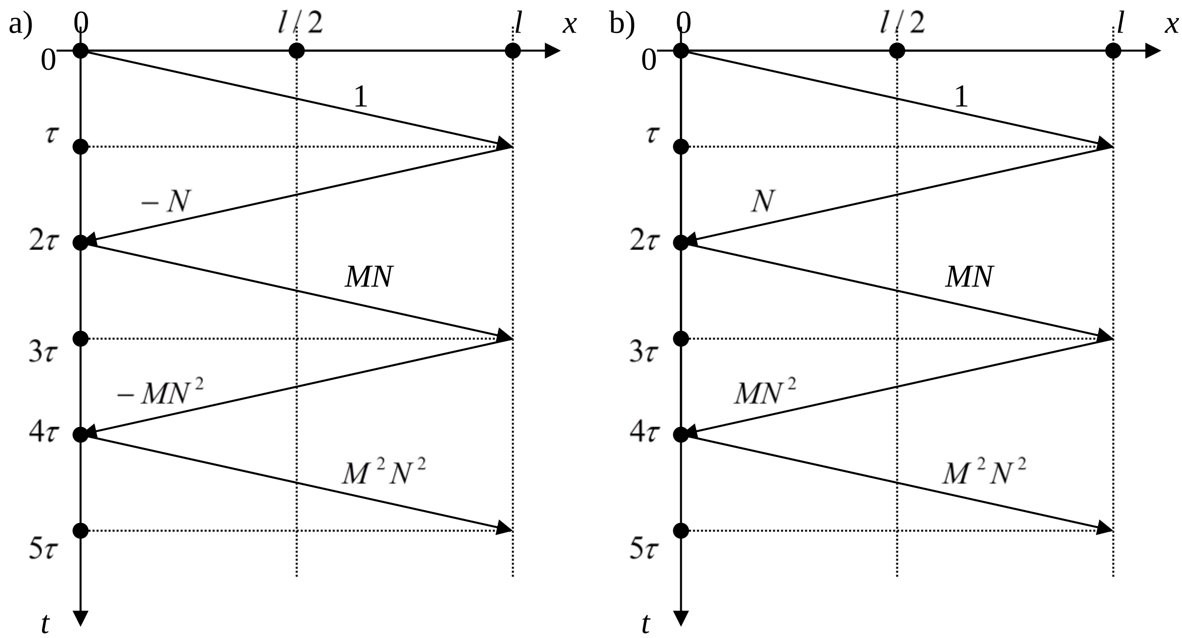


Fig. 5.2.2 Graphical representation of traveling waves, a) voltage, b) current

This representation is very useful in finding gain and delay time of consecutive traveling waves

Example 5.2.1

Find input, half-length and output voltage waveforms of the **matched load line**, after switching on the following source: $E_o = 10$ V, $R_t = 100\,\Omega$. Line length is $l = 10$ m and its secondary parameters are: $\rho = 50\,\Omega$, $\alpha = 10^{-1}$ 1/m, $v = 10^5$ m/s.

÷

For the matched load line

$$R_l = \rho \Rightarrow N = 0 \tag{5.2.16}$$

and there are no reflected waves. The only present 1st incident wave is

$$u_x(t) = E\exp(-\alpha x)\mathbf{1}(t - x/v) \tag{5.2.17}$$

Voltage waveforms at $x = 0,\ l/2,\ l$ are

$$u_0(t) = E\mathbf{1}(t), \tag{5.2.18}$$
$$u_{l/2}(t) = E\exp(-\alpha\tau/2)\mathbf{1}(t-\tau/2)$$
$$u_l(t) = E\exp(-\alpha\tau)\mathbf{1}(t-\tau)$$

where

$$\tau = l/v = l\sqrt{LC} \tag{5.2.19}$$

is time of propagation from the line input to the output or vice-versa. For the assumed source and line parameters: $\tau = 0.1$ ms, $E = E_o/3$. The waveforms (5.2.18) are shown in Fig. 5.2.3, together with waveforms for the lossless line, denoted dashed.

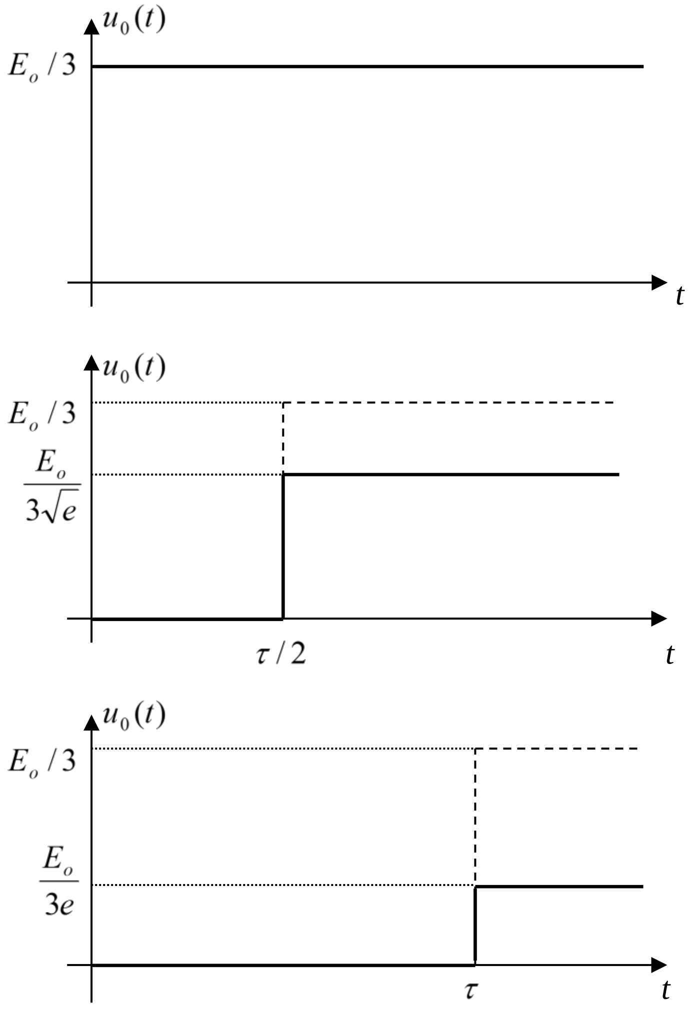


Fig. 5.2.3 Voltage waveforms for Example 5.2.1

♦

Example 5.2.2

Find input, half-length and output voltage waveforms of the lossless **matched generator line**, after switching on the following source: $E_o = 10$ V, $R_t = \rho = 50\,\Omega$. Line length is $l = 10$ m, propagation velocity is $v = 10^5$ m/s and line is: a) open-circuited, b) short-circuited.

÷

For the matched generator line

$$R_t = \rho \Rightarrow M = 0\,,\ E = E_0 / 2 \tag{5.2.20}$$

and there are only two traveling waves: the 1st forward (incident) wave and the 1st backward (reflected) wave, as expressed graphically in Fig. 5.2.4, for both loads.

$$R_l = \infty \Rightarrow N = -1 \tag{5.2.21a}$$

$$R_l = 0 \Rightarrow N = 1 \tag{5.2.21b}$$

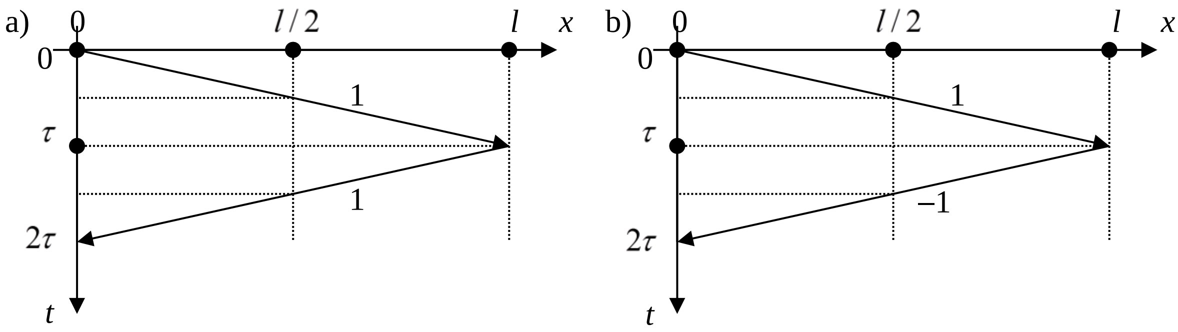


Fig. 5.2.4 Voltage traveling waves for matched generator line, a) opened, b) shorted

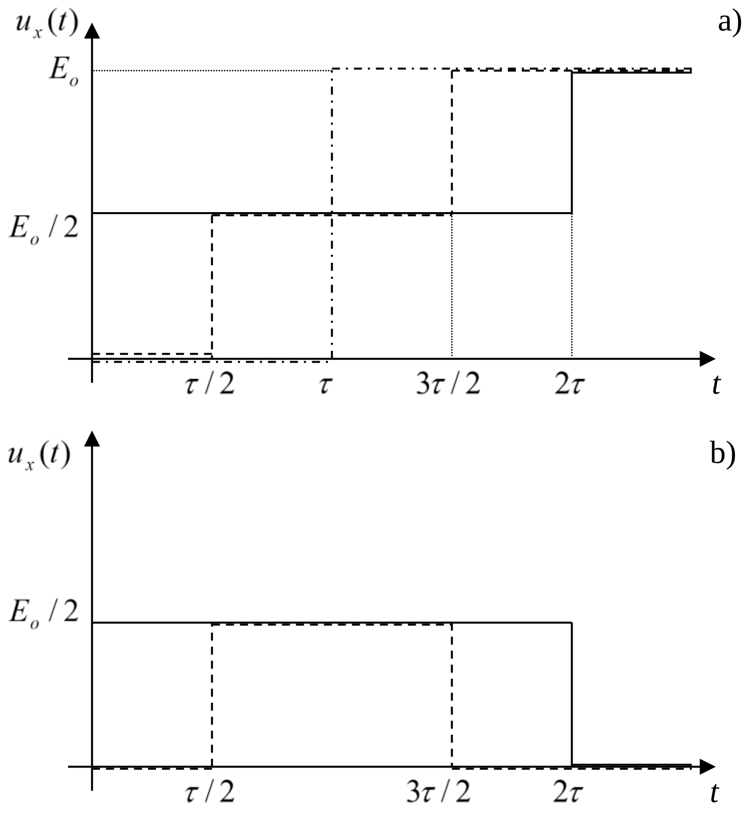


Fig. 5.2.5 Voltage waveforms for matched generator line, a) opened, b) shorted

a) For the opened line, voltage waveforms at $x = 0,\ l/2,\ l$ are

$$u_0(t) = E_0/2 \cdot \mathbf{1}(t) + E_0/2 \cdot \mathbf{1}(t-2\tau)\ , \qquad (5.2.22a)$$
$$u_{l/2}(t) = E_0/2 \cdot \mathbf{1}(t-\tau/2) + E_0/2 \cdot \mathbf{1}(t-3\tau/2)$$
$$u_l(t) = E_0\mathbf{1}(t-\tau)$$

b) For the shorted line, voltage waveforms at $x = 0,\ l/2,\ l$ are

$$u_0(t) = E_0/2 \cdot \mathbf{1}(t) - E_0/2 \cdot \mathbf{1}(t-2\tau)\ , \qquad (5.2.22b)$$
$$u_{l/2}(t) = E_0/2 \cdot \mathbf{1}(t-\tau/2) - E_0/2 \cdot \mathbf{1}(t-3\tau/2)$$
$$u_l(t) = 0$$

The waveforms (5.2.22a) and (5.2.22b) are shown in Fig. 5.2.5a and b, $x = 0$ - solid, $x = l/2$ - dashed, $x = l$ - dot and dash.

♦

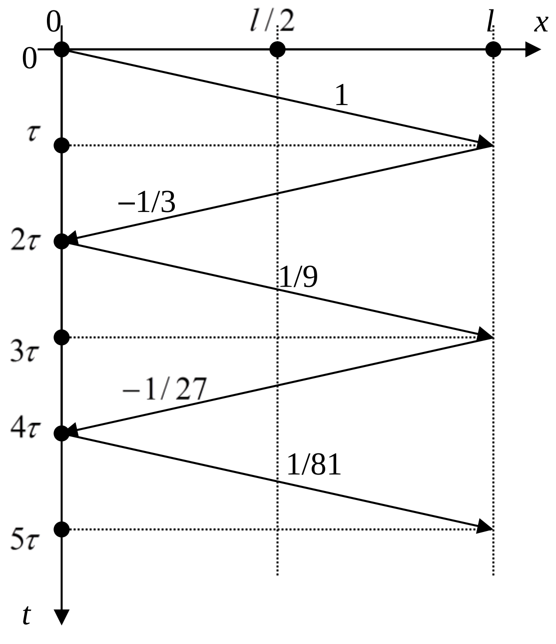
Example 5.2.3

Find input, half-length and output voltage waveforms of the lossless line, after switching on the following source: $E_o = 30\,\text{V}$, $R_t = \rho/2 = 25\,\Omega$. Line length is $l = 10$ m, propagation velocity is $v = 10^5$ m/s and line load is $R_l = \rho/2 = 25\,\Omega$.

÷

For the given source and load resistances: $E = 2E_o/3 = 20\,\text{V}$, reflection coefficients are $M = N = 1/3$. Graphical representation of voltage traveling waves is shown in Fig. 5.2.6.

Fig. 5.2.6 Graphical representation of voltage traveling waves for Example 5.2.3



Voltage waveforms at $x = 0,\ l/2,\ l$ are

(5.2.23)

$$u_0(t) = E \cdot \mathbf{1}(t) + E(-\frac{1}{3}+\frac{1}{9}) \cdot \mathbf{1}(t-2\tau) + E(-\frac{1}{27}+\frac{1}{81}) \cdot \mathbf{1}(t-4\tau) + \cdots =$$
$$= \frac{2}{3}E_o \cdot \mathbf{1}(t) - \frac{4}{27}E_o \cdot \mathbf{1}(t-2\tau) - \frac{4}{243}E_o \cdot \mathbf{1}(t-4\tau) - \cdots$$

$$u_{l/2}(t) = \frac{2}{3} E_o \cdot \mathbf{1}(t - \frac{\tau}{2}) - \frac{2}{9} E_o \cdot \mathbf{1}(t - \frac{3}{2}\tau) + \frac{2}{27} E_o \cdot \mathbf{1}(t - \frac{5}{2}\tau) - \frac{2}{81} E_o \cdot \mathbf{1}(t - \frac{7}{2}\tau) + \cdots$$

$$u_l(t) = E(1 - \frac{1}{3}) \cdot \mathbf{1}(t - \tau) + E(\frac{1}{9} - \frac{1}{27}) \cdot \mathbf{1}(t - 3\tau) + \cdots = \frac{4}{9} E_o \cdot \mathbf{1}(t - \tau) + \frac{4}{81} E_o \cdot \mathbf{1}(t - 3\tau) + \cdots$$

and they are depicted in Fig. 5.2.7

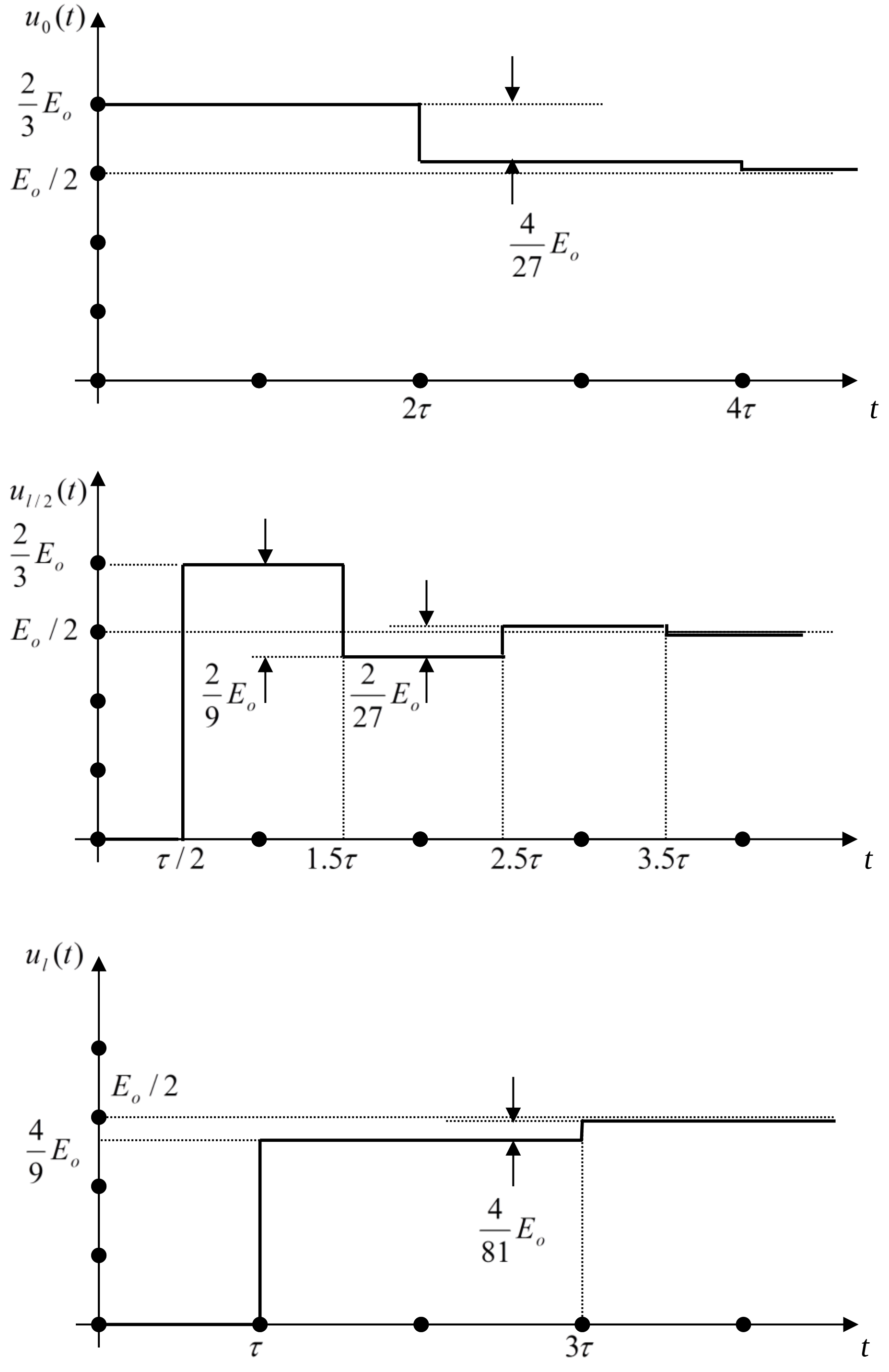


Fig. 5.2.7 Voltage waveforms for Example 5.2.3

♦

Example 5.2.4

A pulse generator is the lossless line source, a pulse counter is the load. Calculate and sketch the output voltage. Define the counter threshold necessary for its proper operation. It is assumed that the subsequent pulse is generated after reaching the steady-state of the previous one. The generator parameters are: pulse magnitude $E_o = 8.1$ V, width $\tau_w = 2$ ms, resistance $R_t = 500\,\Omega$. The counter internal resistance is $R_l = 500\,\Omega$. Line parameters are: $l = 400$ m, $L = 0.01$ H/m, $C = 0.01\,\mu\text{F/m}$.

÷

The line secondary parameters are:

$$\tau = 4\,\text{ms} = 2\tau_w\,,\ \rho = 1000\,\Omega = 2R_t = 2R_l\,.$$

Then, the reflection coefficients are:

$$M = N = 1/3$$

The generator open-circuit voltage is:

$$e_o(t) = E_o\mathbf{1}(t) - E_o\mathbf{1}(t-\tau_w)$$

and

$$e(t) = \frac{2}{3}e_o(t)\,.$$

Graphical representation of traveling waves is identical as in Example 5.2.3, as presented in Fig. 5.2.3. Then, the output voltage is

$$\begin{aligned} u_l(t) &= \frac{4}{9}E_o\big(\mathbf{1}(t-\tau) - \mathbf{1}(t-\tau-\tau_w)\big) + \frac{4}{81}E_o\big(\mathbf{1}(t-3\tau) - \mathbf{1}(t-3\tau-\tau_w)\big) + \cdots = \\ &= 3.6\big(\mathbf{1}(t-\tau) - \mathbf{1}(t-\tau-\tau_w)\big) + 0.4\big(\mathbf{1}(t-3\tau) - \mathbf{1}(t-3\tau-\tau_w)\big) + \cdots \text{ V} \end{aligned} \tag{5.2.24}$$

as depicted in Fig. 5.2.8 – solid.

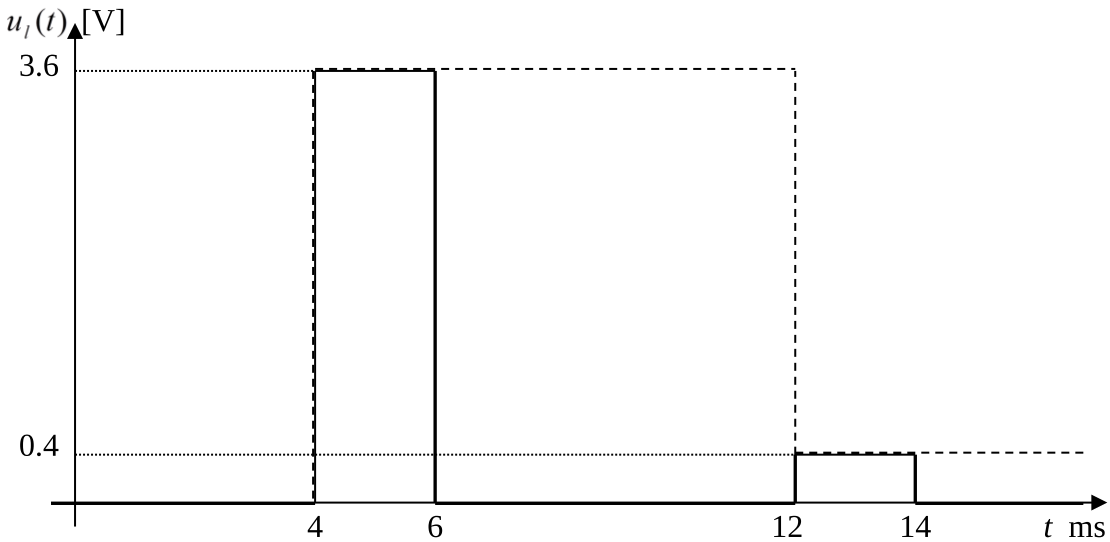


Fig. 5.2.8 Output voltage waveform for Example 5.2.4

Thus, the counter threshold should fall within a range

$$0.4 < U_{\text{thr}} \le 3.6\ \text{V} \tag{5.2.25}$$

Next, consider that pulse width is equal two line propagation times, $\tau_w = 2\tau = 8$ ms. The obtained output waveform is depicted in Fig. 5.2.8 - dashed. As can be observed, gap between the original pulse and the 1st reflected has vanished, and therefore no lower boundary of the threshold is necessary for proper counting of pulses.

♦

Arbitrary termination results in appearance of the reflected waves added to the original incident wave. Therefore, a signal that reaches output device may be significantly distorted, what may cause its malfunctioning. The following general conclusion can be drawn.

For an arbitrary termination of transmission line, the effect of reflected waves can be disregarded if time parameter(s) of the transmitted signal is(are) much greater than the line propagation time τ .

In Example 5.2.4, the effect of reflected waves may be disregarded if $\tau_w \geq 2\tau$. The next example discusses transmission of the practical step for different values of its time parameter, namely the rise time τ_r .

Example 5.2.5

Sketch the output voltage waveform of the lossless line characterized by its length l, propagation time τ and characteristic resistance ρ, for the practical step input given by its magnitude E_o, rise time τ_r and resistance $R_t = 0$, and for the load resistance $R_l = \rho/3$. Assume three different values of the rise time: a) $\tau_r = 0.1\tau$, b) $\tau_r = 2\tau$, c) $\tau_r = 10\tau$.

÷

For the given resistances, the reflection coefficients are

$$M = 1,\ N = 1/2 \tag{5.2.26}$$

Graphical representation of voltage traveling waves is shown in Fig. 5.2.9.

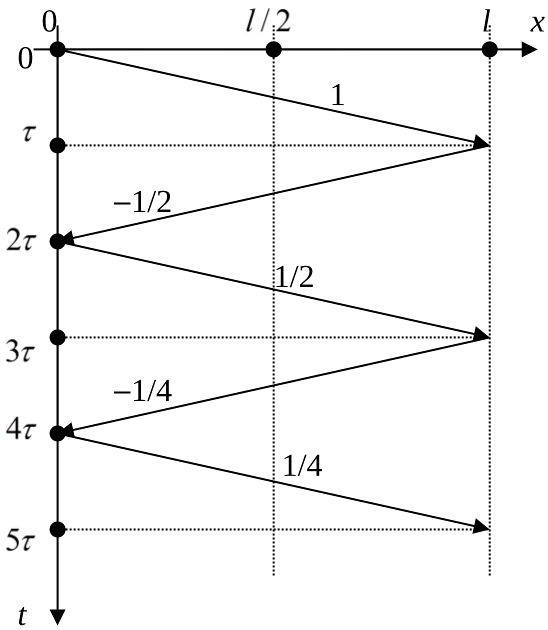


Fig. 5.2.9 Graphical representation of voltage traveling waves for Example 5.2.5

The equivalent voltage is

$$e(t) = e_o(t) = \frac{E_o}{\tau_r} t \cdot \mathbf{1}(t) - \frac{E_o}{\tau_r}(t-\tau_r) \cdot \mathbf{1}(t-\tau_r) \tag{5.2.27}$$

The output waveforms are presented in Fig. 5.2.10 a, b and c - the matched load output is denoted dashed. To compare all three cases, 87.5% of the magnitude has been taken as the reference. For the matched load, this reference level is obviously reached after $t = \tau + 0.875\tau_r$.

a)

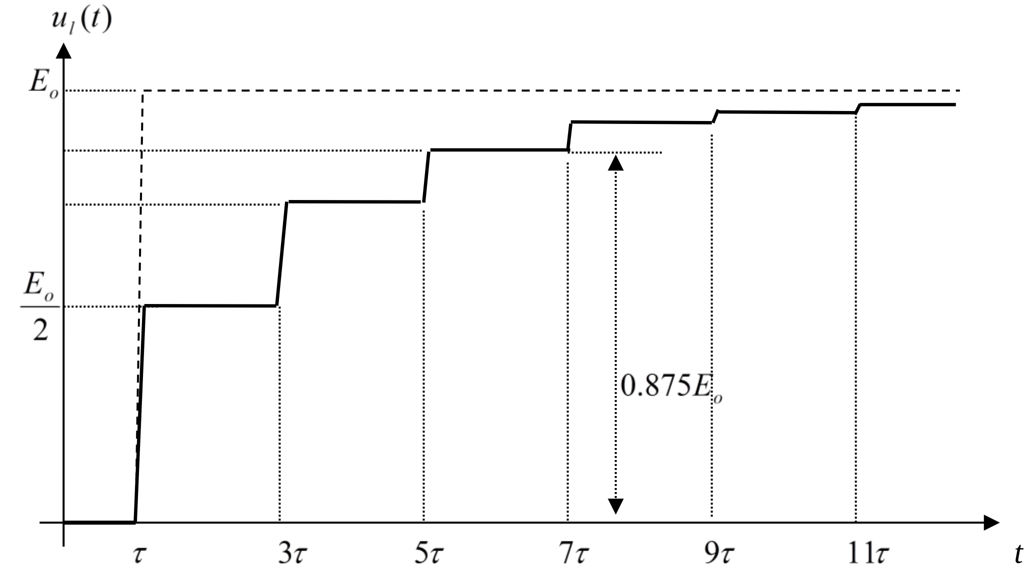


Fig. 5.2.10a Example 5.2.5 output waveform for $\tau_r = 0.1\tau$

1. Rise is of a step character.
2. 87.5% of the magnitude is reached after $t = \tau + (4\tau + \tau_r) = \tau + 41\tau_r$.

b)

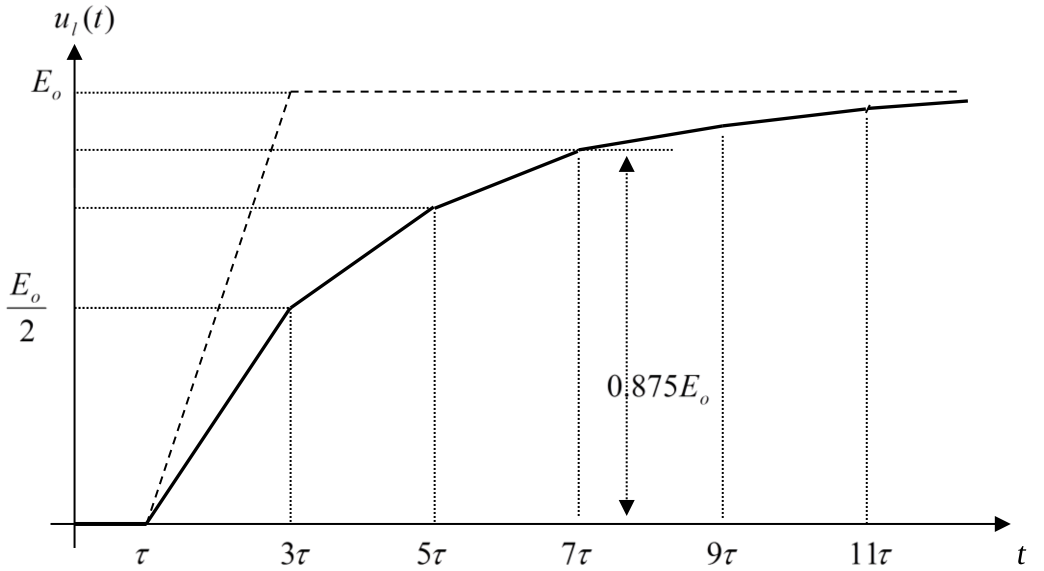


Fig. 5.2.10b Example 5.2.5 output waveform for $\tau_r = 2\tau$

1. Rise is of a PWL character. Steps disappear, rising of one wave ends exactly when the next wave appears.
2. 87.5% of the magnitude is reached after $t = \tau + 6\tau = \tau + 3\tau_r$.

c)

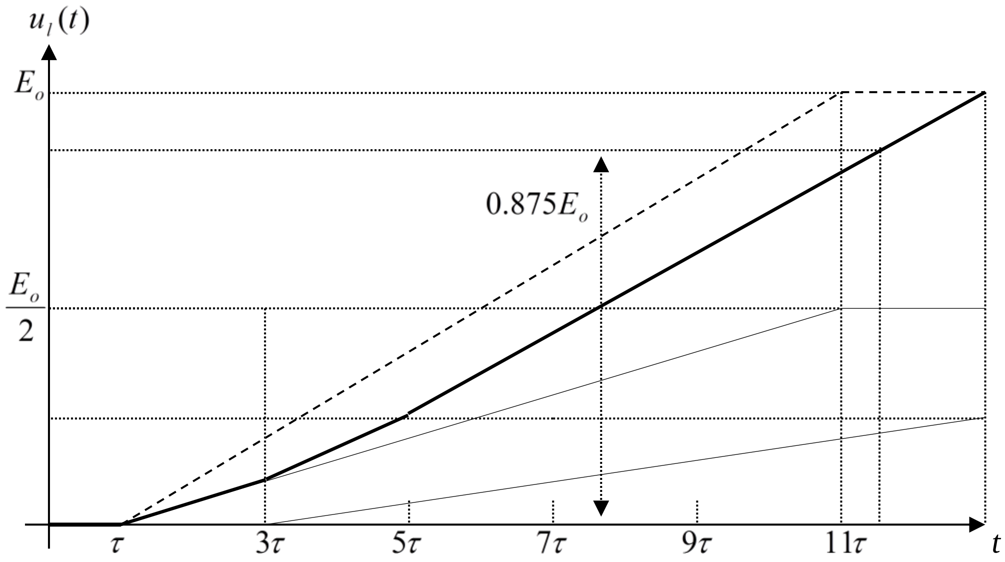


Fig. 5.2.10c Example 5.2.5 output waveform for $\tau_r = 10\tau$

1. Rise is of a PWL character, practically straight line after $t = 3\tau$. At the end of rising of the matched load output, for $t = 11\tau$, five rising waves add up - first two of them are denoted by thin lines.
2. 87.5% of the magnitude is reached after $t \cong \tau + 10.5\tau = \tau + 1.05\tau_r$, the effect of reflected waves is practically unnoticeable.

♦

The reactive load case will be illustrated by the next example.

<u>Example 5.2.6</u>

Find input and output voltage waveforms of the lossless **matched generator line**, after switching on the following source: $E_o = 10$ V, $R_t = \rho$. Line primary parameters are: $L = 2.5$ mH/m, $C = 10$ nF/m, its length is $l = 100$ m and line has capacitive load of $C_l = 0.5\mu\text{F}$.

÷

The line reflection coefficients are:

$$M = 0,\ N(s) = \frac{\rho - \frac{1}{sC_l}}{\rho + \frac{1}{sC_l}} = \frac{sT - 1}{sT + 1};\ T = \rho C_l \tag{5.2.28}$$

The equivalent voltage is

$$E(s) = \frac{E_o}{2}\frac{1}{s} \tag{5.2.29}$$

Then, from equation (5.2.14), setting $\alpha = 0$

$$U_x(s) = \frac{E_o}{2s}\exp\left(-s\frac{x}{v}\right) - \frac{E_o}{2s}\frac{sT+1-2}{sT+1}\exp\left(-s\frac{2l-x}{v}\right) = \tag{5.2.30}$$

$$= \frac{E_o}{2s}\exp\left(-s\frac{x}{v}\right) - \frac{E_o}{2s}\exp\left(-s\frac{2l-x}{v}\right) + \frac{E_o}{s(1+sT)}\exp\left(-s\frac{2l-x}{v}\right)$$

After inverse transformation

$$u_x(t) = \frac{E_o}{2}\mathbf{1}\left(t-\frac{x}{v}\right) - \frac{E_o}{2}\mathbf{1}\left(t-\frac{2l-x}{v}\right) + E_o\left\{1-\exp\left[-\left(t-\frac{2l-x}{v}\right)/T\right]\right\}\mathbf{1}\left(t-\frac{2l-x}{v}\right) \tag{5.2.31}$$

at $x = 0$

$$u_0(t) = \frac{E_o}{2}\mathbf{1}(t) - \frac{E_o}{2}\mathbf{1}(t-2\tau) + E_o\left[1-\exp\left(-\frac{t-2\tau}{T}\right)\right]\mathbf{1}(t-2\tau) \tag{5.2.31a}$$

at $x = l$

$$u_l(t) = E_o\left[1-\exp\left(-\frac{t-\tau}{T}\right)\right]\mathbf{1}(t-\tau) \tag{5.2.31b}$$

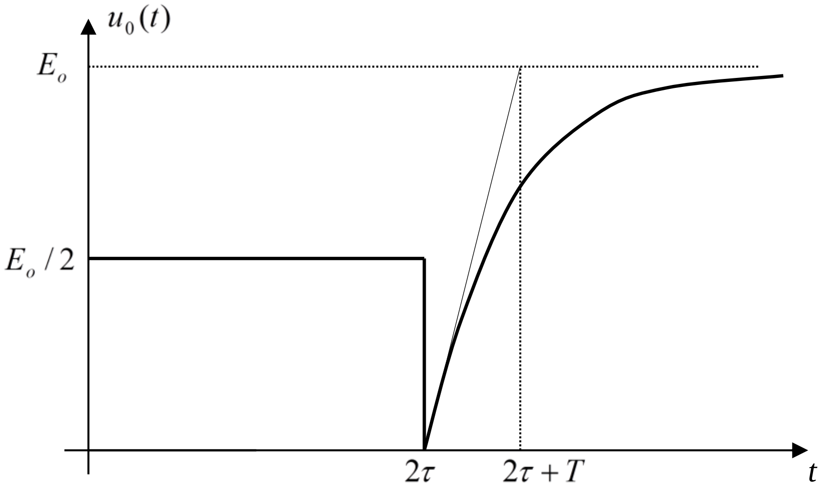


Fig. 5.2.11a Input waveform for Example 5.2.6

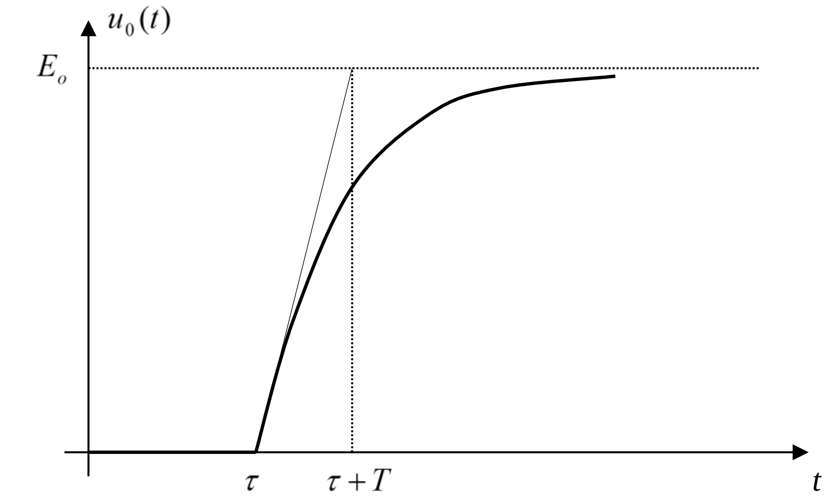


Fig. 5.2.11b Output waveform for Example 5.2.6

For the assumed line and load:

$\rho = 500\,\Omega,\ v = 0.2\cdot 10^3\ \mathrm{m/s},\ \tau = 0.5\,\mathrm{ms},\ T = 0.25\,\mathrm{ms} = \tau/2$

the waveforms are shown in Fig. 5.2.11a and b.

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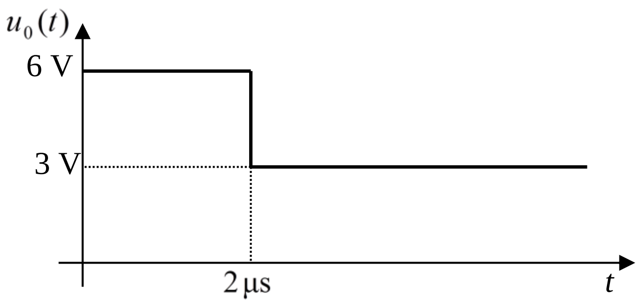
Drill problems 5.2

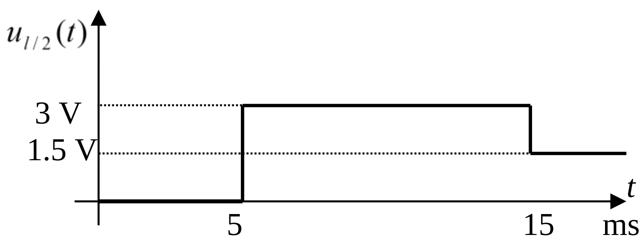
1. Example 5.2.6 line has inductive load of $L = 125$ mH. Sketch the input and output voltage and current waveforms.
2. Outline voltages $u_x(t);\ x = 0,\ l/2,\ l$ of the opened lossless line after inputting a practical source given by the Norton equivalent: $J_s = 1$ mA, $G_t = 1/\rho = 10^{-3}\,\mathrm{S}$. Other line parameters are: $l = 10\,\mathrm{m},\ C = 1\,\mu\mathrm{F/m}$.
3. Outline voltages $u_x(t);\ x = 0,\ l/2,\ l$ of Problem 5.2.2 line after changing its termination to a short-circuit.
4. A lossless matched generator line of $R_t = \rho$, has a resistive load of $R_l = 3\rho = 150\,\Omega$. Sketch the input and output voltages and currents after inputting the step voltage $e_o(t) = 10\cdot\mathbf{1}(t)\,\mathrm{V}$. Other line parameters are: $l = 10\,\mathrm{m},\ C = 1\,\mu\mathrm{F/m}$.
5. Sketch the input and output voltage and current waveforms after connecting an ideal voltage source $e_o(t) = 10\cdot\mathbf{1}(t)\,\mathrm{V}$ to an open-circuited line. Its parameters are: $l = 10\,\mathrm{m}$, $C = 1\,\mu\mathrm{F/m},\ L = 1\,\mathrm{mH/m}$.
6. Sketch the input and output voltage and current waveforms after connecting an ideal current source $j_s(t) = 10\cdot\mathbf{1}(t)\,\mathrm{mA}$ to a short-circuited line of Problem 5.2.5.
7. For the given input voltage waveform find source and load parameters: E_o, R_t, R_l. The line characteristic resistance is $\rho = 75\,\Omega$.

Fig. P.5.2.7



8. For the matched generator lossless line: $E_0 = 6\,\mathrm{V},\ R_t = \rho = 50\,\Omega$, and the given voltage waveform at $x = l/2$, choose the true relationship: a) $R_l > \rho$, b) $R_l < \rho/2$, c) $R_l = \rho/2$, d) $\rho/2 < R_l < \rho$.

Fig. P.5.2.8



5.3 AC ANALYSIS – STANDING WAVES

A transmission line connected to the sinusoidal source and described in the phasor-domain is shown in Fig. 5.3.1.

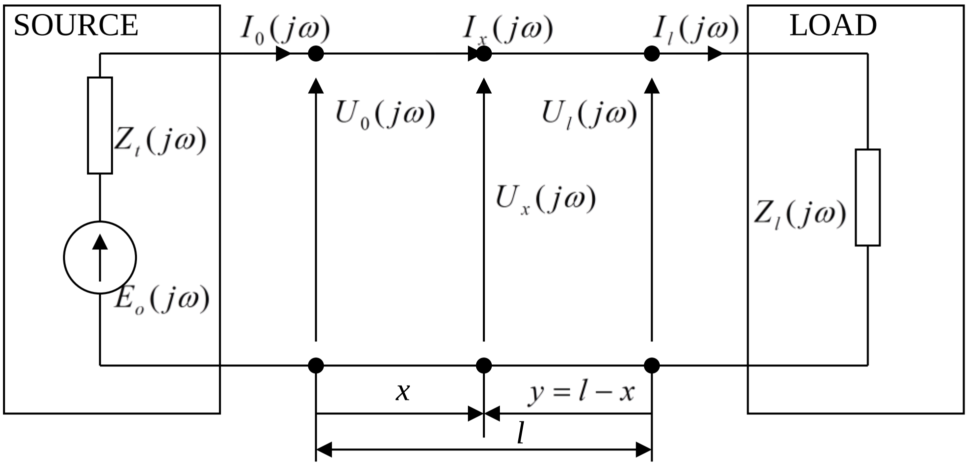


Fig. 5.3.1 Transmission system described in phasor-domain

For the given length l and per-unit-length parameters R, G, L, C, the line secondary parameters can be designated from the s-domain equations, substituting $s = j\omega$.

Characteristic impedance

$$Z(j\omega) = \sqrt{\frac{R + j\omega L}{G + j\omega C}} \tag{5.3.1}$$

For distortionless (lossless) line:

$$Z(j\omega) = \sqrt{\frac{L}{C}} = \rho = const \tag{5.3.1a}$$

Propagation constant

$$\gamma(j\omega) = \sqrt{(R + j\omega L)(G + j\omega C)} \tag{5.3.2}$$

For distortionless line

$$\gamma(j\omega) = \alpha + j\beta \tag{5.3.2a}$$

where

$$\alpha = \sqrt{RG} \tag{5.3.3}$$

is the attenuation constant in [Np/m], for a lossless line it is equal zero,

$$\beta = \omega\sqrt{LC} = \frac{\omega}{v} \tag{5.3.4}$$

is the **phase shift per distance**, a linear function of ω, and

$$v = \frac{1}{\sqrt{LC}} \tag{5.3.5}$$

is the propagation (phase) velocity, in [m/s].

Equations describing line voltage and current in the phasor-domain are obtained from equations (5.2.6) with *s* replaced by $j\omega$.

$$U_x(j\omega) = E_o(j\omega)\frac{Z(j\omega)}{Z(j\omega)+Z_t(j\omega)}\frac{\exp[-\gamma(j\omega)x] - N(j\omega)\exp[-\gamma(j\omega)(2l-x)]}{1-M(j\omega)N(j\omega)\exp[-2\gamma(j\omega)l]} \tag{5.3.6}$$

$$I_x(j\omega) = E_o(j\omega)\frac{1}{Z(j\omega)+Z_t(j\omega)}\frac{\exp[-\gamma(j\omega)x] + N(j\omega)\exp[-\gamma(j\omega)(2l-x)]}{1-M(j\omega)N(j\omega)\exp[-2\gamma(j\omega)l]}$$

where $M(j\omega), N(j\omega)$ are the reflection coefficients, described by equations (5.2.7) with *s* replaced by $j\omega$. For the considered distortionless line, the propagation constant is described by (5.3.2a). The matched load line is discussed at first, then line with arbitrary resistive termination. The equivalent voltage $E(j\omega)$ can be introduced (5.2.9c) and, for simplicity of description, it is assumed that

$$E(j\omega) = E_o(j\omega)\frac{\rho}{\rho + Z_t(j\omega)} = E \tag{5.3.7}$$

MATCHED LOAD LINE

For the matched load line, reflection coefficient is $N(j\omega) = 0$ and $E = U_0$. Then the phasor line voltage and current are

$$U_x(j\omega) = U_0\exp[-\gamma(j\omega)x] = U_0\exp(-\alpha\, x)\exp(-j\beta\, x) \tag{5.3.8}$$

$$I_x(j\omega) = \frac{U_x(j\omega)}{Z(j\omega)}$$

The line voltage in the time-domain is:

$$u_x(t) = U_0\sqrt{2}\exp(-\alpha\, x)\sin(\omega t - \beta\, x) \tag{5.3.8a}$$

For a lossless line:

$$u_x(t) = U_0\sqrt{2}\sin(\omega t - \alpha\, x) \tag{5.3.8b}$$

At a distance *x*, the voltage magnitude and phase shift are:

$$U_x\sqrt{2} = U_0\sqrt{2}\exp(-\alpha\, x) \tag{5.3.8c}$$

$$\beta\, x = \omega\sqrt{LC}\, x \tag{5.3.8d}$$

Attenuation at a distance *x* is designated by the product

$$\alpha\, x = \ln\frac{U_0}{U_x}, \tag{5.3.9}$$

Its unit is **neper** [Np]. Distribution of the voltage magnitude along a line is presented in Fig. 5.3.2, for both lossy line (continuous) and lossless line (dashed). These waves are called the **standing waves**.

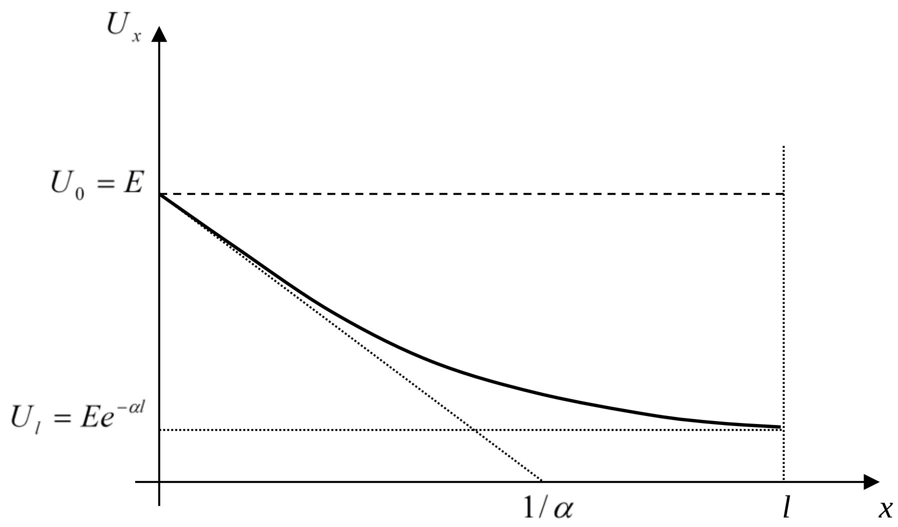


Fig. 5.3.2 Voltage standing waves for matched load lossy and lossless (dashed) line

ARBITRARY TERMINATION

To study an arbitrary resistive load termination, instead of phasor equations (5.3.6), it is more convenient to use equations that express line voltage and current by its output voltage and current, and these equations are obtained from equations (5.2.2b) with s replaced by $j\omega$.

$$U_x(j\omega) = U_l(j\omega)\cosh[\gamma(j\omega)(l-x)] + I_l(j\omega)Z(j\omega)\sinh[\gamma(j\omega)(l-x)] \quad (5.3.10a)$$

$$I_x(j\omega) = I_l(j\omega)\cosh[\gamma(j\omega)(l-x)] + \frac{U_l(j\omega)}{Z(j\omega)}\sinh[\gamma(j\omega)(l-x)]$$

For a lossless line, $Z(j\omega) = \rho,\ \gamma(j\omega) = j\beta$. Then, with $y = l - x$, taking into account $\cosh(j\beta y) = \cos(\beta y),\ \sinh(j\beta y) = j\sin(\beta y)$, the following equations describe the line phasor voltage and current:

$$U_y(j\omega) = U_l(j\omega)\cos(\beta y) + jI_l(j\omega)\rho\sin(\beta y) \quad (5.3.10b)$$

$$I_y(j\omega) = I_l(j\omega)\cos(\beta y) + j\frac{U_l(j\omega)}{\rho}\sin(\beta y)$$

Next, taking into account the load equation

$$U_l(j\omega) = I_l(j\omega)Z_l(j\omega) = I_l(j\omega)R_l \quad (5.3.11)$$

and expressing sin and cos by exponential functions, the following equations are obtained

$$U_y(j\omega) = \frac{1}{2}I_l(j\omega)(\rho + R_l)\exp(j\beta y)\left[1 - N\exp(-j2\beta y)\right] \quad (5.3.12)$$

$$I_y(j\omega) = \frac{1}{2\rho}I_l(j\omega)(\rho + R_l)\exp(j\beta y)\left[1 + N\exp(-j2\beta y)\right]$$

Plots of $U_y = |U_y(j\omega)|$ and $I_y = |I_y(j\omega)|$ are standing waves with their maxima and minima occurring at fixed locations along the line. As two subscripts, *x* and $y = l - x$, have been used, the meaning of subscripts 0 and *l* should be clarified:

$U_0 = U_x\big|_{x=0} = U_y\big|_{y=l}$, etc., $U_l = U_x\big|_{x=l} = U_y\big|_{y=0}$, etc.

For $N > 0 \equiv R_l < \rho$

$U_{y\min}$ and $I_{y\max}$ occur together when $\exp(-j2\beta y) = 1$, i.e. for $2\beta y = 2n\pi$. Then, minima of the voltage standing wave, so called **nodes of standing wave** (maxima or **arrows of** the current **standing wave**) are located at $y = n\lambda/2;\ n = 0,1,2,...$

where

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{\omega\sqrt{LC}} = \frac{1}{f\sqrt{LC}} = \frac{v}{f} \tag{5.3.13}$$

is the **wavelength**.

$U_{y\max}$ and $I_{y\min}$ occur together when $\exp(-j2\beta y) = -1$, i.e. for $2\beta y = 2(n+1)\pi$. Then, maxima of the voltage standing wave, so called arrows (minima or nodes of the current standing wave) are located at $y = (2n+1)\lambda/4;\ n = 0,1,2,...$.

An exemplary voltage and current standing waves, for the single-wave line, $l = \lambda$, are shown in Fig. 5.3.3.

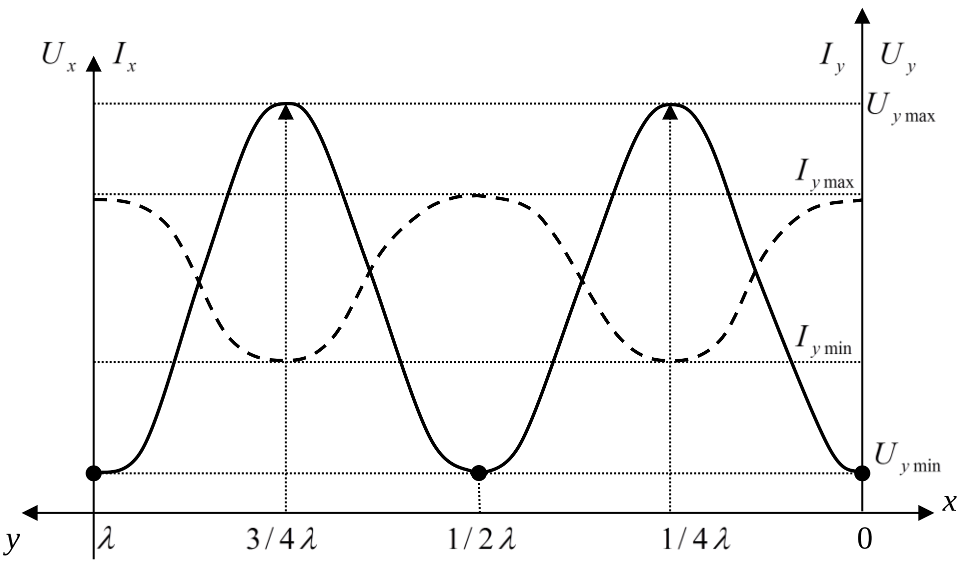


Fig. 5.3.3 Voltage (solid) and current (dashed) standing waves on resistance terminated lossless line, $R_l < \rho$

For $N < 0 \equiv R_l > \rho$

the roles of the voltage and current standing waves are interchanged from those for the case of $R_l < \rho$.

For resistive termination, $Z_l(j\omega) = R_l$, voltage maximum (current minimum) or minimum (current maximum) occurs at the terminating resistance. If the terminating impedance $Z_l(j\omega)$ is not a pure resistance, then a voltage maximum or minimum does not occur at the termination, both are shifted away from termination, however periodic character of standing waves is maintained. The ratio of the maximum to minimum voltages along a line is defined as the **standing wave ratio**, *S*:

$$S = \frac{U_{y\max}}{U_{y\min}} = \frac{1+|N(j\omega)|}{1-|N(j\omega)|} \tag{5.3.14}$$

It is clear that for the special cases:

a) matched load line: $N = 0 \Rightarrow S = 1$, (5.3.15)

b) open-circuited line: $N = -1 \Rightarrow S = 0$,

c) short-circuited line: $N = +1 \Rightarrow S = 0$.

The matched load line has been discussed in the preceding section of this Chapter. For short-circuited or open-circuited lines all the minima (nodes) go to zero. These two special cases will be discussed next.

Open-circuited line

From (5.3.10b), for $I_l(j\omega) = 0$, the voltage and current standing waves are

$$U_y(j\omega) = U_l(j\omega)\cos(\beta y) \Rightarrow U_y = U_l|\cos(\beta y)| \tag{5.3.16}$$

$$I_y(j\omega) = j\frac{U_l(j\omega)}{\rho}\sin(\beta y) \Rightarrow I_y = \frac{U_l}{\rho}|\sin(\beta y)|$$

and they are shown in Fig. 5.3.4, nodes and arrows of the voltage wave are denoted.

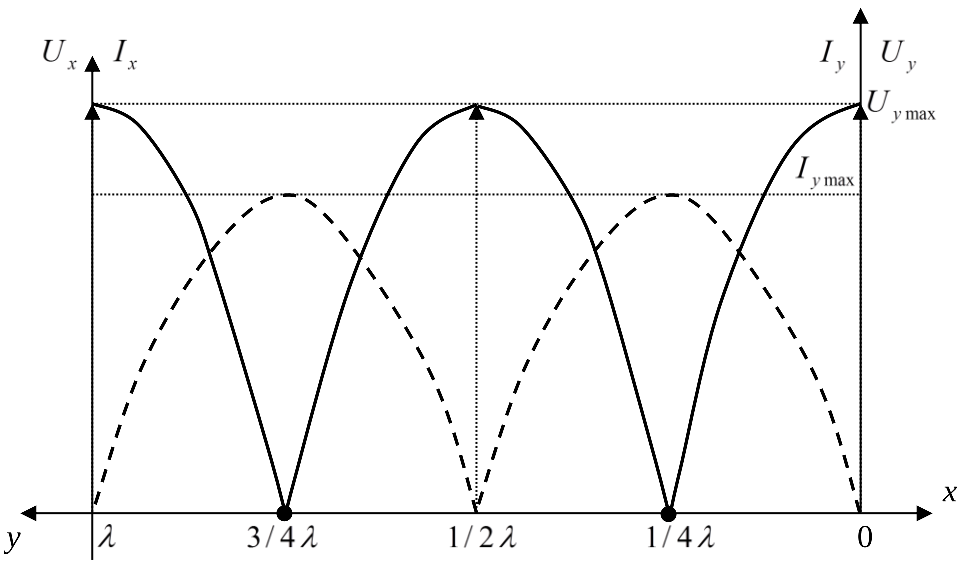


Fig. 5.3.4 Voltage and current (dashed) standing waves on open-circuited line

Short-circuited line

From (5.3.10b), for $U_l(j\omega)=0$, the voltage and current standing waves are

$$U_y(j\omega)= jI_l(j\omega)\rho\sin(\beta y)\Rightarrow U_y = I_l\left|\sin(\beta y)\right| \tag{5.3.17}$$

$$I_y(j\omega)=I_l(j\omega)\cos(\beta y)\Rightarrow I_y = I_l\left|\cos(\beta y)\right|$$

and they are shown in Fig. 5.3.5, nodes and arrows of the voltage wave are denoted.

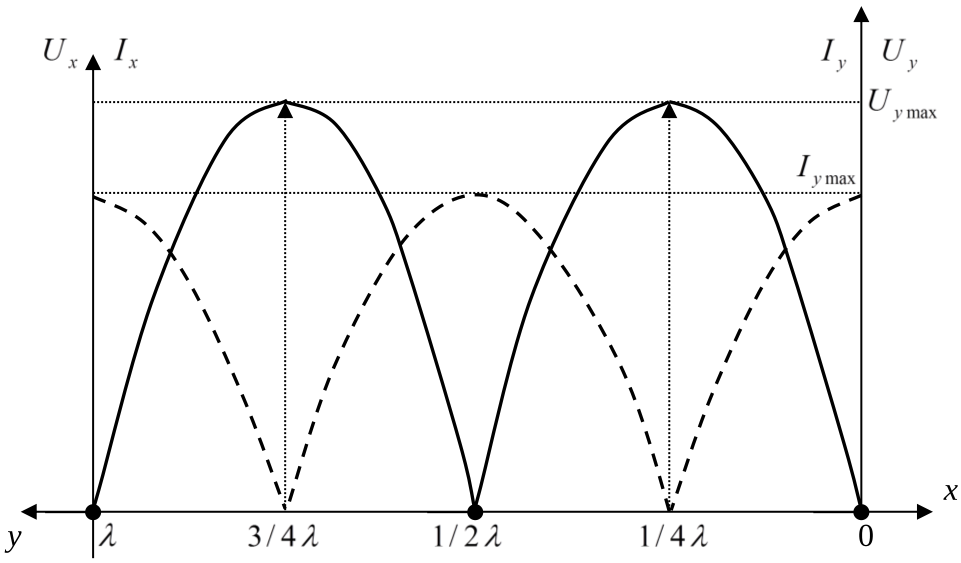


Fig. 5.3.5 Voltage and current (dashed) standing waves on short-circuited line

TRANSMISSION LINE as CIRCUIT ELEMENT, INPUT IMPEDANCE

Not only can transmission line be used as wave-guiding structure for transferring power or information, but it may serve as a circuit element. At ultrahigh frequencies (UHF), ranging from 300 MHz to 3 GHz – wavelength ranging from 1 m to 0.1 m, ordinary lumped elements are difficult to manufacture (see Chapter 4.3). Section of transmission line can be designed to give a pure inductive or capacitive impedance or may be used to match an arbitrary load to the internal impedance of a generator for maximum power transfer.

A transmission line segment can be considered lossless, and then, from equations (5.3.10b), (5.3.11), the input impedance of a lossless line of length *l* terminated in $Z_l(j\omega)$ is

$$Z_i(j\omega)=\left.\frac{U_x(j\omega)}{I_x(j\omega)}\right|_{x=0}=\left.\frac{U_y(j\omega)}{I_y(j\omega)}\right|_{y=l}=\rho\frac{Z_l(j\omega)+j\rho\tan(\beta l)}{\rho+jZ_l(j\omega)\tan(\beta l)} \tag{5.3.18}$$

Three special cases (5.3.15), quarter-wave line and half-wave line will be considered next.

Matched-load line

For the matched-load line, the input impedance is obviously fixed

$$Z_{im}(j\omega)=\rho \tag{5.3.18a}$$

Open-circuited line

For $Z_l(j\omega)=\infty$, the formula in equation (5.3.18) becomes

$$Z_{io}(j\omega)=\frac{\rho}{j\tan(\beta l)}=j[-\rho\cot(\beta l)]=jX_{io}(\omega) \qquad (5.3.18b)$$

As can be seen, the segment impedance can be either capacitive or inductive. Fig. 5.3.6 is a plot of X_{io} versus *l*, for *l* ranging from 0 to λ.

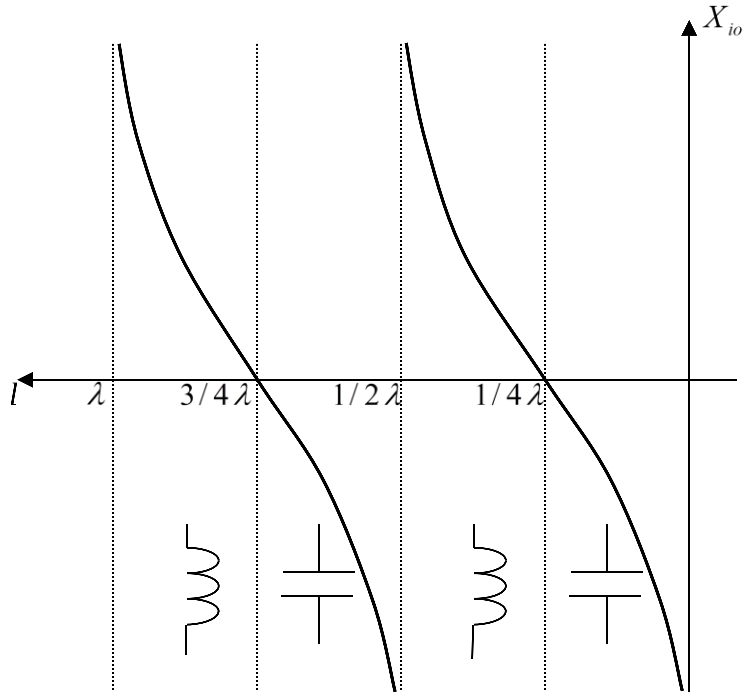


Fig. 5.3.6 Input reactance of open-circuited transmission line

When the length of a short-circuited line is very short in comparison with a wavelength, $\beta l << 2\pi$, then $\tan(\beta l)\cong\beta l$ and a very simple formula for its capacitive reactance is obtained

$$Z_{io}(j\omega)\cong -j\frac{\rho}{\beta l}=-j\frac{\sqrt{L/C}}{\omega\sqrt{LC}\,l}=-j\frac{1}{\omega Cl} \qquad (5.3.18b')$$

That way capacitance of Cl farads is obtained.

Short-circuited line

For $Z_l(j\omega)=0$, the formula in equation (5.3.18) becomes

$$Z_{is}(j\omega)=j\rho\tan(\beta l)]=jX_{is}(\omega) \qquad (5.3.18c)$$

As can be seen, the segment impedance can be either capacitive or inductive and it is worth to note that in the range where X_{io} is capacitive X_{is} is inductive, and vice versa. Fig. 5.3.7 is a plot of X_{is} versus *l*, for *l* ranging from 0 to λ.

When the length of a short-circuited line is very short in comparison with a wavelength, $l << \lambda = 2\pi/\beta \Rightarrow \beta l << 2\pi$, then $\tan(\beta l) \cong \beta l$ and a very simple formula for its inductive reactance is obtained

When the length of a short-circuited line is very short in comparison with a wavelength, $\beta l << 2\pi$, then $\tan(\beta l) \cong \beta l$ and a very simple formula for its inductive reactance is obtained

$$Z_{is}(j\omega) = j\rho\beta l = j\sqrt{L/C}\,\omega\sqrt{LC} = j\omega L l \qquad (5.3.18c')$$

That way inductance of $L l$ henries is obtained.

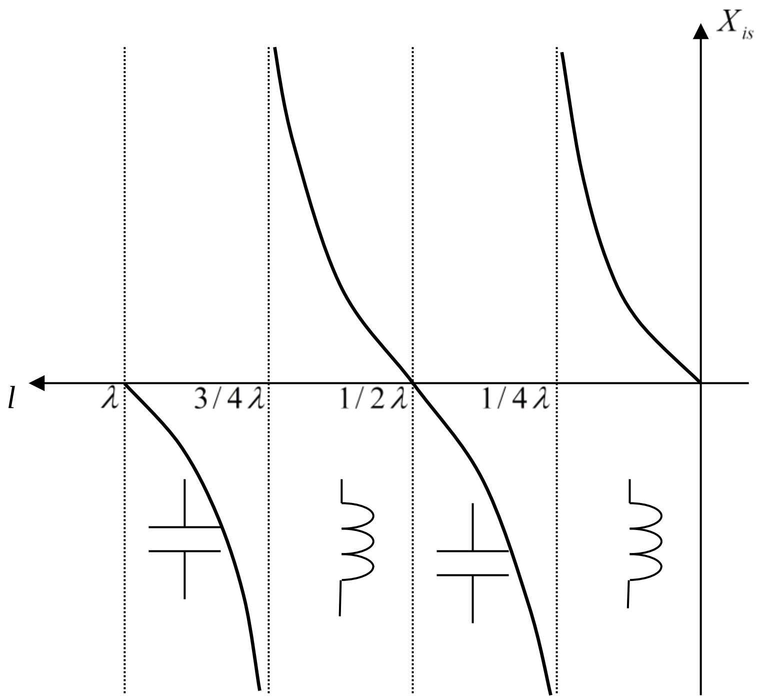


Fig. 5.3.7 Input reactance of short-circuited transmission line

Short-circuit and open circuit are easy provided on a transmission line. By measuring corresponding input impedances, the characteristic impedance and phase constant of the line can be determined. From (5.3.18b) and (5.3.18c), the characteristic impedance of a lossless line is

$$\rho = \sqrt{Z_{io}Z_{is}}\ ;\ Z_{io} = |X_{io}|,\ Z_{is} = |X_{is}| \qquad (5.3.19a)$$

and its phase constant is

$$\beta = \frac{1}{l}\tan^{-1}\sqrt{\frac{Z_{is}}{Z_{io}}} \qquad (5.3.19b)$$

<u>Quarter-wave line</u>

For line arbitrary termination $Z_l(j\omega)$ and its length being a quarter-wave or an odd multiple of $1/4\lambda$

$$\beta l = \frac{2\pi}{\lambda}(2n-1)\frac{\lambda}{4} = (2n-1)\frac{\pi}{2} \Rightarrow \tan(\beta l) = \pm\infty$$

and equation (5.3.18) reduces to

$$Z_i(j\omega) = \frac{\rho^2}{Z_l(j\omega)} \qquad (5.3.20)$$

A quarter-wave lossless line transforms the load impedance to input terminals as its inverse scaled by the square of the characteristic resistance.

<u>Half-wave line</u>

For line arbitrary termination $Z_l(j\omega)$ and its length being a half-wave or a multiple of $1/2\lambda$

$$\beta l = \frac{2\pi}{\lambda} n \frac{\lambda}{2} = n\pi \Rightarrow \tan(\beta l) = 0$$

and equation (5.3.18) reduces to

$$Z_i(j\omega) = Z_l(j\omega) \qquad (5.3.21)$$

A half-wave lossless line transfers the load impedance to input terminals without change.

<u>Example 5.3.1</u>

Consider a lossless line characterized by the following parameters:

$l = 100$ km, $L = 5$ mH/m, $C = 5$ nF/m.

For the given input voltage: $u_0(t) = 50\sin\frac{\pi}{2}t$ V,

find phasors of the input current, output voltage, output current and input impedance for three terminations: a) matched-load, b) open-circuit, c) short circuit.

$\div$

The characteristic resistance is: $\rho = \sqrt{\frac{5\cdot 10^{-3}}{5\cdot 10^{-9}}} = 10^3\,\Omega$.

The phase constant (phase shift per distance) is: $\beta = \frac{\pi}{2}\sqrt{5\cdot 10^{-3}\, 5\cdot 10^{-9}} = 5\frac{\pi}{2}\,10^{-6}$ rad/m.

The line phase shift is: $\beta l = \frac{\pi}{4}$ rad.

The wavelength is: $\lambda = \frac{2\pi}{2.5\pi\cdot 10^{-6}} = 800$ km $\Rightarrow l = \frac{\lambda}{8}$.

a)

$$Z_{im}(j\omega) = \rho = 10^3\,\Omega$$

$$U_l(j\omega) = U_0(j\omega) = 50/\sqrt{2} \text{ V}$$

$$I_l(j\omega) = I_0(j\omega) = \frac{U_0(j\omega)}{\rho} = 50/\sqrt{2} \text{ mA}$$

b)

$$Z_{io}(j\omega) = -j10^3 \cot\frac{\pi}{4} = -j10^3 \ \Omega \Rightarrow C_{io} = \frac{2}{\pi}10^{-3} \ \text{F}$$

$$I_0(j\omega) = \frac{U_0(j\omega)}{Z_{io}(j\omega)} = j50/\sqrt{2} \ \text{mA}$$

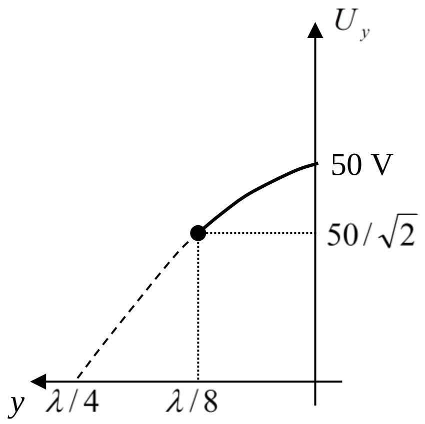
Equation (5.3.10b), for $y = l$ $(x = 0)$ becomes

$$U_y(j\omega)\big|_{y=l} = U_0(j\omega) = U_l(j\omega)\cos(\beta l) + jI_l(j\omega)\rho\sin(\beta l) \qquad (5.3.22)$$

From this equation, taking into account that $I_l(j\omega) = 0$, the output voltage is

$$U_l(j\omega) = \frac{U_0(j\omega)}{\cos(\beta l)} = \frac{50/\sqrt{2}}{\sqrt{2}/2} = 50 \ \text{V}$$

The voltage standing wave and the input reactance are shown in Fig. 5.3.8



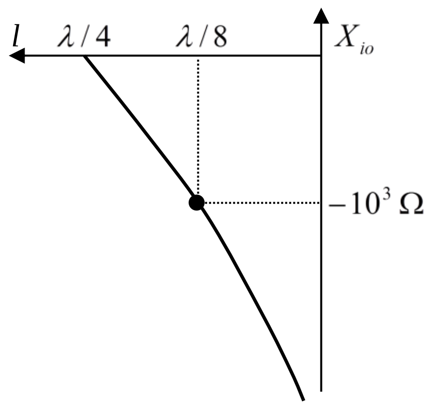


Fig. 5.3.8 Voltage standing wave and input reactance of Example 5.3.1 - open-circuited line

c)

$$Z_{is}(j\omega) = j10^3 \tan\frac{\pi}{4} = j10^3 \ \Omega \Rightarrow L_{is} = \frac{2}{\pi}10^3 \ \text{H}$$

$$I_0(j\omega) = \frac{U_0(j\omega)}{Z_{is}(j\omega)} = -j50/\sqrt{2} \ \text{mA}$$

From equation (5.3.22), taking into account that $U_l(j\omega) = 0$, the output current is

$$I_l(j\omega) = \frac{U_0(j\omega)}{j\rho\sin(\beta l)} = \frac{50/\sqrt{2}}{j10^3\sqrt{2}/2} = -j50 \ \text{mA}$$

The voltage standing wave and input reactance are shown in Fig. 5.3.9

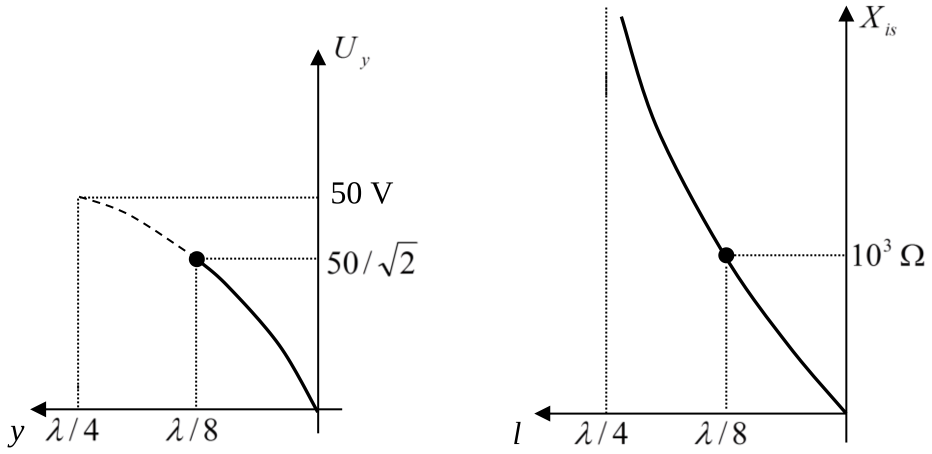


Fig. 5.3.9 Voltage standing wave and input reactance of Example 5.3.1 - short-circuited line

♦

<u>Drill problems 5.3</u>

1. Plot the voltage standing wave $U_y(j\omega)$ for a shorted lossless: a) quarter-wave, b) half-wave, c) single-wave, d) double-wave line. A source voltage is $e_o(t) = 10\sqrt{2}\sin(100t)\,\mathrm{V}$, its resistance is $R_t = 100\,\Omega$ and line parameters are: $l = \lambda = 10\,\mathrm{m}$, $\rho = 50\,\Omega$, $C = 5\,\mu\mathrm{F/m}$. Calculate the standing wave ratio *S* and the rms input current.
2. Outline the input reactance of Problem 5.3.1 line, for its length *l* ranging from 0 to 2λ.
3. Sketch the current standing wave for Problem 5.3.1 line and a source given by the Thevenin equivalent: $e_o(t) = 10\sqrt{2}\sin(100\pi\ t)$ V, $R_t = \rho$.
4. Repeat calculations and plots of Problems 5.3.1-3, after changing the line termination to an open-circuit.
5. Find the rms voltage at a distance of $x = l/2$ of a quarter-wave shorted line if the input voltage is $u_0(t) = 20\sqrt{2}\sin(314t)$ V. Sketch the voltage standing wave and find its ratio *S*.
6. Sketch the current standing wave in a shorted half-wave line of primary parameters: $L = 1\mathrm{mH/m}$, $C = 1\mathrm{nF/m}$. Calculate the output current if the input current is $i_0(t) = 5\sqrt{2}\sin(1000t)$ mA.
7. Find the input impedance of an open-circuited lossless line. Its parameters are: $L = 1\mathrm{mH/m}$, $C = 1\mathrm{nF/m}$, $l = \lambda/8 = 10$ m.
8. Find the input impedance of a short-circuited lossless line. Its parameters are: $L = 1\mathrm{mH/m}$, $C = 1\mathrm{nF/m}$, $l = \lambda/8 = 10$ m.
9. Find the input impedance of a lossless line. Its parameters are: $L = 1\mathrm{mH/m}$, $C = 1\mathrm{nF/m}$, $l = \lambda/4 = 10$ m, and a load impedance is $Z_l(j\omega) = 10 + j10\ \Omega$.

APPENDIX A - LAPLACE TRANSFORM

Laplace transform allows to transform the time-domain function $f(t)$ into the Laplace-domain function $F(s)$:

$$F(s) = \boldsymbol{L}\{f(t)\} \mathrel{\hat{=}} f(t) \tag{A1}$$

where s is the Laplace operator. Then, terms Laplace-domain, operator-domain or simply s-domain are used alternately. The Laplace transformation is the integral transformation defined by equation (A2).

DEFINITION

$$F(s) = \int_0^{\infty} f(t)\exp(-st)dt \tag{A2}$$

Together, $f(t)$ and $F(s)$ are called the Laplace transform pair, while $F(s)$ is called the Laplace transform of $f(t)$ - the image, and $f(t)$ is called the inverse Laplace transform of $F(s)$ - the original.

The Laplace transform is utilized in Circuit Theory to solve circuit transient analysis equations containing time integrals and derivatives, the so called integro-differential equations. After the Laplace transformation, system of linear equations in the s-domain is obtained. Then, its solution, operator response $Y(s)$ or responses $Y_1(s), Y_2(s)$.... are searched for. Inverse transformation is the final step of transient analysis.

When transforming time-domain equations (element $i-u$ relationships and Kirchhoff's laws) into the s-domain equations, the following properties of the Laplace transformation are utilized.

PROPERTIES

P1. Linearity

$$\boldsymbol{L}\{c_1 f_1(t) + c_2 f_2(t)\} = c_1 F_1(s) + c_2 F_2(s) \tag{A3}$$

where c_1, c_2 are real numbers

P2. Integration

$$\boldsymbol{L}\left\{\int_0^t f(t)dt\right\} = \frac{F(s)}{s} \tag{A4}$$

P3. Differentiation

$$\boldsymbol{L}\left\{\frac{d}{dt} f(t)\right\} = sF(s) - f(0) \tag{A5}$$

P4. Time-Shift

$$\boldsymbol{L}\{f(t-\tau)\mathbf{1}(t-\tau)\} = F(s)\exp(-s\tau), \quad \tau \geq 0 \tag{A6}$$

INVERSE TRANSFORMATION - HEAVISIDE'S FORMULA

To find the inverse transform

- integral formula (not presented here), or
- Partial Fraction Expansion (PFE) based, Heaviside's formula (A7)

can be utilized.

If

$$F(s) = \frac{L(s)}{sM(s)} = \frac{s^l + a_1 s^{l-1} + \cdots + a_{l-1}s + a_l}{s(s^m + b_1 s^{m-1} + \cdots + b_{m-1}s + b_m)} = \frac{\prod_{j=1}^{l}(s-q_j)}{s\prod_{k=1}^{m}(s-s_k)} \tag{A7a}$$

where,

q_j, s_k are roots of numerator and denominator polynomial, zeroes and poles of $F(s)$

a_j, b_k are real numbers; $j = 1,...,l; k = 1,...,m$; $l \leq m$

then

$$f(t) = \frac{L(0)}{M(0)} + \sum_{k=1}^{m} \frac{L(s_k)}{s_k M'(s_k)} \exp(s_k t) \tag{A7b}$$

where,

$$M'(s_k) = \frac{d}{ds} M(s)\bigg|_{s = s_k} = \prod_{\substack{i=1 \\ i \neq k}}^{m} (s_k - s_i) \tag{A7c}$$

The first component of $f(t)$, the time-invariant component, is the steady state value F_∞.

The second component is the transient component decaying exponentially to zero. This component is the sum of *m* terms, each term is designated by one pole. Three different types of poles can be distinguished:

1. simple real pole: $s_k = -1/T_k$, (A8a)
2. simple pair of complex conjugate poles: $s_k = -\alpha_k + j\omega_k$ (A8b)
   $s_{k+1} = s_k^* = -\alpha_k - j\omega_k$,
3. repeated (multiple) poles, for two poles: $s_k = s_{k+1} = -\alpha_k$. (A8c)

It should be emphasized, that in a stable circuit all poles lie in the left side of the complex plane. In a stable circuit steady state is always reached, all transients decay to zero and only such circuits are considered. Step response of the ideal integrator, discussed in Chapter 3.3, is the only exception.

For each simple pole (A8a), there will be a term

$$A_k \exp(-t/T_k)\,. \tag{A9a}$$

For each simple pair of complex conjugate poles (A8b), there will be a term

$$A_k \exp(-\alpha_k t)\sin(\omega_k t+\varphi_k) \tag{A9b}$$

where, $\varphi_k = \angle(s-s_k)F(s)\big|_{s=s_k}$

For two repeated poles (A8c), there will be a term

$$A_k t \exp(-t\alpha_k) \tag{A9c}$$

LAPLACE TRANSFORM DICTIONARY

Based on the Laplace transform definition (A2) or Heaviside's formula (A7), dictionary of Laplace transform pairs can be constructed. Four singularity functions, used to describe transient excitation:

- unit step,
- pulse,
- unit impulse,
- unit ramp,

and two ordinary functions, used to describe transient response

- exponential (1st order response),
- 2nd order response,

are discussed.

1. Unit step

in time domain is the function that is equal zero for all negative values of time and that is equal to one for all non-negative values. This dimensionless function is denoted by the bold one:

$$\mathbf{1}(t) = \begin{cases} 0, & t<0 \\ 1, & t\ge 0 \end{cases} \tag{A8}$$

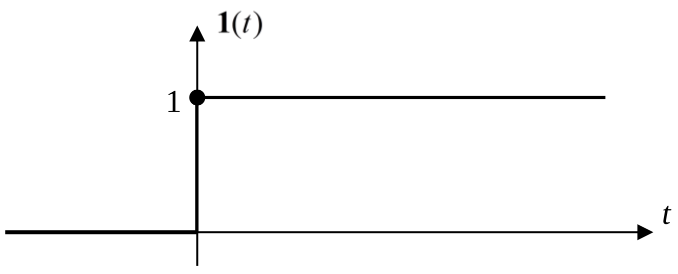


Fig. A1
Graph of the unit step function

The Laplace transform of the unit step function is

$$\boldsymbol{L}\{\mathbf{1}(t)\} = \frac{1}{s} \tag{A8a}$$

1a. Time-shifted unit step
in time domain is the function that is equal zero for all values of time less than τ and that is equal to one for all values greater than or equal τ :

$$\mathbf{1}(t-\tau)=\begin{cases}0, & t<\tau\\ 1, & t\geq\tau\end{cases} \tag{A9}$$

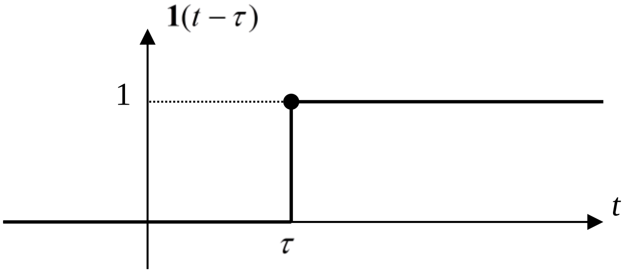


Fig. A1
Graph of the time-shifted unit step function

The Laplace transform of the time-shifted unit step is

$$\boldsymbol{L}\{\mathbf{1}(t-\tau)\}=\frac{1}{s}\exp(-s\tau) \tag{A9a}$$

2. Exponential decay

$$f(t)=\exp(-t/T)\mathbf{1}(t)=\begin{cases}\exp(-t/T), & t\geq 0\\ 0, & t<0\end{cases} \tag{A10}$$

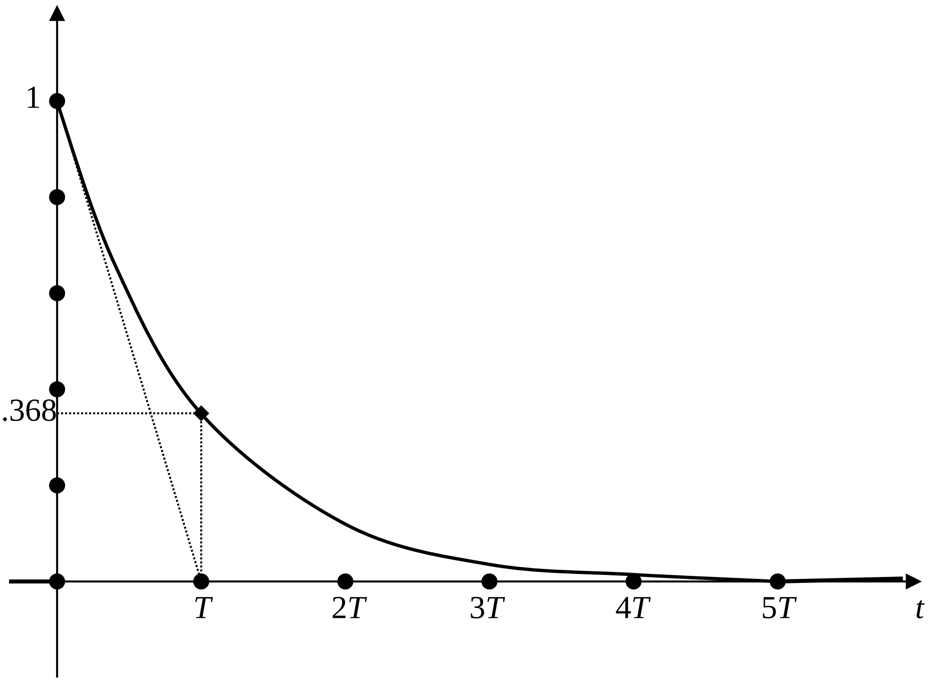


Fig. A2 Graph of the exponentially decaying function

This dimensionless function is defined by one parameter, time-constant *T*. Each time-constant the function is reduced by a factor of 1/*e* relative to its value at the beginning of that one-time-constant interval. Discrete values of the function at multiples of *T* are presented in Table A1.

Table A1

Discrete values of exponential decay at multiples of T

t	T	$2T$	$3T$	$4T$	$5T$
$\exp(-t/T)$	$e^{-1} = 0.368$	$e^{-2} = 0.135$	$e^{-3} = 0.050$	$e^{-4} = 0.018$	$e^{-5} = 0.007$

The exponential decay has the following properties:

1. After five time constants, the function value is less than 1% of its initial value, it is essentially zero.
2. Subtangent at any instance of time is equal to the time-constant T. In particular, a tangent to the curve at $t = 0$ intersects the time axis at $t = T$, what can be utilized when sketching the curve.

The Laplace transform of the exponential decline is

$$\boldsymbol{L}\{\exp(-t/T)\mathbf{1}(t)\} = \frac{T}{1 + sT} \tag{A10a}$$

2a. Exponential rise

$$f(t) = [1 - \exp(-t/T)]\mathbf{1}(t) = \begin{cases} 1 - \exp(-t/T), & t \geq 0 \\ 0, & t < 0 \end{cases} \tag{A11}$$

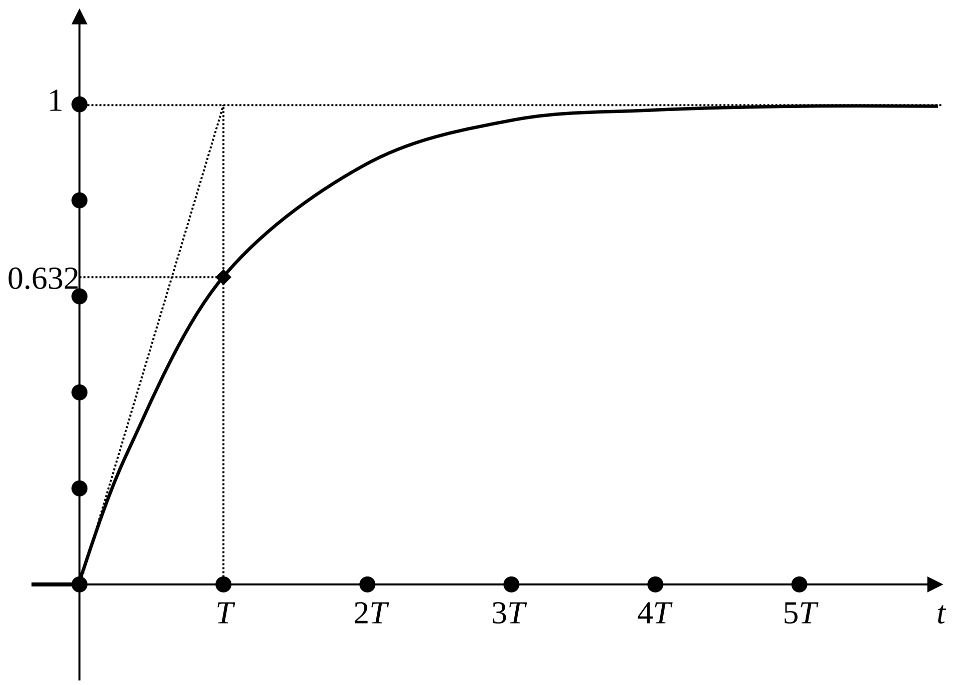


Fig. A2 Graph of the exponentially rising function

The Laplace transform of the exponential rise is

$$\boldsymbol{L}\{[1-\exp(-t/T)]\mathbf{1}(t)\} = \frac{1}{s(1+sT)} \tag{A11a}$$

3. Pulse

$$f(t) = \mathbf{1}(t) - \mathbf{1}(t-\tau) = \begin{cases} 1\,, & 0 \le t \le \tau \\ 0, & t \notin <0, \tau> \end{cases} \tag{A12}$$

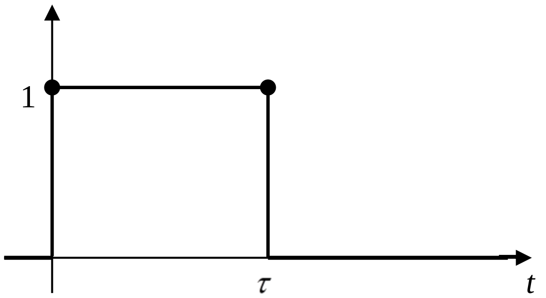


Fig. A3 Graph of the pulse

The Laplace transform of the pulse is

$$\boldsymbol{L}\{\mathbf{1}(t) - \mathbf{1}(t-\tau)\} = \frac{1}{s} - \frac{1}{s}\exp(-s\tau) \tag{A12a}$$

4. Unit impulse or Dirac delta

Consider a pulse $d(t)$ with height $1/2\tau$ and base 2τ centered at $t=0$, as presented in Fig. A4.

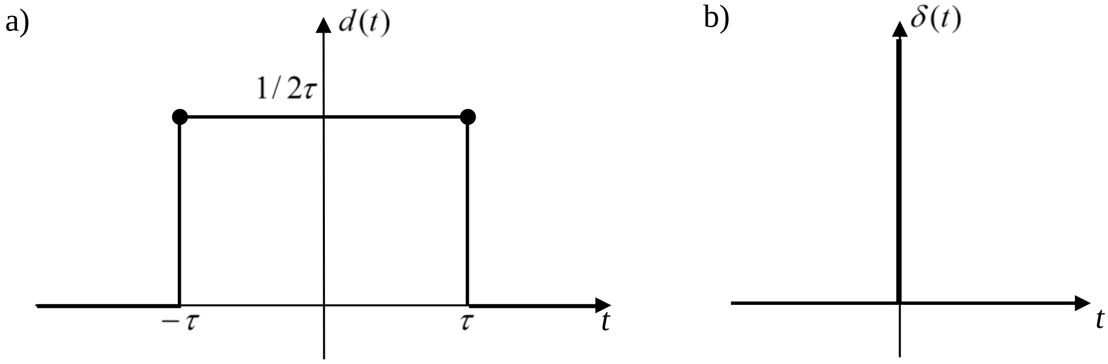


Fig. A4 Graph of a pulse and unit impulse, both of unit area

Area of this pulse is equal to one. Then, unit impulse, the so called Dirac delta, is the limit of $d(t)$ as τ goes to zero:

$$\delta(t) = \begin{cases} \infty\,, & t = 0 \\ 0, & t \neq 0 \end{cases} \tag{A13}$$

The unit impulse can be considered as the first derivative of the unit step. Its unit is [1/s] and its area is equal to 1.

$$\int_{-\infty}^{\infty} \delta(t)dt = 1 \tag{A13a}$$

The Laplace transform of the unit impulse is

$$\boxed{\boldsymbol{L}\{\delta(t)\} = 1} \tag{A13b}$$

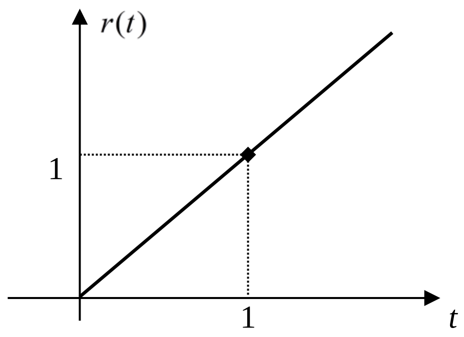
5. Unit ramp

The first integral of the unit step is referred to as the unit ramp function $r(t)$:

$$r(t) = t\mathbf{1}(t) = \begin{cases} t, & t \geq 0 \\ 0, & t < 0 \end{cases} \tag{A14}$$

This function is used to model slope(s) of a practical step(pulse). Its unit is [s] and its graph is presented in Fig. A5

Fig. A5 Graph of the unit ramp



The Laplace transform of the unit ramp is

$$\boxed{\boldsymbol{L}\{r(t)\} = \frac{1}{s^2}} \tag{A14a}$$

6. 2nd order response

The following 2nd order function in the s-domain is considered:

$$F(s) = \frac{1}{s^2 + bs + c} = \frac{1}{(s - s_1)(s - s_2)} = \frac{s}{s(s - s_1)(s - s_2)} \tag{A15}$$

where poles are:

$$s_{1,2} = -\frac{b}{2} \pm \sqrt{\frac{b^2}{4} - c} = -\alpha \pm \beta \tag{A15a}$$

The Heaviside's formula (A7) is utilized to find the inverse transform. Its elements are:

$$L(s) = s,\ L(0) = 0,\ M(s) = (s - s_1)(s - s_2),\ M'(s_1) = (s_1 - s_2),\ M'(s_2) = (s_2 - s_1) \tag{A15b}$$

Then,

$$f(t) = \frac{1}{s_1 - s_2}[\exp(s_1 t) - \exp(s_2 t)] = \frac{1}{2\beta}\exp(-\alpha t)[\exp(\beta t) - \exp(-\beta t)] \qquad \text{(A16)}$$

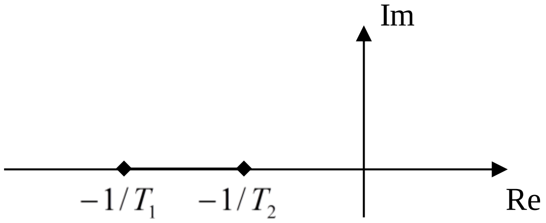
Three different cases have to be considered, subjected by the character of poles (A8):

1. two simple real poles,
2. simple pair of complex conjugate poles,
3. two repeated poles.

1. For $\frac{b^2}{4} > c$: $s_1 = -1/T_1$, $s_2 = -1/T_2$

   Location of poles in the complex plane is presented in Fig. A6a.

Fig. A6a
Complex plane location
of two simple real poles



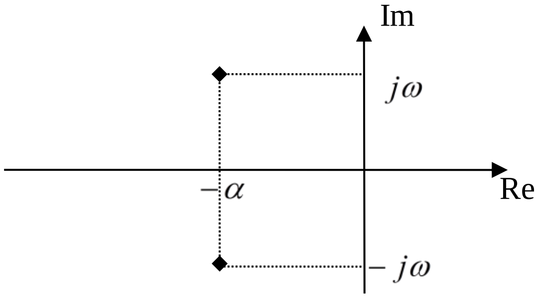
Then,

$$f(t) = \frac{1}{2\beta}[\exp(-t/T_1) - \exp(-t/T_2)] \qquad \text{(A16a)}$$

2. For $\frac{b^2}{4} < c$: $s_1 = -\alpha + j\omega$, $s_2 = -\alpha - j\omega$; $\omega = \sqrt{c - \frac{b^2}{4}}$

   Location of poles in the complex plane is presented in Fig. A6b.

Fig. A6b
Complex plane location
of pair of complex conjugate poles



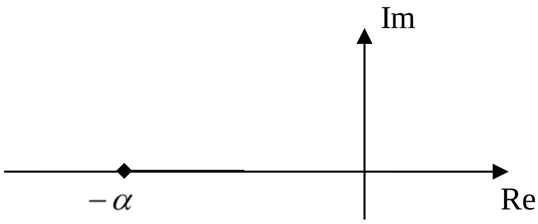
Then,

$$f(t) = \frac{1}{2j\omega}\exp(-\alpha t)[\exp(j\omega t) - \exp(-j\omega t)] = \frac{1}{\omega}\exp(-\alpha t)\sin\omega t \qquad \text{(A16b)}$$

3. For $\frac{b^2}{4} = c$: $s_1 = s_2 = -\alpha$

   Location of poles in the complex plane is presented in Fig. A6c.

Fig. A6c
Complex plane location of
two repeated poles



Then,

$$f(t) = \lim_{\beta \to 0} \frac{1}{2\beta} \exp(-\alpha t)[\exp(\beta t) - \exp(-\beta t /)] = t \exp(-\alpha t) \qquad \text{(A16c)}$$

APPENDIX B – COMPLEX NUMBERS

The **complex number** can be expressed in **rectangular form** as

$$\mathbf{F} = x + jy \tag{B1}$$

and presented in the complex plane, as shown in Fig. B1, where

- $j = \sqrt{-1}$ is the complex number of unit length along the imaginary axis,
- x is the real part of **F**, denoted $x = \mathrm{Re}\,\mathbf{F}$,
- y is the imaginary part of **F**, denoted $y = \mathrm{Im}\,\mathbf{F}$.

The same complex number **F** may be represented in **polar form** as

$$\mathbf{F} = F_m \angle \alpha \tag{B2}$$

where F_m, the magnitude or modulus of **F**, and $\angle\alpha$, the angle of **F**, are given by

$$F_m = \sqrt{x^2 + y^2} \tag{B2a}$$

$$\alpha = \arctan\frac{y}{x} \tag{B2b}$$

The angle is positive when taken in the counterclockwise direction and negative when taken in the clockwise direction.

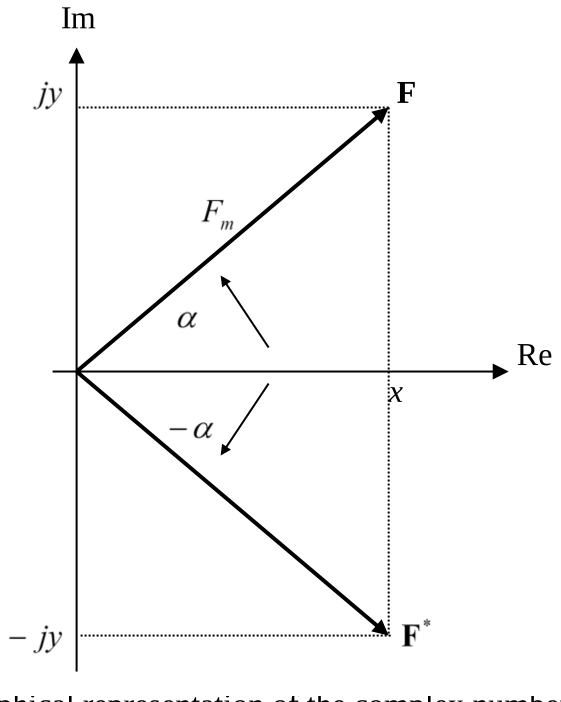


Fig. B1 Graphical representation of the complex number in the complex plane

From **Euler's formula** or **Euler identity**

$$\exp(j\alpha) = \cos\alpha + j\sin\alpha = 1\angle\alpha \tag{B3}$$

a useful alternative way to write complex numbers in the exponential polar form, or simply **exponential form** is obtained

$$\mathbf{F} = F_m \exp(j\alpha) \tag{B4}$$

The **conjugate** of the complex number **F** is defined to be

$$\mathbf{F}^* = x - jy = F_m\angle -\alpha = F_m \exp(-j\alpha) \tag{B5}$$

as presented in Fig. B1. Consider four unity length complex numbers along both axes in the complex plane, in both directions, as shown in Fig. B2.

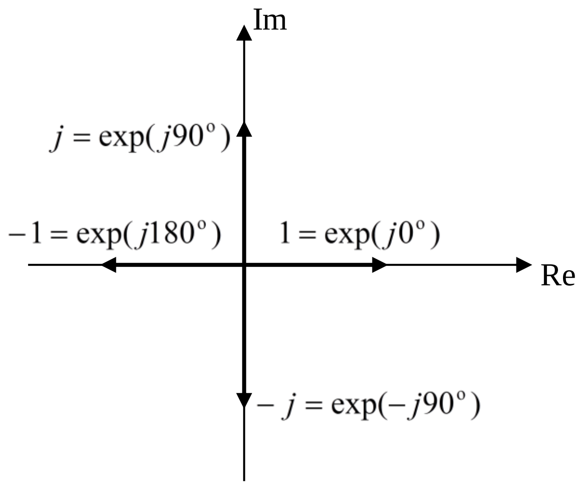


Fig. B2 Unity length complex numbers along both axes in the complex plane

The operations of addition, subtraction, multiplication and division apply to complex numbers exactly as they apply to real numbers, and they will discussed next.

<u>Addition/subtraction</u>

Consider two complex numbers, $\mathbf{F}_1 = F_{1m} \exp(j\alpha_1)$ and $\mathbf{F}_2 = F_{2m} \exp(j\alpha_2)$. Their sum/difference is calculated in the following way

$$\mathbf{F} = \mathbf{F}_1 \pm \mathbf{F}_2 = F_{1m} \cos\alpha_1 + jF_{1m} \sin\alpha_1 \pm (F_{2m} \cos\alpha_2 + jF_{2m} \sin\alpha_2) =$$
$$= x + jy = F_m \exp(j\alpha) \tag{B6}$$

where

$$x = F_{1m} \cos\alpha_1 \pm F_{2m} \cos\alpha_2 \tag{B6a}$$

$$y = F_{1m} \sin\alpha_1 \pm F_{2m} \sin\alpha_2$$

and the rectangular-to-exponential conversion is described by equations (B2). These operations may be carried out geometrically. The result is equivalent to completing the parallelogram or to connecting both vectors in head-to-tail manner, as shown in Fig. B3. Vector subtraction may be considered as vector addition with the subtracted vector shifted by 180 degrees (multiplied by -1).

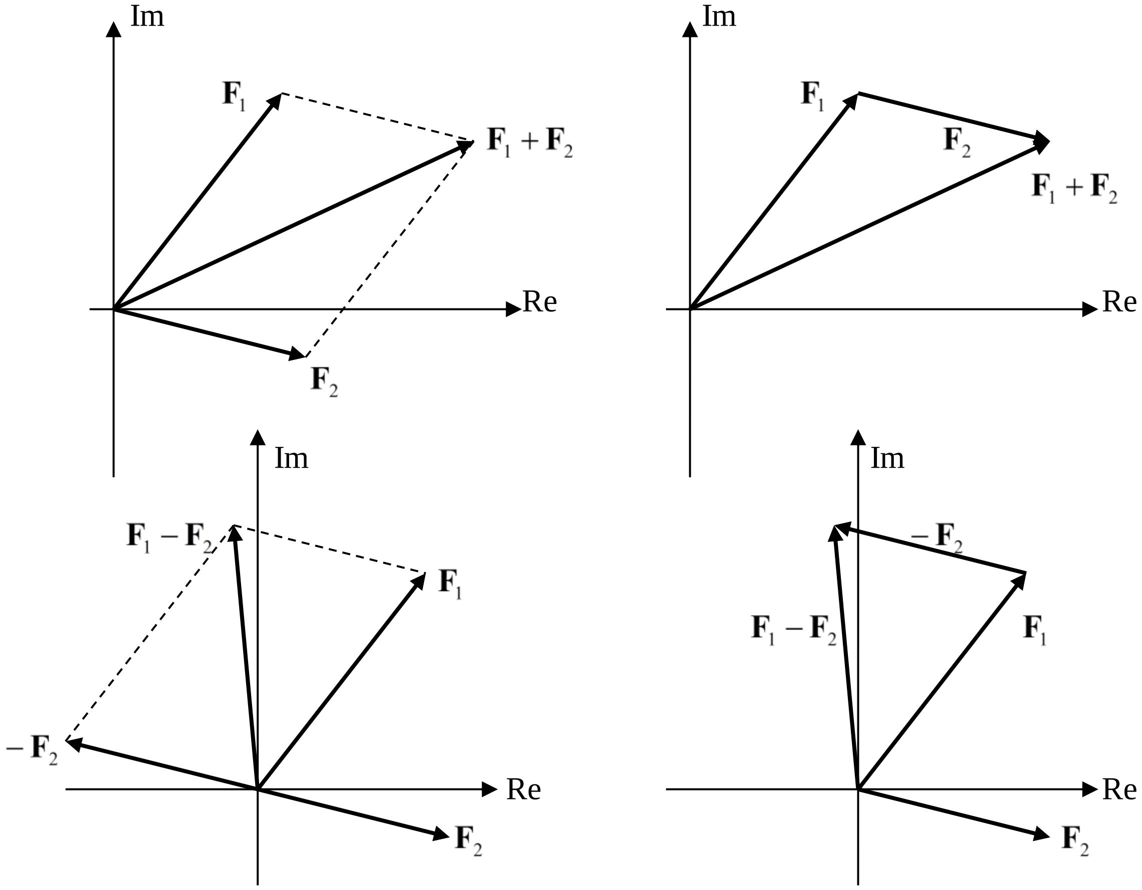


Fig. B3 Two methods of graphical addition/subtraction

<u>Multiplication</u>

Multiplication is normally performed using the exponential form (B4) of complex numbers

$$\mathbf{F} = \mathbf{F}_1\mathbf{F}_2 = F_{1m}F_{2m}\exp(j\alpha_1)\exp(j\alpha_2) \tag{B7}$$

Thus, two complex numbers may be multiplied by multiplying their magnitudes and adding their angles

$$F_m = F_{1m}F_{2m} \tag{B7a}$$

$$\alpha = \alpha_1 + \alpha_2 \tag{B7b}$$

In particular,

- multiplication by j rotates vector in counterclockwise direction by 90°,
  $j\mathbf{F} = F_m \exp[j(\alpha + 90^{\circ})]$
- multiplication by $-j$ rotates vector in clockwise direction by 90°,
  $-j\mathbf{F} = F_m \exp[j(\alpha - 90^{\circ})]$,
- multiplication by -1 rotates vector by 180°,
  $-\mathbf{F} = F_m \exp[j(\alpha + 180^{\circ})]$,

Alternatively, complex numbers may be multiplied in rectangular form

$$\mathbf{F} = (x_1 + jy_1)(x_2 + jy_2) = (x_1x_2 - y_1y_2) + j(x_1y_2 + y_1x_2) = x + jy \tag{B7c}$$

Division

Division can be performed using both rectangular and exponential form.

$$\mathbf{F} = \frac{\mathbf{F}_1}{\mathbf{F}_2} = \frac{F_{1m}}{F_{2m}} \exp[\, j(\alpha_1 - \alpha_2)] \tag{B8}$$

Thus, magnitude of the quotient is quotient of magnitudes, angle of the quotient is difference of angles

$$F_m = \frac{F_{1m}}{F_{2m}} \tag{B8a}$$

$$\alpha = \alpha_1 - \alpha_2 \tag{B8b}$$

In particular,

- division by j, equivalent to multiplication by $-j$, rotates vector in clockwise direction by 90°, $\dfrac{\mathbf{F}}{j} = -j\mathbf{F} = F_m \exp[\, j(\alpha - 90^\circ)]$.

Sometimes, it is more practical to divide two complex numbers in rectangular form. Then, to obtain the quotient in rectangular form, both numerator and denominator are multiplied by denominator conjugate

$$\mathbf{F} = \frac{x_1 + jy_1}{x_2 + jy_2} = \frac{(x_1 + jy_1)(x_2 - jy_2)}{(x_2 + jy_2)(x_2 - jy_2)} = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + j\,\frac{y_1x_2 - x_1y_2}{x_2^2 + y_2^2} = x + jy \tag{B9}$$

Evidently, it is easier to add and subtract complex numbers in rectangular form and to multiply and divide them in exponential form.

Next, consider the **complex exponential function**

$$\exp(\, j\omega t) = 1\angle\omega t \tag{B10a}$$

Examining this function, its magnitude is always unity, while its angle increases uniformly at the rate of ω radians per second. Thus, the complex exponential function traces out unit circles in the complex plane, beginning on the positive real axis at $t = 0$ and moving counterclockwise, completing one full circle every $T = 2\pi / \omega$ seconds, as shown in Fig. B4, together with a general, scaled and phase-shifted, complex exponential $\hat{f}$ (B10b).

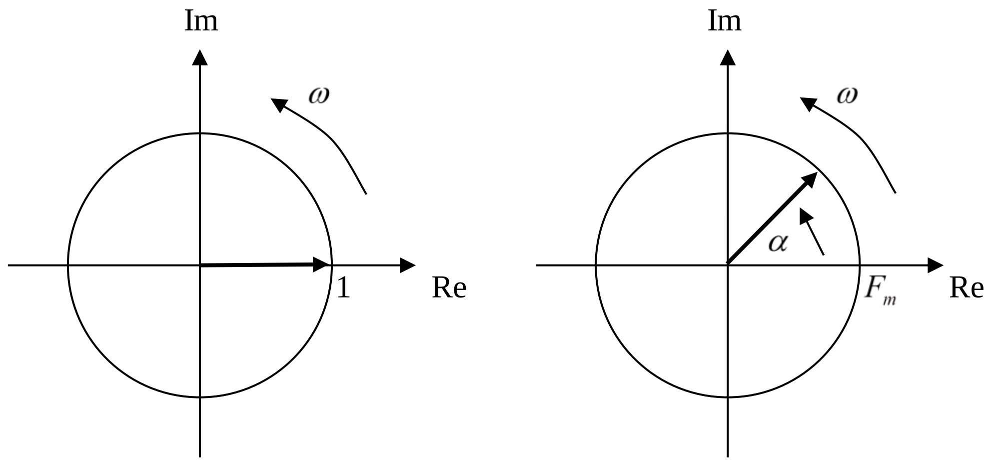


Fig. B4 Complex exponential exp($j\omega t$) and general complex exponential $F_m$exp[j(ωt+α)]

$$\hat{f} = F_m \exp[j(\omega t + \alpha)] = F_m \angle(\omega t + \alpha) \tag{B10b}$$

This **general complex exponential** is similar to the (simple) complex exponential, except that at $t = 0$ its initial phase is α radians and it traces out circles of radius F_m. By vertical projection

$$f = \operatorname{Im} \hat{f} = F_m \sin(\omega t + \alpha) \tag{B11a}$$

and by horizontal projection

$$f' = \operatorname{Re} \hat{f} = F_m \cos(\omega t + \alpha) \tag{B11b}$$

The general complex exponential can be rewritten in the following form

$$\hat{f} = f' + jf = F_m \exp[j(\omega t + \alpha)] = F\sqrt{2} \exp(j\omega t) \exp(j\alpha) \tag{B11c}$$

Then, for the given angular frequency ω, the sinusoidal function is uniquely characterized by the following complex number

$$F_m(j\omega) = F_m \exp(j\alpha) \tag{B12}$$

or, more often

$$F(j\omega) = F \exp(j\alpha) \tag{B12a}$$

This number is called **phasor**.

Any sinusoidal current or voltage, at a distinct frequency ω is uniquely characterized by its phasor (4.1.5). The rms value of the sinusoid is the magnitude of the phasor, and the phase angle of the sinusoid is the angle of the phasor.

In case frequency characteristics of magnitude and phase are considered

$$F(j\omega) = F(\omega) \exp[j\alpha(\omega)] \tag{B12a}$$

Since phasors are complex numbers, they may be represented by vectors in the complex plane also called Argand diagram, where addition or subtraction may be carried out geometrically. This representation is called the **phasor diagram** and may be helpful in analyzing ac steady-state circuits. When two sinusoids are represented as phasors on the same diagram, their phase difference is simply an angle between them, a leading phase angle corresponds to a counterclockwise rotation, according to the usual convention.

APPENDIX C – TERMS AND CONCEPTS

Polish	English	Description (opis)
Admitancja operatorowa	Admittance in *s*-domain $Y(s)$	Reciprocal of $Z(s)$. Odwrotność impedancji.
Admitancja symboliczna	Admittance in phasor-domain $Y(j\omega)$	Reciprocal of $Z(j\omega)$, $Y(j\omega) = G(\omega) + jB(\omega)$
Amper	Ampere A	*See Current*
Amperozwoje	Ampere-turns A_t	*See Magnetomotive force*
Amplituda (wartość szczytowa)	Amplitude (peak value) I_m, U_m	*See Alternating current*
Analiza stanów przejściowych (nieustalonych)	Transient analysis (state)	Analysis of a circuit behavior for a period of time immediately after independent source or sources have been turned on or turned off, at $t = 0$. In stable circuits, transient state vanishes after $5T_{\max}$, where $T_{\max}$ is the maximum time constant. Analiza zachowania obwodu od momentu powstania zaburzenia do momentu ustalenia się odpowiedzi. W obwodzie stabilnym, czas osiągnięcia stanu ustalonego wyznacza pięciokrotność największej stałej czasowej odpowiedzi.
Analiza w stanie ustalonym	Steady-state analysis	Analysis of a circuit behavior resulting after have been on for a long time. Analiza w stanie, w którym wszystkie przebiegi osiągają wartości stałe, niezmienne w czasie.
Aproksymacja odcinkowo-liniowa	Piecewise-Linear Approximation (PWLA):	Approximation of the nonlinear $I-U$ characteristic by linear segments. For each segment its Thevenin or Norton equivalent can be found. Aproksymacja odcinkowa charakterystyki nieliniowej. Każdy segment można zamodelować schematem Thevenina lub Nortona.
Autotransformator	Autotransformer	Transformer that has windings both magnetically and electrically interconnected. Transformator, w którym jest tylko jedno uzwojenie, spełniające jednocześnie rolę pierwotnego i wtórnego.

Polish	English	Description (opis)
Bieguny transmitancji	Poles of transfer function s_k	Roots of the denominator polynomial of the transfer function $K(s)$. Pierwiastki wielomianu mianownika operatorowej funkcji przejścia.
Cewki sprzężone	Coupled coils	*See Mutual inductance*
Charakterystyka zastępcza – połączenie równoległe	Equivalent characteristic – parallel connection	The $I-U$ characteristic for a parallel connection of elements can be obtained by graphically adding the currents of elements at various values of voltage. Charakterystykę zastępczą elementów nieliniowych połączonych równolegle otrzymać można sumując prądy w punktach załamania charakterystyk tych elementów.
Charakterystyka zastępcza – połączenie szeregowe	Equivalent characteristic – series connection	The $I-U$ characteristic for a series connection of elements can be obtained by graphically adding the voltages of elements at various values of current. Charakterystykę zastępczą elementów nieliniowych połączonych szeregowo otrzymać można sumując napięcia w punktach załamania charakterystyk tych elementów.
Ciągłość napięcia na kondensatorze	Continuity of capacitor voltage	Voltage on a capacitor is always continuous, even though the current may be discontinuous. Napięcie na kondensatorze jest ciągłą funkcją czasu, prąd może być funkcją nieciągłą. W szczególności: In particular: $u_C(0_-) = u_C(0_+) = U_{C0}$.
Ciągłość prądu na cewce	Continuity of inductor current	Current through a coil is always continuous, even though the voltage may be discontinuous. Prąd cewki jest ciągłą funkcją czasu, napięcie może być funkcją nieciągłą. W szczególności: In particular: $i_L(0_-) = i_L(0_+) = I_{L0}$.
Czas propagacji	Line propagation time τ	Time of propagation of a signal from the line input to the output or in the reverse direction, $\tau = l/v$. Czas propagacji sygnału od wejścia do wyjścia linii.
Częstotliwość	Frequency f, ω	Frequency of oscillations in a periodic (sinusoidal) waveform, Częstotliwość fali sinusoidalnej, $f = 1/T$ Its unit is hertz $[\text{Hz}] = [1/\text{s}]$. Radian (angular) frequency, Częstotliwość kątowa, $\omega = 2\pi f$ Its unit is [rad/s].
Częstotliwość drgań własnych	Undamped natural frequency	*See Resonant frequency*

Polish	English	Description (opis)
Częstotliwość graniczna	Cut-off or break or corner frequency ω_c	The point where the asymptotic curve for its logarithmic gain exhibits a sharp change in a slope. Punkt, w którym następuje nagła zmiana nachylenia charakterystyki amplitudowej w skali logarytmicznej. *See High/Law Pass Filter*
Częstotliwość graniczna	Half-power frequency ω_l, ω_u	Frequency at which the magnitude response is $1/\sqrt{2}$ times the maximum. Częstotliwość, dla której wzmocnienie wynosi $1/\sqrt{2}$ wartości maksymalnej.
Częstotliwość rezonansowa	Resonant frequency ω_r	Frequency at which a two-terminal circuit becomes purely resistive. In the series or parallel *RLC* circuit, also frequency of the undamped transient response: $\omega_r = 1/\sqrt{LC}$. Częstotliwość dla której dwójnik znajduje się w rezonansie.
Częstotliwościowa funkcja przejścia	Frequency Response $K(j\omega)$	Frequency dependent relation, in both gain and phase, between the input phasor signal and the output phasor signal - transfer function in frequency-domain Funkcja przejścia dla wartości symbolicznych skutecznych $K(j\omega) = K(\omega)\exp[j\varphi(\omega)] = Y(j\omega)/X(j\omega)$.
Częstotliwościowa funkcja przejścia	Transfer function in frequency domain $K(j\omega)$	*See Frequency response*
Czwórnik	Two-port	Four terminal element identified by two distinct pairs of terminals - ports. Element o dwóch wrotach.
Decybel	Decibel dB	log-based measure of gain. *See Logarithmic gain*
Dekada	Decade dec	Frequency band whose endpoint is a factor of 10 larger than its beginning point. Przedział częstotliwości, którego górna wartość graniczna jest 10 razy większa od wartości dolnej.
Delta Diraca	Dirac delta function $\delta(t)$	*See Unit Impulse function*
Długość linii	Line length l	Distance from the line input to its output. Odległość od poczatku do końca linii.

Polish	English	Description (opis)
Dobroć kondensatora, cewki lub obwodu o własnościach selektywnych	Quality factor of practical capacitor practical coil bandpass circuit	Measure of the circuit energy storage property in relation to its energy dissipation property. Miara zdolności obwodu do gromadzenia energii w relacji do zdolności do jej rozpraszania. Capacitor: $Q_C(\omega) = R_C \omega C$. Inductor: $Q_L(\omega) = \omega L / R_L$. Bandpass circuit: Obwód o właściwościach selektywnych: $Q = 2\pi \dfrac{\text{maximum energy stored}}{\text{total energy dissipated per cycle}}$ Series *RLC* circuit: $Q = 1/R\sqrt{L/C}$ Parallel *RLC* circuit: $Q = 1/G\sqrt{C/L}$
Dwójnik	Two-terminal element or circuit	Element/circuit connected at a pair of terminals, described by a single $I-U$ relationship. Element/obwód dwuzaciskowy. *See Element law...*
Dziedzina częstotliwościowa	Frequency (phasor) domain	Mathematical domain where the set of possible values of AC variable (current or voltage) is expressed in terms of frequency. Dziedzina, w której prądy i napięcia wyrażane są w funkcji częstotliwości.
Dzielnik napięcia	Voltage divider	Circuit of a series of resistors that divides the input voltage *U* by the ratio of the R_i to the total series resistance Obwód zbudowany z *n* oporników połączonych szeregowo, dzielący napięcie jak poniżej: $R_t = \sum_{i=1}^{n} R_i$, $U_i = UR_i / R_t$.
Dzielnik prądów	Current divider	Circuit of *n* parallel resistors that divides the input current *I* so that Obwód zbudowany z *n* oporników połączonych równolegle, dzielący prąd jak poniżej: $I_i = I \cdot G_i / \sum_{i=1}^{n} G_i$
Dżul	Joul J	*See Energy*
Elektromagnes	Artificial or temporary magnet	*See Electromagnet or solenoid coil*

Polish	English	Description (opis)
Elektromagnes	Electromagnet or solenoid coil	Wire wound around the soft steel alloy core with the total number of *z* turns. It exhibits the magnetic field of the permanent magnet when energized, i.e. can be called the temporary or artificial magnet. Cewka o *z* zwojach, nawinięta na rdzeń stalowy – zachowuje się jak magnes stały gdy płynie przez nią prąd.
Elektryczność	Electricity	Physical phenomena arising from the existence of interaction of electric charges. Zjawiska jakie występują w wyniku oddziaływania na siebie ładunków.
Element aktywny	Active element	Element that may deliver energy to a circuit. Element, który jest źródłem energii.
Element pasywny	Passive element	Total energy supplied to it from the rest of the circuit is always nonnegative. Such element cannot deliver net power to a circuit. Element, który rozprasza lub magazynuje energię, nie zasila obwodu.
Energia	Energy W or w	Ability to perform work. Zdolność do wykonania pracy. Units: joul [J]; wattsecond $[\mathrm{W \cdot s}]$; [cal] calory, $1[\mathrm{J}] = 1[\mathrm{W \cdot s}] = 0.239$ [cal]. Instantaneous energy dissipated/supplied: Energia chwilowa dostarczana/pobierana: $w = \int_0^t p\,dt$ Instantaneous energy stored: Energia chwilowa zmagazynowana w cewce, kondensatorze: coil: $w_L = \frac{L i_L^2}{2}$, capacitor: $w_C = \frac{C u_C^2}{2}$.
Fale postępujące	Forward waves	Waves traveling from the line input to its output. Fale wędrujace od poczatku linii do jej końca.
Fale powrotne	Backward waves	Reflected waves traveling from the line output to its input. Fala odbita od końca linii.
Fale stojące	Standing waves	Plots of $\|U_x(j\omega)\|$ and $\|I_x(j\omega\|$ with their maxima and minima occurring at fixed locations along the line.

Polish	English	Description (opis)
Fale wędrujące	Traveling waves	Initial and reflected waves traveling from the line input to its output – forward waves and reflected waves traveling in the opposite direction – backward waves. Fala pierwotna i fale odbite, od początku i od końca linii. The reflected waves interference can be disregarded if time parameter(s) of a transmitted signal is(are) much greater than the line propagation time τ . Efekt nakładania sie fal odbitych na falę pierwotną można pominąć jeśli czas charakterystyczny trsansmitowanego sygnału jest znacznie wiekszy od czasu propagacji τ.
Farad	Farad F	*See Capacitance*
Faza początkowa	Initial phase angle α_i, α_u	*See Alternating current*
Filtr dolno-przezepustowy	Low-Pass Filter (LPF)	Filter that passes all frequencies up to the cut-off frequency ω_c and rejects all frequencies above it. Filtr przepuszczający wszystkie częstotliwości aż do częstotliwości granicznej.
Filtr elektryczny	Electric Filter	Circuit designated to provide a magnitude gain or loss over a predefined range of frequencies. Obwód (czwórnik) przepuszczający lub tłumiący sygnał wejściowy w zadanym paśmie częstotliwości.
Filtr górno-przepustowy	High-Pass Filter (HPF)	Filter that passes all frequencies above the cut-off frequency ω_c and rejects all frequencies below the cut-off frequency. Filtr, który przepuszcza wszystkie częstotliwości powyżej częstotliwości granicznej.
Filtr pasmowo-przepustowy	Band-Pass Filter (BPF)	Circuit that passes unimpeded all frequencies in a selected range of frequencies and rejects all frequencies outside this range. Obwód (czwórnik), który przepuszcza częstotliwości z zadanego pasma a nie przepuszcza wszystkich pozostałych.
Filtr pasmowo-zaporowy	Band-Stop Filter (BSF)	Circuit that rejects all frequencies in a selected range of frequencies and passes unimpeded all frequencies outside this range. Obwód (czwórnik), który nie przepuszcza częstotliwości z zadanego pasma a przepuszcza wszystkie pozostałe.

Polish	English	Description (opis)
Funkcja impulsowa	Unit Impulse (Dirac delta) function $\delta(t)$	Infinitely short pulse of infinitely large magnitude - its value is zero for $t \neq 0$, infinity at $t = 0$ and its area is equal to 1. Unit: [1/s]. Nieskończenie krótki impuls o nieskończonej amplitudzie i polu jednostkowym.
Funkcja liniowo narastająca	Unit Ramp function $r(t)$	$r(t) = t\mathbf{1}(t)$, an integral of the unit step function. całka skoku jednostkowego.
Gałąź	Branch	Element or string of two-terminal elements connected between two nodes. Number of circuit branches is denoted as *b*. Element dwuzaciskowy lub kilka połączonych w szereg, włączony między dwa węzły. Liczba wszystkich gałęzi: *b*.
Generator trójfazowy	Three-phase source	Three voltage sources of the same frequency and magnitude, and the phase shift or 120° between any two of them, connected in the form of Y or Δ. Trzy źródła napięcia sinusoidalnego o tej samej częstotliwości, amplitudzie i przesunięciem 120° między każdą parą.
Graf obwodu	Circuit graph	Graphical representation of the circuit structure (component interconnections). Graph consists of branches connected in nodes. Graficzne zobrazowanie struktury obwodu (bez wnikania w rodzaj elementów) – *b* gałęzi połączonych w *n* węzłach.
Granice obszaru tolerancji	Tolerance margins of circuit parameter X^-, X^+	Margins specifying the allowed (by the design) variation of parameter *X* from its nominal value X^n: Marginesy odchyłek parametru (górnej i dolnej) od wartości nominalnej, dopuszczalnych tolerancjami projektowymi: $X^- = X^n - \Delta X \quad X^+ = X^n + \Delta X$.
Henr	Henry H	*See Inductance*
Hertz	Hertz Hz	*See Frequency*

Polish	English	Description (opis)
I prawo Kirchhoffa	Kirchhoff's Current Law – KCL	The algebraic sum of currents (constant or varying in time, in the *s*-domain or in the phasor-domain) entering and leaving the node (cutset) equals zero, "+" if current arrowhead is directed to the node, "–" otherwise. Algebraiczna suma prądów (stałych, zmiennych, operatorowych, symbolicznych) wpływajacych do odciecia jest równa zero, „+" jeśli strzałka prądu do odciecia, „–" gdy od odcięcia.
Idealne (niezależne) źródło napięciowe	Independent ideal voltage source E or e	Source that provides a voltage independent of other circuit variables, (electromotive force – emf). Źródło wymuszające napięcie niezależne od reszty obwodu, do którego zostało dołączone (siła elektromotoryczna – SEM).
Idealne (niezależne) źródło prądowe	Independent ideal current source J or j	Source that provides a current independent of other circuit variables. Źródło wymuszające przepływ prądu niezależnego od reszty obwodu, do którego zostało dołączone (siła prądomotoryczna – SPM).
II prawo Kirchhoffa	Kirchhoff's Voltage Law – KVL	The algebraic sum of voltages (constant or varying in time, in the *s*-domain or in the phasor-domain) around a loop (any closed path) equals zero, "+" if voltage arrowhead is directed clockwise, "–" otherwise. Algebraiczna suma napięć (stałych, zmiennych, operatorowych, symbolicznych) zamkniętej ścieżki (oczka) jest równa zero, „+" jeśli strzałka zgodna z ruchem wskazówek zegara, „–" gdy przeciwna.
Impedancja falowa	Characteristic impedance (resistance) $Z(s)$ or $Z(j\omega)$ (ρ)	$Z(s)=\sqrt{(R+sL)/(G+sC)}$ For a distortionless line: Dla linii bezstratnej: $Z(s)=\rho=\sqrt{L/C}=const$, in [Ω].
Impedancja operatorowa	Impedance in *s*-domain $Z(s)$	Ratio of the voltage $U(s)$ at a pair of element terminals to the current $I(s)$ flowing into the positive voltage terminal. Stosunek operatorowej wartości napięcia na zaciskach dwójnika do prądu,

Polish	English	Description (opis)
Impedancja symboliczna	Impedance in phasor-domain $Z(j\omega)$	Ratio of the phasor voltage $U(j\omega)$ at a pair of element terminals to the phasor current $I(j\omega)$ flowing into the positive voltage terminal, Stosunek wartości symbolicznej skutecznej napięcia na zaciskach dwójnika do prądu, $Z(j\omega) = R(\omega) + jX(\omega) = Z(\omega)\exp\varphi(\omega)$.
Impedancja wejściowa	Input impedance $Z_{in}(j\omega)$	Impedance seen at port 1 (input) of a possibly terminated two-port. Impedancja widziana z zacisków wejściowych czwórnika.
Impedancja zastępcza lub wewnętrzna	Equivalent or total or Thevenin or internal impedance $Z_t(j\omega)$	Impedance of the two-terminal circuit when internal independent source is deactivated, also impedance that appears in the Thevenin equivalent of a practical source Impedancja widziana z zacisków obwodu dwuzaciskowego po wyzerowaniu jego źródeł, rezystancja schematu zastępczego Thevenina.
Impuls prostokątny	Pulse	Function of time, built of step functions, that is zero for $t < 0$, has magnitude 1 for $0 \le t \le \tau$, and is equal to zero for $t > \tau$. Funkcja różna od zera w przedziale od 0 do τ.
Indukcja magnetyczna	Magnetic flux density B	Ratio of the magnetic flux that passes perpendicularly through an area S to this area. Stosunek strumienia przenikajacego prostopadle dany przekrój do tego przekroju. Unit: tesla $[\mathrm{T}] = [\mathrm{V \cdot s/m^2}]$.
Indukcyjność doprowadzeń	Stray or parasitic inductance	Unwanted inductance of element connections. Pasożytnicza indukcyjność doprowadzeń elementu.
Indukcyjność wzajemna	Mutual inductance M	Coefficient of proportionality relating current passing through one coil and flux caused by this current in the second (coupled) coil: Współczynnik proporcjonalności między prądem jednej cewki a strumieniem jaki ten prąd wywołuje w cewce sprzężonej: $M = k\sqrt{L_1 L_2}$ k – coupling coefficient (współczynnik sprzężenia)

Polish	English	Description (opis)
Induktancja	Inductance L	Constant of proportionality interrelating current passing a coil and the total flux: Współczynnik proporcjonalności pomiędzy prądem płynacym przez cewkę a strumieniem całkowitym jaki ten prąd wywołuje: $\phi_t = Li$; $L = z^2 / R_m$ Unit: henry [H] =$[\mathrm{V\,s/A}]$.
Kondensator	Capacitor	Two-terminal energy storage element, described by the equation: $Q = CU$. Element dwuzaciskowy magazynujący energię (ładunek). Jego równanie i-u oraz energia: Its law and energy stored are: $i = C\,du/dt$, $w = Cu^2/2$.
Konduktancja	Conductance in phasor-domain $G(\omega)$	*See Admittance in phasor-domain*
Krzywa magnesowania	Magnetization or *B-H* curve $B = f(H)$	For ferromagnetic materials, $B = f(H)$. For diamagnetic or after linearization of the curve: Dla diamagnetyka po linearyzacji krzywej $B = \mu_r \mu_0 H$ μ_r - magnetic permeability of the material, przenikalność magnetyczna materiału $\mu_0 = 4\pi 10^{-7}$ $[\mathrm{V \cdot s/A \cdot m}]$ magnetic permeability of the free space. przenikalność magnetyczna w próżni
kulomb	Coulomb C	*See Charge*
Ładunek	Charge Q or q	Fundamental unit of matter responsible for electric phenomena, $Q = Q^+$ is the positive charge, Q^- is the negative charge. Like charges repel and unlike charges attract each other. Podstawowa jednostka materii odpowiedzialna za zjawiska elektryczne. Mamy ładunki dodatnie i ujemne: różnoimienne odpychają się, jednoimienne przyciągają. Units: coulomb [C] $= [\mathrm{A \cdot s}]$.
Linia bezstratna	Loss-less line	Line with no energy loss, $R = 0$, $G = 0$. Linia, w której nie ma strat energetycznych.
Linia długa	Transmission line	Two-wire line connecting the input circuit with the output circuit. Linia dwuprzewodowa łącząca źródło z obciążeniem.

Polish	English	Description (opis)
Linia dopasowana na wejściu	Matched generator line	Line with the generator (input) resistance equal to the characteristic resistance, for such line M=0. Linia, której oporność charakterystyczna równa jest oporności wewnętrznej generatora.
Linia dopasowana na wyjściu	Matched load line	Line with the load (output) resistance equal to the characteristic resistance, for such line N=0. Linia, której oporność charakterystyczna równa jest oporności obciążenia.
Linia nieznieksztalcajaca	Distortionless line	Line with parameters that satisfy Linia, której parametry spełniaja zależność $R/L = G/C$.
Liniowość	Linearity	When responses to inputs X_1, X_2, each acting alone, are Y_1, Y_2, then the response to the scaled inputs K_1X_1, K_2X_2 applied simultaneously is Jeśli odpowiedzi na wejściowe sygnały X_1, X_2 działające niezależnie są Y_1, Y_2, to po ich przeskalowaniu i podaniu na wejście jednocześnie $Y = Y_1 + Y_2 = K_1X_1 + K_2X_2$. Linearity implies both superposition and proportionality. Liniowość implikuje tak superpozycję jak i proporcjonalność.
Liniowy obwód rezystorowy	Linear resistive circuit	Circuit consisting of only linear resistors and independent sources. Such circuit is a reciprocal circuit. Obwód zbudowany z liniowych oporników i źródeł niezależnych. Taki obwód podlega zasadzie wzajemności.
Macierz konduktancyjna	Conductance matrix G	*See Nodal analysis*
Macierz konduktancyjna wielobiegunnika	Conductance matrix of *m*-terminal element G	Matrix that relates the terminal currents with the terminal voltages: Macierz wiążąca prący zaciskowe i napięcia zaciskowe: $\mathbf{I} = \mathbf{G} \cdot \mathbf{U}$.
Magnes trwały	Permanent (natural) magnet	Magnet made of the iron compound magnetite (Fe_3O_4). Magnes zbudowany z magnetytu.

Polish	English	Description (opis)
Metoda potencjałów węzłowych	Nodal analysis	KCL equations with currents expressed by node voltages and branch parameters: Równania I prawa Kircchoffa, w których prądy wyrażono przez potencjały węzłowe i parametry układu: $\mathbf{GV} = \mathbf{I}_s$ G – conductance matrix (macierz konduktancyjna) $\mathbf{I}_s$ - vector of source currents of individual nodes wektor pradów źródłowych poszczególych węzłów.
Moc	Power P or p	Energy per unit period of time, Energia przypadająca na jednostkę czasu, $p = dw/dt$; $w = \int_0^t pdt$. In the DC case: $P = UI$; $w = UI \cdot t$. Unit: watt $[\mathrm{W}] = [\mathrm{J/s}]$.
Moc bierna	Reactive power Q	Power oscillating between the circuit reactive elements (capacitors and inductors) and the power source, $Q = UI \sin\varphi$. Moc oscylująca między reaktancjami (cewka, kondensator) a źródłem. Unit: var, volt-ampere-reactive [VAr].
Moc chwilowa	Instantaneous power p	Product of the voltage u and the current i flowing into the positive voltage terminal of two-terminal element, $p = ui$. Iloczyn wartości chwilowych prądu i napięcia dwójnika.
Moc czynna	Average or real power P	Average value of the instantaneous power in the AC circuit. Represents the power delivered by the source or absorbed by the circuit. Unit: watt [W]. Wartość średnia mocy chwilowej. Moc, którą odbiornik pobiera ze źródła i zamienia na pracę lub ciepło. $P = 1/T \int_0^T pdt = UI \cos\varphi$, $\cos\varphi = pf$ is the so called power factor.
Moc pozorna	Apparent power S	Power that defines the maximum capacity of the sinusoidal source, $S = P_{\mid pf=1} = UI$. Moc określająca wydajność źródła (moc znamionowa). Unit: volt-ampere $[\mathrm{V} \cdot \mathrm{A}]$.

Polish	English	Description (opis)
Moc zespolona	Complex power $S(j\omega)$	Sum of the average power and the reactive power expressed as a complex number, Suma mocy czynnej i biernej, przedstawiona jako liczba zespolona. $S(j\omega) = P(\omega) + jQ(\omega) = S(\omega)\exp[j\varphi(\omega)]$
Model obwodu	Circuit model	Approximation of a real circuit, interconnection of ideal elements (practical elements are modeled by ideal elements). Aproksymacja obwodu rzeczywistego, w którym elementy rzeczywiste reprezentują modele zbudowane z elementów idealnych.
Moduł	Magnitude or modulus	Magnitude (modulus) *F* of a complex number: Moduł liczby zepolonej: $\mathbf{F} = Fe^{j\alpha}$.
Najgorszy przypadek	*Worst Case*	Case when deviation of circuit variable, caused by the design tolerances, reaches its maximum $\Delta F_{\max}$. Przypadek największej odchyłki funkcji układowej od wartości nominalnej, spowodowanej tolerancjami projektowymi parametrów obwodu.
Napięcie (różnica potencjałów)	Voltage (potential difference) U or u	Work required to move a unit charge $Q = 1$ [C] from one point A to another B, Praca niezbędna do przemieszczenia ładunku jednostkowego z punktu A do punktu B, $U_{\text{AB}} = W_{\text{AB}}\big\|_{Q=1}$. Unit: volt [V].
Napięcie biegu luzem	Open-circuit voltage E_o	Voltage that appears between two terminals of a circuit or element in the open-circuit condition. Napięcie na zaciskach dwójnika dla biegu luzem.
Napięcie fazowe	Phase voltage V_i	Voltage appearing at a phase impedance. For the four-wire system, the voltage between line 1 or 2 or 3 and the neutral. Napięcie na impedancji fazowej. Dla układu cztero-przewodowego, napięcie między przewodem fazowym a przewodem zerowym.
Napięcie lub prąd symboliczny	Phasor voltage or current $U(j\omega)$ or $I(j\omega)$	Complex number associated with sinusoidal voltage or current, Liczba zespolona opisująca sinusoidalny prąd lub napięcie. $U(j\omega) = U(\omega)\exp\alpha_u(\omega)$ or $I(j\omega) = I(\omega)\exp\alpha_i(\omega)$

Polish	English	Description (opis)
Napięcie między-przewodowe	Line (-to-line) voltage U_{ij}	Voltage between any two lines *i* and *j*, except the neutral one. For wye connection: $U = \sqrt{3}E$. Napięcie między przewodami fazowymi.
Natężenie pola elektrycznego	Electric field intensity	Vector uniquely defining the electric field in its every point, Wektor jednoznacznie określający pole elektryczne w każdym jego punkcie, $K = F / Q$. Unit: $[\text{N/C}] = [\text{V/m}]$.
Natężenie pola magnetycznego	Magnetic field intensity (magnetizing force) *H*	Force per unit pole (magnetic body), number of ampere-turns per length of magnetic element Siła działająca na dipol jednostkowy, stosunek amperozwojów do długości elementu obwodu magnetycznego $H = F_m / M = I\,z / l$ Unit: $[\text{N/Wb}] = [\text{At/m}]$.
Neper	Neper *Np*	*See Propagation constant*
Nieliniowy obwód rezystorowy	Noninear resistive circuit	Circuit that contains at least one nonlinear resistor. Obwód zawierający przynajmniej jeden element nieliniowy.
Obciążenie symetryczne	Balanced load (circuit)	Load that has three identical impedances connected in a Y or Δ configuration. Obciążenie składające się z trzech identycznych impedancji skojarzonych w gwiazdę lub trójkąt.
Obszar sprawności	Acceptability region	Region in the parameter space $\Re^P$ with boundaries designated by the design constraints on circuit variables: $F_j^{\min}, F_j^{\max}$. Region w przestrzeni parametrów, którego ograniczenia wyznaczone są specyfikacjami projektowymi.
Obszar tolerancji	Tolerance region (box)	Parallelepiped in the parameter space $\Re^P$ with planes parallel with the coordinate axes, designated by the tolerance margins of all circuit parameters. Równoległościan w przestrzeni parametrów wyznaczony tolerancjami projektowymi.
Obwód aktywny	Active circuit	Circuit that contains at least one active element, independent source. Obwód zawierający przynajmniej jedno źródło energii (niesterowaną SEM i/lub SPM)
Obwód idealny	Ideal circuit	Circuit built of elements as given by the design and nominal values of parameters. Obwód podany w projekcie o nominalnych wartościach parametrów.

Polish	English	Description (opis)
Obwód pasywny	Passive circuit	Circuit consisting of resistors, capacitors and inductors, that can only store and/or dissipate energy. Obwód zbudowany z elementów *RLC*, który może tylko rozpraszać energię lub/i ją magazynować.
Obwód pierwszego rzędu	First-order circuit	Circuit that contains only one energy storage element, either capacitor or inductor. Obwód zawierający tylko jeden element magazynujacy energię, cewkę lub kondensator.
Obwód planarny	Planar circuit	Circuit whose diagram (graph) can be drawn on a plane without branches crossing each other. Obwód, którego graf można tak narysować by gałęzie się nie przecinały.
Obwód podlegający zasadzie wzajemności	Reciprocal circuit	Circuit whose node equations have symmetric conductance matrix, $G_{ij} = G_{ji}$. Obwód o symetrycznej macierzy konduktancyjnej. *See Linear resistive circuit.*
Obwód rzeczywisty	Practical circuit	Circuit built of practical elements, with parameters given by the design tolerances. Obwód uwzględniający modele elementów o parametrach zadanych tolerancjami projektowymi.
Obwód tłumiony krytycznie	Critically damped response	Nonoscillatory response of the *RLC* circuit, but on the verge of becoming oscillatory – condition that exists when two poles of the response are identical. Aperiodyczna odpowiedź obwodu *RLC*, na granicy odpowiedzi periodycznej – pierwiastek podwójny odpowiedzi.
Obwód zastępczy	Equivalent circuit	Circuit whose terminal characteristics remain identical to those of the original circuit. The original circuit can be substituted by the equivalent without affecting the voltages and the currents in any attached circuit. Obwód, dwuzaciskowy, którego charakterystyka *I-U* jest identyczna z charakterystyką obwodu oryginalnego. Jego zastąpienie obwodem zastępczym nie wpływa na pracę reszty obwodu.
Oczko	Mesh	Loop that does not contain any other loop within it. Pętla, która nie zawiera wewnątrz żadnej innej pętli.
Odcięcie	Cutset	Closed line around one or more nodes, crossing two or more branches, each branch only once. Linia zamknięta okalająca jeden lub więcej węzłów, przecinająca dwie lub więcej gałęzi, każdą tylko raz.

Polish	English	Description (opis)
Odpowiedź amplitudowa	Magnitude response (gain ratio) $K(\omega)$	Frequency characteristic – ratio of effective values of the output to the input phasor signals, Charakterystyka częstotliwościowa – stosunek wartości symbolicznych sygnału wyjściowego do wejściowego $K(\omega) = Y(\omega) / X(\omega)$
Odpowiedź impulsowa	Impulse response $k = k(t)$	Inverse transform of the transfer function $K(s)$, output signal of a circuit when the input is the unit impulse, with no initial stored energy in a circuit. Transformata odwrotna operatorowej funkcji przejścia, odpowiedź układu na jednostkowy impuls Dirac'a.
Odpowiedź naturalna	Natural or zero-input response	Response to the initial condition, when all source excitations are set to zero. Odpowiedź na warunek(ki) początkowe, po odłączeniu źródła(eł).
Odpowiedź nie tłumiona	Undamped response	Transient response in *LC* (resistiveless) circuit. Odpowied obwodu *LC* (bezoporowego).
Odpowiedź silnie tłumiona	Overdamped response	Nonoscillatory response of the *RLC* - circuit condition that exists when all poles of the response are real and distinct. Aperiodyczna odpowiedź układu *RLC* – warunek spełniony gdy oba bieguny odpowiedzi są rzeczywiste i różne od siebie.
Odpowiedź słabo tłumiona	Underdamped response	Periodic response of the 2nd order circuit – condition that exists when two poles of the response are complex conjugates. Odpowiedź periodyczna obwodu słabo tłumionego, ma miejsce gdy w odpowiedzi występują dwa bieguny zespolone sprzężone.
Odpowiedź wymuszona	Forced or zero-state response	Response to the source excitation, when all initial conditions are set to zero. Odpowiedź obwodu z zerowymi warunkami początkowymi po dołączeniu źródła.
Odpowiedź zupełna	Complete response	Sum of natural and forced responses. Suma odpowiedzi naturalnej i wymuszonej.
Okres oscylacji	Period of oscillations T	Time between two subsequent maximum points of a periodic (sinusoidal) waveform. Czas pomiędzy kolejnymi maksimami fali sinusoidalnej.
Opór (przewodność) krytyczny (a)	Critical resistance (conductance) $R_c\,(G_c)$	Resistance (conductance) of the critically damped series (parallel) *RLC* circuit: Oporność (przewodność) krytycznie tłumionego szeregowego (równoległego) obwodu *RLC*: $R_c = 2\sqrt{L/C}$ ($G_c = 2\sqrt{C/L}$).

Polish	English	Description (opis)
Opór magnetyczny	Reluctance or magnetic resistance R_m	Parameter describing linear magnetic element, ratio of the magnetic voltage drop to the flux flowing Parametr opisujacy liniowy element magnetyczny $R_m = l/(\mu_r \mu_0 S)$ l - mean length of a core, S - its cross-section area. średnia droga magnetyczna, pole przekroju
Opornik	Resistor	Element whose primary purpose is to introduce resistance, i.e. to impede current flow and voltage drop into a circuit. Resistor converts electric energy into heat. Element zamieniający energię elektryczną na ciepło.
Oporność (opór)	Resistance R	Coefficient of proportionality between the voltage and the current of linear resistor. Unit: ohm [Ω]=[V/A]. Współczynnik proporcjonalności między prądem a napięciem na liniowym oporniku.
Oporność zastępcza lub wewnętrzna	Equivalent or total or Thevenin or internal resistance R_t	Resistance of the two-terminal circuit when all internal independent sources are deactivated, also resistance that appears in the Thevenin equivalent of a practical source Oporność widziana z zacisków obwodu dwuzaciskowego po wyzerowaniu jego źródeł, rezystancja schematu zastępczego Thevenina.
Parametr (stała) obwodu	Circuit parameter (constant) P or X	Parameter that defines the circuit element, such as resistance R, capacitance C, inductance L, etc. Parametr definiujący element, np. rezystancja, pojemność, itd.
Parametry jednostkowe linii	Line primary parameters	Per unit length parameters: Parametry na jednostkę długości: R [Ω/m], G [S/m], L [H/m], C [F/m].
Parametry wtórne linii	Line secondary parameters	Functions of primary parameters, such as: characteristic impedance, propagation constant, etc. Parametry charakterystyczne linii, wyrażone przez parametry jednostkowe.
Pasmo częstotliwości	Bandwidth $\Delta\omega$	Range of frequencies that lie between the two frequencies where the magnitude of the gain is equal to $1/\sqrt{2}$ of the maximum. Pasmo częstotliwości wyznaczone częstotliwościami, dla których wzmocnienie czwórnika wynosi $1/\sqrt{2}$ maksimum.
Pętla	Loop	Closed path formed by two or more branches. Ścieżka zamknięta zbudowana z dwóch lub wiecej gałęzi.

Polish	English	Description (opis)
Pojemność	Capacitance C	Constant of proportionality between the capacitor charge and the voltage: $Q = CU$. Współczynnik proporcjonalności między ładunkiem zgromadzonym w kondensatorze a napięciem na nim. Unit: farad $[\mathrm{F}] = [\mathrm{A \cdot s/V}]$.
Pojemność bocznikująca	Stray or parasitic or shunting capacitance	Unwanted capacitance that exists between element terminals or between a terminal and ground. Pasożytnicza pojemność między zaciskami elementu.
Pojemność między węzłem a masą	Stray capacitance	Unwanted capacitance between a circuit node (element terminal) and ground. Pasożytnicza pojemność między zaciskiem elementu a masą układu.
Połączenie równoległe	Parallel connection	Arrangement of elements so that each element has the same voltage appearing across it. Połączenie, dla którego wszystkie elementy podłączone są na to samo napięcie.
Połączenie szeregowe	Series connection	Circuit of a series of elements connected so that the same current passes through each element. Obwód zbudowany z elementów połączonych tak by płynął przez nie ten sam prąd.
Pole elektryczne	Electric field	Region in space wherein a test charge Q experiences an electric force F_e. Przestrzeń, w której na umieszczony ładunek działa siła.
Pole magnetyczne	Magnetic field	Region in space where a force F_m acts upon a magnetic body M. Przestrzeń, w której na umieszczone dipole magnetyczne działają siły.
Potencjał	Potential (node voltage) V or v	Voltage between the reference point P and the other one A: Napięcie między węzłem odniesienia a danym węzłem. $V_\mathrm{A} = U_\mathrm{AP}$; $U_\mathrm{AB} = V_\mathrm{A} - V_\mathrm{B}$.
Prąd	Current I or i	Rate of flow of charge, $I = \Delta Q / \Delta t$ or $i = dq/dt$. Szybkość przepływu ładunków. Units: ampere $[\mathrm{A}] = [\mathrm{C/s}]$.
Prąd konwencjonalny	Conventional current flow	Flow of positive charges. Przepływ ładunków dodatnich.
Prąd sinusoidalny	Alternating current i	Sinusoidal time-varying current $i = I_m \sin(\omega t + \alpha_i)$, I_m - amplitude or peak value, ω - angular frequency, α_i - initial phase angle.

Polish	English	Description (opis)
Prąd stały	Direct current (DC) I	Current constant in time. Prąd nie zmieniający się w czasie.
Prąd zwarcia	Short-circuit current J_s	Current passing an active element (practical source) in the short-circuit condition. Prąd płynący przez zaciski dwójnika w stanie zwarcia.
Prawo Faradaya	Faraday's law	When the magnetic flux linking a coil changes, a voltage proportional to the rate of flux change is induced in the coil: Zmiana strumienia magnetycznego przenikającego cewkę powoduje wyindukowanie napięcia proporcjonalnego do szybkości zmian: $u = z\,d\phi / dt = d\phi_t / dt$.
Prawo Ohma	Ohm's Law	The voltage across the terminals of a resistor is related to the current flowing into the positive terminal as: $U = RI$. Napięcie na zaciskach opornika proporcjonalne jest do pradu, ze współczynnikiem *R*.
Prędkość rozchodzenia się fali	Propagation velocity v	*See Propagation function*
Proporcjonalność	Proportionality	When an input to a linear resistive circuit is acting alone, then scaling the input by a constant *K* implies that the response is also scaled by *K*. Dla pojedynczego pobudzenia ukadu, jego przemnożenie przez stałą *K* powoduje przemnożenie odpowiedzi przez tą sama stałą. *See Linearity*
Przekładnia	Turns ratio n	Ratio $n = z_2 / z_1$, where z_2 and z_1 are turns in secondary and primary coil of an ideal transformer. Stosunek liczby zwojów uzwojenia wtórnego do liczby zwojów uzwojenia pierwotnego.
Przenikalność magnetyczna	Magnetic permeability μ_r	*See Magnetization curve*
Przesunięcie fazowe	Phase shift φ	Phase angle between an element voltage and its current, with current as the reference, $\varphi = \alpha_u - \alpha_i$. Przesunięcie fazowe miedzy napięciem na elemencie a prądem, liczone od prądu.
Przesunięcie fazowe	Phase shift per distance β	*See Propagation constant*

Polish	English	Description (opis)
Przewód zerowy	Neutral line	For a star connection, line connecting a common junction point of a generator and a load star. Dla skojarzenia gwiazda-gwiazda, przewód łączący środki gwiazd.
Przewodność (konduktancja)	Conductance G	Reciprocal of the resistance, $G = 1/R$. Odwrotność oporności. Unit: siemens $[\mathrm{S}] = [\mathrm{A/V}]$.
Pulsacja (częstotliwość kątowa)	Angular (radian) frequency ω	*See Alternating current, Frequency*
Punkt pracy	Operating or Q-point (Quiescent point)	The point on an element $I - U$ characteristic at which the circuit Kirchhoff's laws are satisfied. The coordinates at this point are the operating voltage U^Q and the operating current I^Q. Punkt na charakterystyce elementu, dla którego spełnione są w obwodzie prawa Kircchoffa.
Reaktancja	Reactance $X(\omega)$	*See Impedance in phasor-domain*
Reguła prawej dłoni	Right hand rule	If a current-carrying conductor is grasped in the right hand with the thumb pointing in the direction of the conventional current, the fingers will then point in the direction of the magnetic lines of flux. Jeśli objąć przewód prawą dłonią tak by kciuk wskazywał przepływ prądu, to pozostałe palce wskażą kierunek wytworzonego strumienia magnetycznego.
Rezonans	Resonance	Condition in a two-terminal circuit, occurring at the resonant frequency, when the equivalent impedance $Z(j\omega)$ or admittance $Y(j\omega)$ becomes a real number (circuit becomes non-reactive). Stan pracy dwójnika, w którym impedancja zastępcza posiada tylko część rzeczywistą (urojona jest równa zero).
Rezystancja	Resistance in phasor-domain $R(\omega)$	*See Impedance in phasor-domain*
Równanie elementu	Element law or $i - u$ relationship	Graphical or functional representation of a two-terminal element: Graficzny lub algebraiczny opis element dwuzaciskowego. $i = f(u), u = f^{-1}(i)$.

Polish	English	Description (opis)
Równległy obwód rezonansowy	Parallel resonant circuit	Circuit with a resistor, capacitor and inductor in parallel. Obwód, w którym element *R*,*L* i *C* połączone są równolegle.
Rozwarcie, bieg luzem	Open-circuit (oc)	Condition that exists when the current between two terminals is zero, irrespective of the voltage across the terminals. Stan pracy, przy którym zaciski dwójnika są rozwarte = nie płynie przezeń prąd.
Schemat obwodu	Circuit diagram	Drawing that shows schematically the inter-connection of circuit components represented by their graphic standard symbols. Schemat połączeń elementów obwodu, z zachowaniem standardowych oznaczeń.
Schemat zastępczy Nortona	Norton equivalent	Independent current source J_s or $J_s(j\omega)$ in parallel with a conductance G_t or admittance $Y_t(j\omega)$. Źródło prądowe (prąd zwarcia) połączone równolegle z opornością wewnętrzną.
Schemat zastępczy Thevenina	Thevenin equivalent	Independent voltage source E_o or $E_o(j\omega)$ in series with a resistance R_t or impedance $Z_t(j\omega)$ Schemat zastępczy powstały z szeregowego połączenia SEM biegu luzem z opornością wewnętrzną dwójnika.
Siemens	Siemens S	*See Conductance*
Siła elektromotoryczna	Electromotive force (emf)	*See Independent ideal voltage source*
Siła magnetomotoryczna	Magnetomotive force (MMF) *F*	Product of the current *I* passing through a coil and number of its turns: $F = I \cdot z$. Iloczyn prądu płynącego przez cewkę i liczby zwojów. Unit: ampere-turns [At].
Skojarzenie w gwiazdę	Star or wye (Y) connection	Individual phase windings/loads are joined in a common junction point. Wye (Y) connection in case of a three-phase system. Wszystkie fazy mają wspólny zacisk.
Skojarzenie w trójkąt	Mesh or delta (Δ) connection	Individual phase windings/loads are connected to form a closed path. Delta (Δ) connection in case of a three-phase system. Wszystkie fazy tworzą oczko.
Skok jednostkowy	Unit Step function $\mathbf{1}(t)$	Dimensionless function of time that is zero for $t < 0$ and unity for $t \geq 0$. Bezwymiarowa funkcja przyjmująca wartość 1 dla czasów większych od 0, wartość 0 dla pozostałych czasów.

Polish	English	Description (opis)
Sprawność przy przekazywaniu mocy	Efficiency of power transfer	Ratio of the power delivered to the load P_l to the power supplied by the source P_E: Stosunek mocy pobieranej przez odbiornik do mocy wydawanej przez źródło: $\eta = P_l / P_E = R_l /(R_l + R_t)$. At the maximum power transfer: $\eta\% = 50\%$. 50% w warunkach dopasowania energetycznego.
Sprzężenie idealne	Unity coupling	Coupling with $k = 1$. Sprzężenie ze współczynnikiem jednostkowym. *See Mutual inductance*
Stała czasowa	Time constant T	Parameter of exponentially decaying or rising response. After one time constant the response drops to $\approx 38\%$ of its initial value or rises to $\approx 62\%$ of its end value, Parametr krzywej wykładniczej. Po upływie jednej stałej czasowej krzywa zanika do 38% wartości początkowej lub narasta do 62% wartości końcowej. for *RL* circuit: $T = L / R_t$, for *RC* circuit: $T = R_t C$.
Stała propagacji	Propagation constant $\gamma(s)$ or $\gamma(j\omega)$	$\gamma(s) = \sqrt{(R + sL)(G + sC)}$ For the distortionless line: Dla linii bezstratnej: $\gamma(s) = \alpha + s / v$ or $\gamma(j\omega) = \alpha + j\beta$ $\alpha = \sqrt{RG}$ - attenuation constant tłumienność in [1/m] or neper per meter [Np/m], $v = 1/\sqrt{LC}$ - propagation (phase) velocity szybkość propagacji in [m/s], $\beta = \omega\sqrt{LC} = \omega / v$ - phase shift per distance przesunięcie fazowe.
Strumień całkowity	Flux linkage	*See Total flux*
Strumień całkowity	Total flux or flux linkage Φ_t	The total flux linked by the *z* turns of the coil, Iloczyn strumienia i liczby zwojów, $\Phi_t = z\Phi$. *See Magnetic flux*

Polish	English	Description (opis)
Strumień magnetyczny	Magnetic flux Φ or ϕ	Total number of lines of magnetic force $\Phi = B \cdot S$ Unit: weber $[\mathrm{Wb}] = [\mathrm{V} \cdot \mathrm{s}]$. *See Magnetic flux density*
Superpozycja	Superposition	When a number of inputs are applied simultaneously to a linear circuit, the response is the sum of responses due to each input acting alone. W liniowym układzie o wielu pobudzeniach odpowiedź można wyznaczyć sumując odpowiedzi na każde z pobudzeń z osobna, przy pozostałych wyzerowanych. *See Linearity*
Susceptancja	Susceptance $B(\omega)$	*see Admittance in phasor-domain*
Sygnał elektryczny *u* lub *i*	Electric signal *u* or *i*	Voltage or current varying in time in a manner that conveys information. Napięcie lub prąd zmienny w czasie, niosący pewna informację.
Sygnał wejściowy	Input signal X or x or $X(s)$ or $X(j\omega)$	Excitation of a system. Pobudzenie układu.
Sygnał wyjściowy	Output signal Y or y or $Y(s)$ or $Y(j\omega)$	Response of a system. Odpowiedź układu.
Sygnał zmienny w czasie	Signal variable in time $f(t) = f$	Real valued function of time; waveform that conveys information, denoted by a small letter. Funkcja czasu, oznaczana mała literą.
System	System	Interconnection of electrical elements and circuits to achieve a desired objective. Połączenie elementów i obwodów dla uzyskania pożądanego celu.
Szeregowy obwód rezonansowy	Series Resonant Circuit	Circuit with a series connection of a resistor, capacitor and inductor. Szeregowy obwód *RLC*.
Tesla	Tesla T	*See Magnetic flux density*
Tłumienność	Attenuation constant α	*See Propagation constant*

Polish	English	Description (opis)
Tłumiona częstotliwość drgań własnych	Damped resonant frequency ω_d	Frequency of oscillation of the underdamped response: Częstotliwość oscylacji odpowiedzi słabo tłumionej: $\omega_d = \sqrt{\omega_r^2 - \alpha^2}$.
Tolerancja parametru	Parameter tolerance tol_X	Ratio of the parameter design deviation to its nominal value: Stosunek odchyłki projektowej parametru do wartości nominalnej: $tol_X = \Delta X / X^n$.
Tożsamość Eulera	Euler identity	$e^{j\alpha} = \cos\alpha + j\sin\alpha$
Transformacja	Transformation	Conversion of a set of equations from one domain to another, e.g. from the *t*-domain to the *s*-domain. Konwersja równań z dziedziny czasu w dziedzinę operatorową
Transformacja impedancji	Impedance transformation	When the secondary of an ideal transformer is terminated in an impedance $Z_l(j\omega)$, the input impedance across the primary is Impedancja wejściowa transformatora idealnego obciążonego impedancją $Z_l(j\omega)$ wynosi $Z_{in}(j\omega) = Z_l(j\omega)/n^2$.
Transformata Laplace'a	Laplace transform $F(s)$	Transform of $f(t)$ into its *s*-domain form Transformata funkcji czasowej $F(s) = \boldsymbol{L}\{f(t)\}$.
Transformator	Transformer	Magnetic circuit with two or more multi-turn coils wound on a common core. Obwód magnetyczny o dwóch uzwojeniach.
Transformator idealny	Ideal transformer	Model of a transformer with i) resistiveless windings, ii) unity coupling, iii) primary and secondary reactances infinitely large compared to impedances connected to the transformer terminals. Transformator o bezoporowych uzwojeniach, idealnym sprzężeniu i nieskończenie dużych reaktancjach (przenikalności magnetycznej).
Transformator obniżający	Step-down transformer	Transformer of the turns ratio less than one. Transformator o przekładni mniejszej od 1.
Transformator podwyższający	Step-up transformer	Transformer of the turns ratio greater than one. Transformator o przekładni większej od 1.
Transmitancja	Transfer function (gain) for DC signals K	Ratio of the response (output signal) of a circuit to an excitation (input signal) Stosunek odpowiedzi układu do pobudzenia DC.

Polish	English	Description (opis)
Transmitancja operatorowa	Transfer function in *s*-domain $K(s)$	Ratio of the response (output signal) of a circuit to an excitation (input signal) expressed as a function of *s* (initial conditions are assumed to be zero). Stosunek operatorowej postaci sygnału wejściowego do sygnału wejściowego.
Twierdzenie Nortona	Norton's theorem	For any linear active two-terminal circuit its linear equivalent circuit can be found. This circuit consists of the parallel connection of a current source and total (equivalent) conductance (admittance), the current source is the short circuit current of the circuit, the conductance (admittance) is the conductance (admittance) at the terminals when all the independent sources are deactivated. Liniowy dwójnik aktywny zastąpić można schematem zastępczym Nortona: równoległym połączeniem SPM zwarcia i oporności widzianej z zacisków po wyzerowaniu źródeł. *See Deactivation of independent source*, *Norton equivalent*
Twierdzenie o splocie	Convolution theorem	Convolution of the impulse response and the input signal: Splot odpowiedzi impulsowej oraz sygnału wejściowego. $k(t)*x(t)=\int_0^{\infty} k(t-\tau)x(\tau)d\tau \ = \boldsymbol{L}\{K(s)\cdot X(s)\}$
Twierdzenie Thevenina	Thevenin's theorem	For any linear active two-terminal subcircuit its linear equivalent circuit can be found. This circuit consists of the series connection of a voltage source and total (equivalent) resistance (impedance): the voltage source is the open-circuit voltage of the subcircuit the resistance (impedance) is the resistance (impedance) at the terminals when all the independent sources are deactivated. Liniowy dwójnik aktywny zastąpić można schematem zastępczym Thevenina: szeregowym połączeniem SEM biegu luzem i oporności widzianej z zacisków po wyzerowaniu źródeł. *See Deactivation of independent source*
Układ bez strat	Lossless device	Device, such as ideal coil or capacitor or lossless line, in which energy can only be stored and never dissipated. Układ/element, który nie rozprasza energii, może ją tylko magazynować, taki jak idealna cewka lub kondensator.

Polish	English	Description (opis)
Układ całkujący	Integrator	Circuit (system) that provides integration of the input voltage (signal). Transfer function of an ideal integrator is Układ całkujący sygnał wejściowy. Funkcja przejścia idealnego układu całkującego: $K(s) = 1/sT$ T is the integration constant. T jest stałą całkowania.
Układ różniczkujący	Differentiator	Circuit (system) that provides differentiation of the input voltage (signal). Transfer function of an ideal differentiator is: Układ różniczkujący sygnał wejściowy. Funkcja przejścia idealnego układu różniczkującego: $K(s) = sT$ T is the differentiation constant T jest stałą różniczkowania
Układ ze stratami	Lossy device	Device that dissipates energy, such as resistor or lossy two-port (line). Układ/element, który rozprasza energię, taki jak rezystor, linia ze stratami.
Uzwojenie pierwotne	Primary coil (winding)	Coil shown on the left-hand side of the model of a transformer. Winding connected to a source. Cewka z lewej strony modelu transformatora – jej uzwojenie podłączone jest do źródła.
Uzwojenie wtórne	Secondary coil (winding)	Coil shown on the right-hand side of the model of a transformer. Winding connected to a load. Cewka z prawej strony modelu transformatora. Jej uzwojenie połączone jest z obciążeniem.
Var	Var	*See Reactive power*
Wartość skuteczna	Effective or rms (root-mean-square) value of voltage or current I or U	The DC voltage or current that delivers the same energy as the periodically varying voltage or current, a value for periodic waveform relating its heating effect to the DC value. Prąd (napięcie) stały, który powoduje wydzielanie tej samej energii jaką wydziela prąd periodyczny. $F = \sqrt{1/T \int_0^T f(t)^2 \, dt}$; $F = U$ or I, $f = u$ or i.
Warunek początkowy	Initial condition $i_L(0) = I_{L0}$ $u_C(0) = U_{C0}$	Current that flows through a coil at $t = 0$. Voltage drop across a capacitor at $t = 0$. Prąd cewki (napięcie kondensatora) w chwili rozpoczęcia stanu nieustalonego.

Polish	English	Description (opis)
Warunek przekazywania maksymalnej mocy	Maximum power transfer – DC case	The maximum power delivered by a source represented by its Thevenin equivalent is attained when the load resistance R_l is equal to the Thevenin (equivalent) resistance R_t . Dla źródła opisanego schematem Thevenina, w warunkach dopasowania energetycznego oporność obciążenia jest równa oporności wewnętrznej źródła.
Warunek przekazywania maksymalnej mocy czynnej	Maximum power transfer – AC case	If the source has the Thevenin equivalent impedance $Z_t(j\omega)$, then the maximum power is delivered to the load when its impedance is Dla źródła opisanego schamatem Thevenina warunek dopasowania energetycznego $Z_l(j\omega) = Z_t(j\omega)^*$
Weber	Weber Wb	*See Magnetic flux*
Węzeł	Node	Connection point between two or more branches. Number of circuit nodes is denotes as *n*. Punkt połączenia dwóch lub więcej gałęzi. Liczba wszystkich węzłów: *n*.
Wielobiegunnik	Multi-terminal element or circuit	Element or circuit with *m* terminals available for external connections. Element lub obwód o *m* zaciskach zewnętrznych.
Wrażliwość funkcji układowej	Sensitivity of circuit variable S_X^F	Sensitivity of *F* with respect to *X*: $S_X^F = (\partial F / \partial X)\big\|_{\mathbf{X}=\mathbf{X}^n}$
Wrażliwość względna	Relative sensitivity Sr_X^F	$Sr_X^F = (\partial F / \partial X)\big\|_{\mathbf{X}=\mathbf{X}^n} /(F^n / X^n)$ *See Sensitivity of circuit variable*
Wrota	Port	Pair of circuit terminals to which another subcircuit may be attached. Current entering one terminal is equal to the current leaving the other. Para zacisków – prąd wpływający do jednego z nich wypływa z drugiego.
Współczynnik fali stojącej	Standing wave ratio *S*	Ratio of the maximum to the minimum rms voltages along a line Stosunek maksymalnej amplitudy do minimalnej amplitudy fali stojącej. $S = U_{x\max} / U_{x\min} = (1+\|N\|)/(1-\|N\|)$

Polish	English	Description (opis)
Współczynnik mocy czynnej	Power factor *pf*	Ratio of an average power to an apparent power: Stosunek mocy czynnej do mocy pozornej: $pf = \cos\varphi = P/S$ *See Average power, Apparent power*
Współczynnik sprzężenia	Coupling coefficient *k*	*See Mutual inductance*
Współczynnik tłumienia (tłumienność)	Damping coefficient α	Coefficient that designates the rapidity of decay of the series (parallel) *RLC* circuit response, Współczynnik określający szybkość zanikania składowej zaburzeniowej odpowiedzi szeregowego (równoległego) obwodu *RLC*, $\alpha = R/2L$ ($\alpha = G/2C$).
Współczynniki odbicia	Reflection coefficients $M(s)$, $N(s)$	Coefficients of the incident and reflected waves, Współczynniki, z jakimi odbiją się fale wędrujące, $M(s) = \dfrac{Z(s) - Z_t(s)}{Z(s) - Z_t(s)}$; $N(s) = \dfrac{Z(s) - Z_l(s)}{Z(s) + Z_l(s)}$
Wykres Bodego	Bode (gain) plot	Plot of logarithmic-gain values in dB on a log-frequency base. Logarytmiczna zależność wzmocnienia układu (czwórnika) od częstotliwości.
Wykres wektorowy	Phasor diagram	Phasors expressed graphically in a complex plane. Wartości symboliczne skuteczne zobrazowane graficznie na płaszczyźnie fazowej.
Wyłączenie źródeł niezależnych	Deactivation of independent source	Zeroing of a source, $E = 0$ or $J = 0$: short-circuiting the voltage source, open-circuiting the current source. Wyzerowanie źródła: zwarcie SEM, rozwarcie SPM.
Wzmocnienie w skali logarytmicznej	Logarithmic gain $K_L(\omega)$	Gain in the logarithmic scale: $K_{\mathrm{dB}}(\omega) = 20\log_{10} K(\omega)$. Unit: decibel [dB]. *See Bode plot*.
Zaciski jednoimienne	Dot Convention	Currents entering the dotted ends are creating additive fluxes. Dotted ends have a positive voltage at the same time. Prądy wpływające do zacisków jednoimiennych powodują powstanie zgodnych strumieni. Napięcia zastrzałkowane są do tych zacisków.

Polish	English	Description (opis)
Zasada superpozycji	Superposition principle	For a linear circuit containing independent sources, the voltage across (or the current through) any element may be obtained by adding algebraically all the individual voltages (or currents) caused by each independent source acting alone with all other sources deactivated. Remark: Power can't be found by superposing power losses. W obwodzie liniowym, w którym działa wiele źródeł niezależnych, dowolny prąd lub napięcie wyznaczyć można sumując składowe wywołane przez każde ze źródeł z osobna, przy pozostałych wyzerowanych. Dla mocy zasada superpozycji nie obowiązuje. *See Deactivation of independent source*
Zasada wyodrębnienia	Separation (voltage/current substitution) principle	Two subcircuits connected in *m* nodes can be separated by means of $m-1$ pairs of voltage or current sources connected between the arbitrarily selected reference node and each of other $m-1$ nodes. Value of the voltage source connected between two nodes is equal to the original circuit voltage. Value of the current source equals the total current entering/leaving the node from/to one of the subcircuits. Dwa obwody połączone w *m* węzłach można odseparować od siebie włączając *m*–1 par sił elektromotorycznych między kolejne węzły a *m*-ty węzeł odniesienia, o wartościach jak przed wyodrębnieniem.
Zasada zachowania mocy/energii	Energy/power conservation principle	In any circuit the algebraic sum of DC powers, or instantaneous powers, in any linear circuit the algebraic sum of average powers, or reactive powers, or complex powers absorbed by all elements, is zero (negative power absorbed is equivalent to positive power supplied). W liniowym obwodzie algebraiczna suma mocy DC, mocy chwilowych, mocy czynnych, mocy biernych jest równa zero (ujemna moc pobierana jest dodatnią mocą wydawaną).
Zbiór oczek niezależnych	Set of independent loops	All meshes of a circuit. Their number: $l=b-n+1$ Zbiór wszystkich oczek.
Zbiór odcięć niezależnych	Set of independent cutsets	Cutsets around all individual nodes except the reference one, their number: $t=n-1$ Zbiór odcięć wokół wszystkich węzłów za wyjątkiem węzła odniesienia.

Polish	English	Description (opis)
Zera transmitancji	Zeros of transfer function q_k	Roots of the numerator polynomial of the transfer function $K(s)$. Pierwiastki wielomianu licznika operatorowej funkcji przejścia. *See Transfer function in s-domain*
Zmienna obwodowa	Circuit variable F	Any voltage, current, power, gain, etc. - a nonlinear function of circuit parameters. Napięcie, prąd, moc, etc. – nieliniowa funkcja parametrów obwodu.
Zmodyfikowana metoda potencjałów węzłowych	Modified nodal analysis	Modification in which the unknowns are not only the usual nodal voltages but also currents of resistiveless branches (ideal voltage sources and short-circuit elements). Modyfikacja polegająca na pozostawieniu w równaniach prądów gałęzi bezoporowych (SEM, amperomierz idealny).
Źródła sterowane	Dependent or controlled source	Current or voltage source that provides a current or voltage that depends on another voltage or current elsewhere in the circuit, Źródło prądowe lub napięciowe, którego wartość zależy od innego prądu lub napięcia, Voltage Controlled Voltage Source – VCVS Current Controlled Voltage Source – CCVS Voltage Controlled Current Source – VCCS Current Controlled Current Source – CCCS.
Zwarcie	Short-circuit (sc)	Condition that exists when the voltage across two terminals is zero, irrespective of the current between the two terminals. Stan pracy, przy którym zaciski dwójnika są zwarte = napięcie między nimi jest równe zero..

English	Polish	Description (opis)
Acceptability region	Obszar sprawności	Region in the parameter space $\Re^P$ with boundaries designated by the design constraints on circuit variables: $F_j^{\min}, F_j^{\max}$. Region w przestrzeni parametrów, którego ograniczenia wyznaczone są specyfikacjami projektowymi.
Active circuit	Obwód aktywny	Circuit that contains at least one active element, independent source. Obwód zawierający przynajmniej jedno źródło energii (niesterowaną SEM i/lub SPM)

English	Polish	Description (opis)
Active element	Element aktywny	Element that may deliver energy to a circuit. Element, który jest źródłem energii.
Admittance in *s*-domain $Y(s)$	Admitancja operatorowa	Reciprocal of $Z(s)$. Odwrotność impedancji
Admittance in phasor-domain $Y(j\omega)$	Admitancja symboliczna	Reciprocal of $Z(j\omega)$, $Y(j\omega) = G(\omega) + jB(\omega)$
Alternating current i	Prąd sinusoidalny	Sinusoidal time-varying current $i = I_m \sin(\omega t + \alpha_i)$, I_m - amplitude or peak value, ω - angular frequency, α_i - initial phase angle.
Ampere **A**	Amper	*See Current*
Ampere-turns $\mathbf{A_t}$	Amperozwoje	*See Magnetomotive force*
Amplitude (peak value) I_m, U_m	Amplituda (wartość szczytowa)	*See Alternating current*
Angular (radian) frequency ω	Pulsacja (częstotliwość kątowa)	*See Alternating current, Frequency*
Apparent power S	Moc pozorna	Power that defines the maximum capacity of the sinusoidal source, $S = P_{\|pf=1} = UI$. Moc określająca wydajność źródła (moc znamionowa). Unit: volt-ampere $[\mathrm{V} \cdot \mathrm{A}]$.
Artificial or temporary magnet	Elektromagnes	*See Electromagnet or solenoid coil*
Attenuation constant α	Tłumienność	*See Propagation constant*
Autotransformer	Autotransformator	Transformer that has windings both magnetically and electrically interconnected. Transformator, w którym jest tylko jedno uzwojenie, spełniające jednocześnie rolę pierwotnego i wtórnego.

English	Polish	Description (opis)
Average or real power ***P***	Moc czynna	Average value of the instantaneous power in the AC circuit. Represents the power delivered by the source or absorbed by the circuit. Unit: watt [W]. Wartość średnia mocy chwilowej. Moc, którą odbiornik pobiera ze źródła i zamienia na pracę lub ciepło. $P = 1/T\int_0^T p dt = UI\cos\varphi$, $\cos\varphi = pf$ is the so called power factor.
Backward waves	Fale powrotne	Reflected waves traveling from the line output to its input. Fala odbita od końca linii.
Balanced load (circuit)	Obciążenie symetryczne	Load that has three identical impedances connected in a Y or Δ configuration. Obciążenie składające się z trzech identycznych impedancji skojarzonych w gwiazdę lub trójkąt.
Band-Pass Filter (BPF)	Filtr pasmowo-przepustowy	Circuit that passes unimpeded all frequencies in a selected range of frequencies and rejects all frequencies outside this range. Obwód (czwórnik), który przepuszcza częstotliwości z zadanego pasma a nie przepuszcza wszystkich pozostałych.
Band-Stop Filter (BSF)	Filtr pasmowo-zaporowy	Circuit that rejects all frequencies in a selected range of frequencies and passes unimpeded all frequencies outside this range. Obwód (czwórnik), który nie przepuszcza częstotliwości z zadanego pasma a przepuszcza wszystkie pozostałe.
Bandwidth $\Delta\omega$	Pasmo częstotliwości	Range of frequencies that lie between the two frequencies where the magnitude of the gain is equal to $1/\sqrt{2}$ of the maximum. Pasmo częstotliwości wyznaczone częstotliwościami, dla których wzmocnienie czwórnika wynosi $1/\sqrt{2}$ maksimum.
Bode (gain) plot	Wykres Bodego	Plot of logarithmic-gain values in dB on a log-frequency base. Logarytmiczna zależność wzmocnienia układu (czwórnika) od częstotliwości.
Branch	Gałąź	Element or string of two-terminal elements connected between two nodes. Number of circuit branches is denoted as *b*. Element dwuzaciskowy lub kilka połączonych w szereg, włączony między dwa węzły. Liczba wszystkich gałęzi: *b*.

English	Polish	Description (opis)
Capacitance ***C***	Pojemność	Constant of proportionality between the capacitor charge and the voltage: $Q = CU$. Współczynnik proporcjonalności między ładunkiem zgromadzonym w kondensatorze a napięciem na nim. Unit: farad $[\mathrm{F}] = [\mathrm{A \cdot s/V}]$.
Capacitor	Kondensator	Two-terminal energy storage element, described by the equation: $Q = CU$. Element dwuzaciskowy magazynujący energię (ładunek). Jego równanie i-u oraz energia: Its law and energy stored are: $i = C\,du/dt$, $w = Cu^2/2$.
Characteristic impedance (resistance) $Z(s)$ or $Z(j\omega)$ **(ρ)**	Impedancja falowa	$Z(s) = \sqrt{(R + sL)/(G + sC)}$ For a distortionless line: Dla linii bezstratnej: $Z(s) = \rho = \sqrt{L/C} = const$, in [Ω].
Charge Q or q	Ładunek	Fundamental unit of matter responsible for electric phenomena, $Q = Q^{+}$ is the positive charge, Q^{-} is the negative charge. Like charges repel and unlike charges attract each other. Podstawowa jednostka materii odpowiedzialna za zjawiska elektryczne. Mamy ładunki dodatnie i ujemne: różnoimienne odpychają się, jednoimienne przyciągają. Units: coulomb [C] $= [\mathrm{A \cdot s}]$.
Circuit diagram	Schemat obwodu	Drawing that shows schematically the inter-connection of circuit components represented by their graphic standard symbols. Schemat połączeń elementów obwodu, z zachowaniem standardowych oznaczeń.
Circuit graph	Graf obwodu	Graphical representation of the circuit structure (component interconnections). Graph consists of branches connected in nodes. Graficzne zobrazowanie struktury obwodu (bez wnikania w rodzaj elementów) – *b* gałęzi połączonych w *n* węzłach.
Circuit model	Model obwodu	Approximation of a real circuit, interconnection of ideal elements (practical elements are modeled by ideal elements). Aproksymacja obwodu rzeczywistego, w którym elementy rzeczywiste reprezentują modele zbudowane z elementów idealnych.

English	Polish	Description (opis)
Circuit parameter (constant) ***P*** **or** ***X***	Parametr (stała) obwodu	Parameter that defines the circuit element, such as resistance *R*, capacitance *C*, inductance *L*, etc. Parametr definiujący element, np. rezystancja, pojemność, itd.
Circuit variable ***F***	Zmienna obwodowa	Any voltage, current, power, gain, etc. - a nonlinear function of circuit parameters. Napięcie, prąd, moc, etc. – nieliniowa funkcja parametrów obwodu.
Complete response	Odpowiedź zupełna	Sum of natural and forced responses. Suma odpowiedzi naturalnej i wymuszonej.
Complex power $S(j\omega)$	Moc zespolona	Sum of the average power and the reactive power expressed as a complex number, Suma mocy czynnej i biernej, przedstawiona jako liczba zespolona. $S(j\omega) = P(\omega) + jQ(\omega) = S(\omega)\exp[j\varphi(\omega)]$
Conductance in phasor-domain $G(\omega)$	Konduktancja	*See Admittance in phasor-domain*
Conductance ***G***	Przewodność (konduktancja)	Reciprocal of the resistance, $G = 1/R$. Odwrotność oporności. Unit: siemens $[\mathrm{S}] = [\mathrm{A/V}]$.
Conductance matrix **G**	Macierz konduktancyjna	*See Nodal analysis*
Conductance matrix of ***m*****-terminal element** **G**	Macierz konduktancyjna wielobiegunnika	Matrix that relates the terminal currents with the terminal voltages: Macierz wiążąca prący zaciskowe i napięcia zaciskowe: $\mathbf{I} = \mathbf{G} \cdot \mathbf{U}$.
Continuity of capacitor voltage	Ciągłość napięcia na kondensatorze	Voltage on a capacitor is always continuous, even though the current may be discontinuous. Napięcie na kondensatorze jest ciągłą funkcją czasu, prąd może być funkcją nieciągłą. W szczególności: In particular: $u_C(0_-) = u_C(0_+) = U_{C0}$.
Continuity of inductor current	Ciągłość prądu na cewce	Current through a coil is always continuous, even though the voltage may be discontinuous. Prąd cewki jest ciągłą funkcją czasu, napięcie może być funkcją nieciągłą. W szczególności: In particular: $i_L(0_-) = i_L(0_+) = I_{L0}$.
Conventional current flow	Prąd konwencjonalny	Flow of positive charges. Przepływ ładunków dodatnich.

English	Polish	Description (opis)
Convolution theorem	Twierdzenie o splocie	Convolution of the impulse response and the input signal: Splot odpowiedzi impulsowej oraz sygnału wejściowego. $k(t)*x(t)=\int_0^\infty k(t-\tau)x(\tau)d\tau \ =\boldsymbol{L}\{K(s)\cdot X(s)\}$
Coulomb **C**	kulomb	*See Charge*
Coupled coils	Cewki sprzężone	*See Mutual inductance*
Coupling coefficient ***k***	Współczynnik sprzężenia	*See Mutual inductance*
Critical resistance (conductance) $R_c\,(G_c)$	Opór (przewodność) krytyczny (a)	Resistance (conductance) of the critically damped series (parallel) *RLC* circuit: Oporność (przewodność) krytycznie tłumionego szeregowego (równoległego) obwodu *RLC*: $R_c=2\sqrt{L/C}$ ($G_c=2\sqrt{C/L}$).
Critically damped response	Obwód tłumiony krytycznie	Nonoscillatory response of the *RLC* circuit, but on the verge of becoming oscillatory – condition that exists when two poles of the response are identical. Aperiodyczna odpowiedź obwodu *RLC*, na granicy odpowiedzi periodycznej – pierwiastek podwójny odpowiedzi.
Current ***I* or *i***	Prąd	Rate of flow of charge, $I=\Delta Q/\Delta t$ or $i=dq/dt$. Szybkość przepływu ładunków. Units: ampere [A]=[C/s].
Current divider	Dzielnik prądów	Circuit of *n* parallel resistors that divides the input current *I* so that Obwód zbudowany z *n* oporników połączonych równolegle, dzielący prąd jak poniżej: $I_i=I\cdot G_i/\sum_{i=1}^{n}G_i$
Cut-off or break or corner frequency ω_c	Częstotliwość graniczna	The point where the asymptotic curve for its logarithmic gain exhibits a sharp change in a slope. Punkt, w którym następuje nagła zmiana nachylenia charakterystyki amplitudowej w skali logarytmicznej. *See High/Law Pass Filter*
Cutset	Odcięcie	Closed line around one or more nodes, crossing two or more branches, each branch only once. Linia zamknięta okalająca jeden lub więcej węzłów, przecinająca dwie lub więcej gałęzi, każdą tylko raz.

English	Polish	Description (opis)
Damped resonant frequency ω_d	Tłumiona częstotliwość drgań własnych	Frequency of oscillation of the underdamped response: Częstotliwość oscylacji odpowiedzi słabo tłumionej: $\omega_d = \sqrt{\omega_r^2 - \alpha^2}$.
Damping coefficient α	Współczynnik tłumienia (tłumienność)	Coefficient that designates the rapidity of decay of the series (parallel) *RLC* circuit response, Współczynnik określający szybkość zanikania składowej zaburzeniowej odpowiedzi szeregowego (równoległego) obwodu *RLC*, $\alpha = R/2L$ ($\alpha = G/2C$).
Deactivation of independent source	Wyłączenie źródeł niezależnych	Zeroing of a source, $E = 0$ or $J = 0$: short-circuiting the voltage source, open-circuiting the current source. Wyzerowanie źródła: zwarcie SEM, rozwarcie SPM.
Decade dec	Dekada	Frequency band whose endpoint is a factor of 10 larger than its beginning point. Przedział częstotliwości, którego górna wartość graniczna jest 10 razy większa od wartości dolnej.
Decibel dB	Decybel	log-based measure of gain. *See Logarithmic gain*
Dependent or controlled source	Źródła sterowane	Current or voltage source that provides a current or voltage that depends on another voltage or current elsewhere in the circuit, Źródło prądowe lub napięciowe, którego wartość zależy od innego prądu lub napięcia, Voltage Controlled Voltage Source – VCVS Current Controlled Voltage Source – CCVS Voltage Controlled Current Source – VCCS Current Controlled Current Source – CCCS.
Differentiator	Układ różniczkujący	Circuit (system) that provides differentiation of the input voltage (signal). Transfer function of an ideal differentiator is: Układ różniczkujący sygnał wejściowy. Funkcja przejścia idealnego układu różniczkującego: $K(s) = sT$ *T* is the differentiation constant *T* jest stałą różniczkowania
Dirac delta function $\delta(t)$	Delta Diraca	*See Unit Impulse function*

English	Polish	Description (opis)
Direct current (DC) ***I***	Prąd stały	Current constant in time. Prąd nie zmieniający się w czasie.
Distortionless line	Linia nieznieksztalcajaca	Line with parameters that satisfy Linia, której parametry spełniaja zależność $R/L = G/C$.
Dot Convention	Zaciski jednoimienne	Currents entering the dotted ends are creating additive fluxes. Dotted ends have a positive voltage at the same time. Prądy wpływające do zacisków jednoimiennych powodują powstanie zgodnych strumieni. Napięcia zastrzałkowane są do tych zacisków.
Effective or rms (root-mean-square) value of voltage or current ***I* or *U***	Wartość skuteczna	The DC voltage or current that delivers the same energy as the periodically varying voltage or current, a value for periodic waveform relating its heating effect to the DC value. Prąd (napięcie) stały, który powoduje wydzielanie tej samej energii jaką wydziela prąd periodyczny. $F = \sqrt{1/T\int_0^T f(t)^2\,dt}$; $F = U$ or *I*, $f = u$ or *i*.
Efficiency of power transfer	Sprawność przy przekazywaniu mocy	Ratio of the power delivered to the load P_l to the power supplied by the source P_E: Stosunek mocy pobieranej przez odbiornik do mocy wydawanej przez źródło: $\eta = P_l / P_E = R_l / (R_l + R_t)$. At the maximum power transfer: $\eta\% = 50\%$. 50% w warunkach dopasowania energetycznego.
Electric field	Pole elektryczne	Region in space wherein a test charge *Q* experiences an electric force F_e. Przestrzeń, w której na umieszczony ładunek działa siła.
Electric field intensity	Natężenie pola elektrycznego	Vector uniquely defining the electric field in its every point, Wektor jednoznacznie określający pole elektryczne w każdym jego punkcie, $K = F/Q$. Unit: $[\mathrm{N/C}] = [\mathrm{V/m}]$.
Electric Filter	Filtr elektryczny	Circuit designated to provide a magnitude gain or loss over a predefined range of frequencies. Obwód (czwórnik) przepuszczający lub tłumiący sygnał wejściowy w zadanym paśmie częstotliwości.

English	Polish	Description (opis)
Electric signal **u or i**	Sygnał elektryczny u lub i	Voltage or current varying in time in a manner that conveys information. Napięcie lub prąd zmienny w czasie, niosący pewna informację.
Electricity	Elektryczność	Physical phenomena arising from the existence of interaction of electric charges. Zjawiska jakie występują w wyniku oddziaływania na siebie ładunków.
Electromagnet or solenoid coil	Elektromagnes	Wire wound around the soft steel alloy core with the total number of z turns. It exhibits the magnetic field of the permanent magnet when energized, i.e. can be called the temporary or artificial magnet. Cewka o z zwojach, nawinięta na rdzeń stalowy – zachowuje się jak magnes stały gdy płynie przez nią prąd.
Electromotive force (emf)	Siła elektromotoryczna	*See Independent ideal voltage source*
Element law or $i-u$ **relationship**	Równanie elementu	Graphical or functional representation of a two-terminal element: Graficzny lub algebraiczny opis element dwuzaciskowego. $i = f(u),\, u = f^{-1}(i)$.
Energy W or w	Energia	Ability to perform work. Zdolność do wykonania pracy. Units: joul [J]; wattsecond $[\mathrm{W}\cdot\mathrm{s}]$; [cal] calory, $1[\mathrm{J}] = 1[\mathrm{W}\cdot\mathrm{s}] = 0.239$ [cal]. Instantaneous energy dissipated/supplied: Energia chwilowa dostarczana/pobierana: $w = \int_0^t p\,dt$ Instantaneous energy stored: Energia chwilowa zmagazynowana w cewce, kondensatorze: coil: $w_L = \frac{L i_L^2}{2}$, capacitor: $w_C = \frac{C u_C^2}{2}$.

English	Polish	Description (opis)
Energy/power conservation principle	Zasada zachowania mocy/energii	In any circuit the algebraic sum of DC powers, or instantaneous powers, in any linear circuit the algebraic sum of average powers, or reactive powers, or complex powers absorbed by all elements, is zero (negative power absorbed is equivalent to positive power supplied). W liniowym obwodzie algebraiczna suma mocy DC, mocy chwilowych, mocy czynnych, mocy biernych jest równa zero (ujemna moc pobierana jest dodatnią mocą wydawaną).
Equivalent characteristic – parallel connection	Charakterystyka zastępcza – połączenie równoległe	The $I-U$ characteristic for a parallel connection of elements can be obtained by graphically adding the currents of elements at various values of voltage. Charakterystykę zastępczą elementów nieliniowych połączonych równolegle otrzymać można sumując prądy w punktach załamania charakterystyk tych elementów.
Equivalent characteristic – series connection	Charakterystyka zastępcza – połączenie szeregowe	The $I-U$ characteristic for a series connection of elements can be obtained by graphically adding the voltages of elements at various values of current. Charakterystykę zastępczą elementów nieliniowych połączonych szeregowo otrzymać można sumując napięcia w punktach załamania charakterystyk tych elementów.
Equivalent circuit	Obwód zastępczy	Circuit whose terminal characteristics remain identical to those of the original circuit. The original circuit can be substituted by the equivalent without affecting the voltages and the currents in any attached circuit. Obwód, dwuzaciskowy, którego charakterystyka *I-U* jest identyczna z charakterystyką obwodu oryginalnego. Jego zastąpienie obwodem zastępczym nie wpływa na pracę reszty obwodu.
Equivalent or total or Thevenin or internal resistance R_t	Oporność zastępcza lub wewnętrzna	Resistance of the two-terminal circuit when all internal independent sources are deactivated, also resistance that appears in the Thevenin equivalent of a practical source Oporność widziana z zacisków obwodu dwuzaciskowego po wyzerowaniu jego źródeł, rezystancja schematu zastępczego Thevenina.

English	Polish	Description (opis)
Equivalent or total or Thevenin or internal impedance $Z_t(j\omega)$	Impedancja zastępcza lub wewnętrzna	Impedance of the two-terminal circuit when internal independent source is deactivated, also impedance that appears in the Thevenin equivalent of a practical source Impedancja widziana z zacisków obwodu dwuzaciskowego po wyzerowaniu jego źródeł, rezystancja schematu zastępczego Thevenina.
Euler identity	Tożsamość Eulera	$e^{j\alpha} = \cos\alpha + j\sin\alpha$
Farad **F**	Farad	*See Capacitance*
Faraday's law	Prawo Faradaya	When the magnetic flux linking a coil changes, a voltage proportional to the rate of flux change is induced in the coil: Zmiana strumienia magnetycznego przenikającego cewkę powoduje wyindukowanie napięcia proporcjonalnego do szybkości zmian: $u = z\, d\phi / dt = d\phi_t / dt$.
First-order circuit	Obwód pierwszego rzędu	Circuit that contains only one energy storage element, either capacitor or inductor. Obwód zawierający tylko jeden element magazynujacy energię, cewkę lub kondensator.
Flux linkage	Strumień całkowity	*See Total flux*
Forced or zero-state response	Odpowiedź wymuszona	Response to the source excitation, when all initial conditions are set to zero. Odpowiedź obwodu z zerowymi warunkami początkowymi po dołączeniu źródła.
Forward waves	Fale postępujące	Waves traveling from the line input to its output. Fale wędrujace od poczatku linii do jej końca.
Frequency f, ω	Częstotliwość	Frequency of oscillations in a periodic (sinusoidal) waveform, Częstotliwość fali sinusoidalnej, $f = 1/T$ Its unit is hertz $[\mathrm{Hz}] = [1/\mathrm{s}]$. Radian (angular) frequency, Częstotliwość kątowa, $\omega = 2\pi f$ Its unit is [rad/s].
Frequency (phasor) domain	Dziedzina częstotliwościowa	Mathematical domain where the set of possible values of AC variable (current or voltage) is expressed in terms of frequency. Dziedzina, w której prądy i napięcia wyrażane są w funkcji częstotliwości.

English	Polish	Description (opis)
Frequency Response $K(j\omega)$	Częstotliwościowa funkcja przejścia	Frequency dependent relation, in both gain and phase, between the input phasor signal and the output phasor signal - transfer function in frequency-domain Funkcja przejścia dla wartości symbolicznych skutecznych $K(j\omega) = K(\omega)\exp[j\varphi(\omega)] = Y(j\omega)/X(j\omega)$.
Half-power frequency ω_l, ω_u	Częstotliwość graniczna	Frequency at which the magnitude response is $1/\sqrt{2}$ times the maximum. Częstotliwość, dla której wzmocnienie wynosi $1/\sqrt{2}$ wartości maksymalnej.
Henry **H**	Henr	*See Inductance*
Hertz **Hz**	Hertz	*See Frequency*
High-Pass Filter (HPF)	Filtr górno-przepustowy	Filter that passes all frequencies above the cut-off frequency ω_c and rejects all frequencies below the cut-off frequency. Filtr, który przepuszcza wszystkie częstotliwości powyżej częstotliwości granicznej.
Ideal circuit	Obwód idealny	Circuit built of elements as given by the design and nominal values of parameters. Obwód podany w projekcie o nominalnych wartościach parametrów.
Ideal transformer	Transformator idealny	Model of a transformer with i) resistiveless windings, ii) unity coupling, iii) primary and secondary reactances infinitely large compared to impedances connected to the transformer terminals. Transformator o bezoporowych uzwojeniach, idealnym sprzężeniu i nieskończenie dużych reaktancjach (przenikalności magnetycznej).
Impedance in phasor-domain $Z(j\omega)$	Impedancja symboliczna	Ratio of the phasor voltage $U(j\omega)$ at a pair of element terminals to the phasor current $I(j\omega)$ flowing into the positive voltage terminal, Stosunek wartości symbolicznej skutecznej napięcia na zaciskach dwójnika do prądu, $Z(j\omega) = R(\omega) + jX(\omega) = Z(\omega)\exp\varphi(\omega)$.
Impedance in *s*-domain $Z(s)$	Impedancja operatorowa	Ratio of the voltage $U(s)$ at a pair of element terminals to the current $I(s)$ flowing into the positive voltage terminal. Stosunek operatorowej wartości napięcia na zaciskach dwójnika do prądu,

English	Polish	Description (opis)
Impedance transformation	Transformacja impedancji	When the secondary of an ideal transformer is terminated in an impedance $Z_l(j\omega)$, the input impedance across the primary is Impedancja wejściowa transformatora idealnego obciążonego impedancją $Z_l(j\omega)$ wynosi $Z_{in}(j\omega) = Z_l(j\omega)/n^2$.
Impulse response $k = k(t)$	Odpowiedź impulsowa	Inverse transform of the transfer function $K(s)$, output signal of a circuit when the input is the unit impulse, with no initial stored energy in a circuit. Transformata odwrotna operatorowej funkcji przejścia, odpowiedź układu na jednostkowy impuls Dirac'a.
Independent ideal current source J or j	Idealne (niezależne) źródło prądowe	Source that provides a current independent of other circuit variables. Źródło wymuszające przepływ prądu niezależnego od reszty obwodu, do którego zostało dołączone (siła prądomotoryczna – SPM).
Independent ideal voltage source E or e	Idealne (niezależne) źródło napięciowe	Source that provides a voltage independent of other circuit variables, (electromotive force – emf). Źródło wymuszające napięcie niezależne od reszty obwodu, do którego zostało dołączone (siła elektromotoryczna – SEM).
Inductance L	Induktancja	Constant of proportionality interrelating current passing a coil and the total flux: Współczynnik proporcjonalności pomiędzy prądem płynacym przez cewkę a strumieniem całkowitym jaki ten prąd wywołuje: $\phi_t = Li;\ L = z^2 / R_m$ Unit: henry [H] =[V s/A] .
Initial condition $i_L(0) = I_{L0}$ $u_C(0) = U_{C0}$	Warunek początkowy	Current that flows through a coil at $t = 0$. Voltage drop across a capacitor at $t = 0$. Prąd cewki (napięcie kondensatora) w chwili rozpoczęcia stanu nieustalonego.
Initial phase angle α_i, α_u	Faza początkowa	*See Alternating current*
Input impedance $Z_{in}(j\omega)$	Impedancja wejściowa	Impedance seen at port 1 (input) of a possibly terminated two-port. Impedancja widziana z zacisków wejściowych czwórnika.
Input signal X or x **or** $X(s)$ **or** $X(j\omega)$	Sygnał wejściowy	Excitation of a system. Pobudzenie układu.

English	Polish	Description (opis)
Instantaneous power ***p***	Moc chwilowa	Product of the voltage *u* and the current *i* flowing into the positive voltage terminal of two-terminal element, $p = ui$. Iloczyn wartości chwilowych prądu i napięcia dwójnika.
Integrator	Układ całkujący	Circuit (system) that provides integration of the input voltage (signal). Transfer function of an ideal integrator is Układ całkujący sygnał wejściowy. Funkcja przejścia idealnego układu całkującego: $K(s) = 1/sT$ *T* is the integration constant. *T* jest stałą całkowania.
Joul **J**	Dżul	*See Energy*
Kirchhoff's Current Law – KCL	I prawo Kirchhoffa	The algebraic sum of currents (constant or varying in time, in the *s*-domain or in the phasor-domain) entering and leaving the node (cutset) equals zero, "+" if current arrowhead is directed to the node, "–" otherwise. Algebraiczna suma prądów (stałych, zmiennych, operatorowych, symbolicznych) wpływajacych do odciecia jest równa zero, „+" jeśli strzałka prądu do odciecia, „–" gdy od odcięcia.
Kirchhoff's Voltage Law – KVL	II prawo Kirchhoffa	The algebraic sum of voltages (constant or varying in time, in the *s*-domain or in the phasor-domain) around a loop (any closed path) equals zero, "+" if voltage arrowhead is directed clockwise, "–" otherwise. Algebraiczna suma napięć (stałych, zmiennych, operatorowych, symbolicznych) zamkniętej ścieżki (oczka) jest równa zero, „+" jeśli strzałka zgodna z ruchem wskazówek zegara, „–" gdy przeciwna.
Laplace transform $F(s)$	Transformata Laplace'a	Transform of $f(t)$ into its *s*-domain form Transformata funkcji czasowej $F(s) = \boldsymbol{L}\{f(t)\}$.
Line (-to-line) voltage U_{ij}	Napięcie między-przewodowe	Voltage between any two lines *i* and *j*, except the neutral one. For wye connection: $U = \sqrt{3}E$. Napięcie między przewodami fazowymi.
Line length ***l***	Długość linii	Distance from the line input to its output. Odległość od poczatku do końca linii.

English	Polish	Description (opis)
Line primary parameters	Parametry jednostkowe linii	Per unit length parameters: Parametry na jednostkę długości: *R* [Ω/m], *G* [S/m], *L* [H/m], *C* [F/m].
Line propagation time τ	Czas propagacji	Time of propagation of a signal from the line input to the output or in the reverse direction, $\tau = l/v$. Czas propagacji sygnału od wejścia do wyjścia linii.
Line secondary parameters	Parametry wtórne linii	Functions of primary parameters, such as: characteristic impedance, propagation constant, etc. Parametry charakterystyczne linii, wyrażone przez parametry jednostkowe.
Linear resistive circuit	Liniowy obwód rezystorowy	Circuit consisting of only linear resistors and independent sources. Such circuit is a reciprocal circuit. Obwód zbudowany z liniowych oporników i źródeł niezależnych. Taki obwód podlega zasadzie wzajemności.
Linearity	Liniowość	When responses to inputs X_1, X_2, each acting alone, are Y_1, Y_2, then the response to the scaled inputs K_1X_1, K_2X_2 applied simultaneously is Jeśli odpowiedzi na wejściowe sygnały X_1, X_2 działające niezależnie są Y_1, Y_2, to po ich przeskalowaniu i podaniu na wejście jednocześnie $Y = Y_1 + Y_2 = K_1X_1 + K_2X_2$. Linearity implies both superposition and proportionality. Liniowość implikuje tak superpozycję jak i proporcjonalność.
Logarithmic gain $K_L(\omega)$	Wzmocnienie w skali logarytmicznej	Gain in the logarithmic scale: $K_{\mathrm{dB}}(\omega) = 20\log_{10} K(\omega)$. Unit: decibel [dB]. *See Bode plot*.
Loop	Pętla	Closed path formed by two or more branches. Ścieżka zamknięta zbudowana z dwóch lub wiecej gałęzi.
Lossless device	Układ bez strat	Device, such as ideal coil or capacitor or lossless line, in which energy can only be stored and never dissipated. Układ/element, który nie rozprasza energii, może ją tylko magazynować, taki jak idealna cewka lub kondensator.
Loss-less line	Linia bezstratna	Line with no energy loss, $R = 0$, $G = 0$. Linia, w której nie ma strat energetycznych.
Lossy device	Układ ze stratami	Device that dissipates energy, such as resistor or lossy two-port (line). Układ/element, który rozprasza energię, taki jak rezystor, linia ze stratami.

English	Polish	Description (opis)
Low-Pass Filter (LPF)	Filtr dolno-przezepustowy	Filter that passes all frequencies up to the cut-off frequency ω_c and rejects all frequencies above it. Filtr przepuszczający wszystkie częstotliwości aż do częstotliwości granicznej.
Magnetic field	Pole magnetyczne	Region in space where a force F_m acts upon a magnetic body *M*. Przestrzeń, w której na umieszczone dipole magnetyczne działają siły.
Magnetic field intensity (magnetizing force) ***H***	Natężenie pola magnetycznego	Force per unit pole (magnetic body), number of ampere-turns per length of magnetic element Siła działająca na dipol jednostkowy, stosunek amperozwojów do długości elementu obwodu magnetycznego $H = F_m / M = I z / l$ Unit: $[\mathrm{N/Wb}] = [\mathrm{At/m}]$.
Magnetic flux Φ or ϕ	Strumień magnetyczny	Total number of lines of magnetic force $\Phi = B \cdot S$ Unit: weber $[\mathrm{Wb}] = [\mathrm{V \cdot s}]$. *See Magnetic flux density*
Magnetic flux density ***B***	Indukcja magnetyczna	Ratio of the magnetic flux that passes perpendicularly through an area *S* to this area. Stosunek strumienia przenikajacego prostopadle dany przekrój do tego przekroju. Unit: tesla $[\mathrm{T}] = [\mathrm{V \cdot s/m^2}]$.
Magnetic permeability μ_r	Przenikalność magnetyczna	*See Magnetization curve*
Magnetization or *B-H* curve $B = f(H)$	Krzywa magnesowania	For ferromagnetic materials, $B = f(H)$. For diamagnetic or after linearization of the curve: Dla diamagnetyka po linearyzacji krzywej $B = \mu_r \mu_0 H$ μ_r - magnetic permeability of the material, przenikalność magnetyczna materiału $\mu_0 = 4\pi 10^{-7}$ $[\mathrm{V \cdot s/A \cdot m}]$ magnetic permeability of the free space. przenikalność magnetyczna w próżni

English	Polish	Description (opis)
Magnetomotive force (MMF) ***F***	Siła magnetomotoryczna	Product of the current I passing through a coil and number of its turns: $F = I \cdot z$. Iloczyn prądu płynącego przez cewkę i liczby zwojów. Unit: ampere-turns [At].
Magnitude or modulus	Moduł	Magnitude (modulus) F of a complex number: Moduł liczby zepolonej: $\mathbf{F} = Fe^{j\alpha}$.
Magnitude response (gain ratio) $K(\omega)$	Odpowiedź amplitudowa	Frequency characteristic – ratio of effective values of the output to the input phasor signals, Charakterystyka częstotliwościowa – stosunek wartości symbolicznych sygnału wyjściowego do wejściowego $K(\omega) = Y(\omega) / X(\omega)$
Matched generator line	Linia dopasowana na wejściu	Line with the generator (input) resistance equal to the characteristic resistance, for such line M=0. Linia, której oporność charakterystyczna równa jest oporności wewnętrznej generatora.
Matched load line	Linia dopasowana na wyjściu	Line with the load (output) resistance equal to the characteristic resistance, for such line N=0. Linia, której oporność charakterystyczna równa jest oporności obciążenia.
Maximum power transfer – AC case	Warunek przekazywania maksymalnej mocy czynnej	If the source has the Thevenin equivalent impedance $Z_t(j\omega)$, then the maximum power is delivered to the load when its impedance is Dla źródła opisanego schamatem Thevenina warunek dopasowania energetycznego $Z_l(j\omega) = Z_t(j\omega)^*$
Maximum power transfer – DC case	Warunek przekazywania maksymalnej mocy	The maximum power delivered by a source represented by its Thevenin equivalent is attained when the load resistance R_l is equal to the Thevenin (equivalent) resistance R_t . Dla źródła opisanego schematem Thevenina, w warunkach dopasowania energetycznego oporność obciążenia jest równa oporności wewnętrznej źródła.
Mesh	Oczko	Loop that does not contain any other loop within it. Pętla, która nie zawiera wewnątrz żadnej innej pętli.
Mesh or delta (Δ) connection	Skojarzenie w trójkąt	Individual phase windings/loads are connected to form a closed path. Delta (Δ) connection in case of a three-phase system. Wszystkie fazy tworzą oczko.

English	Polish	Description (opis)
Modified nodal analysis	Zmodyfikowana metoda potencjałów węzłowych	Modification in which the unknowns are not only the usual nodal voltages but also currents of resistiveless branches (ideal voltage sources and short-circuit elements). Modyfikacja polegająca na pozostawieniu w równaniach prądów gałęzi bezoporowych (SEM, amperomierz idealny).
Multi-terminal element or circuit	Wielobiegunnik	Element or circuit with *m* terminals available for external connections. Element lub obwód o *m* zaciskach zewnętrznych.
Mutual inductance ***M***	Indukcyjność wzajemna	Coefficient of proportionality relating current passing through one coil and flux caused by this current in the second (coupled) coil: Współczynnik proporcjonalności między prądem jednej cewki a strumieniem jaki ten prąd wywołuje w cewce sprzężonej: $M = k\sqrt{L_1 L_2}$ *k* – coupling coefficient (współczynnik sprzężenia)
Natural or zero-input response	Odpowiedź naturalna	Response to the initial condition, when all source excitations are set to zero. Odpowiedź na warunek(ki) początkowe, po odłączeniu źródła(eł).
Neper ***Np***	Neper	*See Propagation constant*
Neutral line	Przewód zerowy	For a star connection, line connecting a common junction point of a generator and a load star. Dla skojarzenia gwiazda-gwiazda, przewód łączący środki gwiazd.
Nodal analysis	Metoda potencjałów węzłowych	KCL equations with currents expressed by node voltages and branch parameters: Równania I prawa Kircchoffa, w których prądy wyrażono przez potencjały węzłowe i parametry układu: $\mathbf{GV} = \mathbf{I}_s$ G – conductance matrix (macierz konduktancyjna) $\mathbf{I}_s$ - vector of source currents of individual nodes wektor pradów źródłowych poszczególych węzłów.
Node	Węzeł	Connection point between two or more branches. Number of circuit nodes is denotes as *n*. Punkt połączenia dwóch lub więcej gałęzi. Liczba wszystkich węzłów: *n*.
Noninear resistive circuit	Nieliniowy obwód rezystorowy	Circuit that contains at least one nonlinear resistor. Obwód zawierający przynajmniej jeden element nieliniowy.

English	Polish	Description (opis)
Norton equivalent	Schemat zastępczy Nortona	Independent current source J_s or $J_s(j\omega)$ in parallel with a conductance G_t or admittance $Y_t(j\omega)$. Źródło prądowe (prąd zwarcia) połączone równolegle z opornością wewnętrzną.
Norton's theorem	Twierdzenie Nortona	For any linear active two-terminal circuit its linear equivalent circuit can be found. This circuit consists of the parallel connection of a current source and total (equivalent) conductance (admittance), the current source is the short circuit current of the circuit, the conductance (admittance) is the conductance (admittance) at the terminals when all the independent sources are deactivated. Liniowy dwójnik aktywny zastąpić można schematem zastępczym Nortona: równoległym połączeniem SPM zwarcia i oporności widzianej z zacisków po wyzerowaniu źródeł. *See Deactivation of independent source, Norton equivalent*
Ohm's Law	Prawo Ohma	The voltage across the terminals of a resistor is related to the current flowing into the positive terminal as: $U = RI$. Napięcie na zaciskach opornika proporcjonalne jest do pradu, ze współczynnikiem *R*.
Open-circuit (oc)	Rozwarcie, bieg luzem	Condition that exists when the current between two terminals is zero, irrespective of the voltage across the terminals. Stan pracy, przy którym zaciski dwójnika są rozwarte = nie płynie przezeń prąd.
Open-circuit voltage E_o	Napięcie biegu luzem	Voltage that appears between two terminals of a circuit or element in the open-circuit condition. Napięcie na zaciskach dwójnika dla biegu luzem.
Operating or Q-point (Quiescent point)	Punkt pracy	The point on an element $I-U$ characteristic at which the circuit Kirchhoff's laws are satisfied. The coordinates at this point are the operating voltage U^Q and the operating current I^Q . Punkt na charakterystyce elementu, dla którego spełnione są w obwodzie prawa Kircchoffa.
Output signal Y or y **or** $Y(s)$ **or** $Y(j\omega)$	Sygnał wyjściowy	Response of a system. Odpowiedź układu.

English	Polish	Description (opis)
Overdamped response	Odpowiedź silnie tłumiona	Nonoscillatory response of the *RLC* - circuit condition that exists when all poles of the response are real and distinct. Aperiodyczna odpowiedź układu *RLC* – warunek spełniony gdy oba bieguny odpowiedzi są rzeczywiste i różne od siebie.
Parallel connection	Połączenie równoległe	Arrangement of elements so that each element has the same voltage appearing across it. Połączenie, dla którego wszystkie elementy podłączone są na to samo napięcie.
Parallel resonant circuit	Równległy obwód rezonansowy	Circuit with a resistor, capacitor and inductor in parallel. Obwód, w którym element *R,L* i *C* połączone są równolegle.
Parameter tolerance tol_X	Tolerancja parametru	Ratio of the parameter design deviation to its nominal value: Stosunek odchyłki projektowej parametru do wartości nominalnej: $tol_X = \Delta X / X^n$.
Passive circuit	Obwód pasywny	Circuit consisting of resistors, capacitors and inductors, that can only store and/or dissipate energy. Obwód zbudowany z elementów *RLC*, który może tylko rozpraszać energię lub/i ją magazynować.
Passive element	Element pasywny	Total energy supplied to it from the rest of the circuit is always nonnegative. Such element cannot deliver net power to a circuit. Element, który rozprasza lub magazynuje energię, nie zasila obwodu.
Period of oscillations ***T***	Okres oscylacji	Time between two subsequent maximum points of a periodic (sinusoidal) waveform. Czas pomiędzy kolejnymi maksimami fali sinusoidalnej.
Permanent (natural) magnet	Magnes trwały	Magnet made of the iron compound magnetite (Fe_3O_4). Magnes zbudowany z magnetytu.
Phase shift φ	Przesunięcie fazowe	Phase angle between an element voltage and its current, with current as the reference, $\varphi = \alpha_u - \alpha_i$. Przesunięcie fazowe miedzy napięciem na elemencie a prądem, liczone od prądu.
Phase shift per distance β	Przesunięcie fazowe	*See Propagation constant*

English	Polish	Description (opis)
Phase voltage V_i	Napięcie fazowe	Voltage appearing at a phase impedance. For the four-wire system, the voltage between line 1 or 2 or 3 and the neutral. Napięcie na impedancji fazowej. Dla układu cztero-przewodowego, napięcie między przewodem fazowym a przewodem zerowym.
Phasor diagram	Wykres wektorowy	Phasors expressed graphically in a complex plane. Wartości symboliczne skuteczne zobrazowane graficznie na płaszczyźnie fazowej.
Phasor voltage or current $U(j\omega)$ or $I(j\omega)$	Napięcie lub prąd symboliczny	Complex number associated with sinusoidal voltage or current, Liczba zespolona opisująca sinusoidalny prąd lub napięcie. $U(j\omega) = U(\omega)\exp\alpha_u(\omega)$ or $I(j\omega) = I(\omega)\exp\alpha_i(\omega)$
Piecewise-Linear Approximation (PWLA):	Aproksymacja odcinkowo-liniowa	Approximation of the nonlinear $I-U$ characteristic by linear segments. For each segment its Thevenin or Norton equivalent can be found. Aproksymacja odcinkowa charakterystyki nieliniowej. Każdy segment można zamodelować schematem Thevenina lub Nortona.
Planar circuit	Obwód planarny	Circuit whose diagram (graph) can be drawn on a plane without branches crossing each other. Obwód, którego graf można tak narysować by gałęzie się nie przecinały.
Poles of transfer function s_k	Bieguny transmitancji	Roots of the denominator polynomial of the transfer function $K(s)$. Pierwiastki wielomianu mianownika operatorowej funkcji przejścia.
Port	Wrota	Pair of circuit terminals to which another subcircuit may be attached. Current entering one terminal is equal to the current leaving the other. Para zacisków – prąd wpływający do jednego z nich wypływa z drugiego.
Potential (node voltage) V or v	Potencjał	Voltage between the reference point P and the other one A: Napięcie między węzłem odniesienia a danym węzłem. $V_{\mathrm{A}} = U_{\mathrm{AP}}$; $U_{\mathrm{AB}} = V_{\mathrm{A}} - V_{\mathrm{B}}$.

English	Polish	Description (opis)
Power P or p	Moc	Energy per unit period of time, Energia przypadająca na jednostkę czasu, $p = dw/dt$; $w = \int_0^t pdt$. In the DC case: $P = UI$; $w = UI \cdot t$. Unit: watt $[\mathrm{W}] = [\mathrm{J/s}]$.
Power factor ***pf***	Współczynnik mocy czynnej	Ratio of an average power to an apparent power: Stosunek mocy czynnej do mocy pozornej: $pf = \cos\varphi = P/S$ *See Average power, Apparent power*
Practical circuit	Obwód rzeczywisty	Circuit built of practical elements, with parameters given by the design tolerances. Obwód uwzględniający modele elementów o parametrach zadanych tolerancjami projektowymi.
Primary coil (winding)	Uzwojenie pierwotne	Coil shown on the left-hand side of the model of a transformer. Winding connected to a source. Cewka z lewej strony modelu transformatora – jej uzwojenie podłączone jest do źródła.
Propagation constant $\gamma(s)$ **or** $\gamma(j\omega)$	Stała propagacji	$\gamma(s) = \sqrt{(R+sL)(G+sC)}$ For the distortionless line: Dla linii bezstratnej: $\gamma(s) = \alpha + s/v$ or $\gamma(j\omega) = \alpha + j\beta$ $\alpha = \sqrt{RG}$ - attenuation constant tłumienność in [1/m] or neper per meter [Np/m], $v = 1/\sqrt{LC}$ - propagation (phase) velocity szybkość propagacji in [m/s], $\beta = \omega\sqrt{LC} = \omega/v$ - phase shift per distance przesunięcie fazowe.
Propagation velocity ***v***	Prędkość rozchodzenia się fali	*See Propagation function*

English	Polish	Description (opis)
Proportionality	Proporcjonalność	When an input to a linear resistive circuit is acting alone, then scaling the input by a constant *K* implies that the response is also scaled by *K*. Dla pojedynczego pobudzenia ukadu, jego przemnożenie przez stałą *K* powoduje przemnożenie odpowiedzi przez tą sama stałą. *See Linearity*
Pulse	Impuls prostokątny	Function of time, built of step functions, that is zero for $t < 0$, has magnitude 1 for $0 \le t \le \tau$, and is equal to zero for $t > \tau$. Funkcja różna od zera w przedziale od 0 do τ.
Quality factor of practical capacitor practical coil bandpass circuit	Dobroć kondensatora, cewki lub obwodu o własnościach selektywnych	Measure of the circuit energy storage property in relation to its energy dissipation property. Miara zdolności obwodu do gromadzenia energii w relacji do zdolności do jej rozpraszania. Capacitor: $Q_C(\omega) = R_C \omega C$. Inductor: $Q_L(\omega) = \omega L / R_L$. Bandpass circuit: Obwód o właściwościach selektywnych: $Q = 2\pi \dfrac{\text{maximum energy stored}}{\text{total energy dissipated per cycle}}$ Series *RLC* circuit: $Q = 1/R\sqrt{L/C}$ Parallel *RLC* circuit: $Q = 1/G\sqrt{C/L}$
Reactance $X(\omega)$	Reaktancja	*See Impedance in phasor-domain*
Reactive power ***Q***	Moc bierna	Power oscillating between the circuit reactive elements (capacitors and inductors) and the power source, $Q = UI \sin\varphi$. Moc oscylująca między reaktancjami (cewka, kondensator) a źródłem. Unit: var, volt-ampere-reactive [VAr].
Reciprocal circuit	Obwód podlegający zasadzie wzajemności	Circuit whose node equations have symmetric conductance matrix, $G_{ij} = G_{ji}$. Obwód o symetrycznej macierzy konduktancyjnej. *See Linear resistive circuit.*
Reflection coefficients $M(s)$, $N(s)$	Współczynniki odbicia	Coefficients of the incident and reflected waves, Współczynniki, z jakimi odbiją się fale wędrujące, $M(s) = \dfrac{Z(s) - Z_t(s)}{Z(s) - Z_t(s)}$; $N(s) = \dfrac{Z(s) - Z_l(s)}{Z(s) + Z_l(s)}$

English	Polish	Description (opis)
Relative sensitivity Sr_X^F	Wrażliwość względna	$Sr_X^F = (\partial F / \partial X)\vert_{X=X^n} /(F^n / X^n)$ *See Sensitivity of circuit variable*
Reluctance or magnetic resistance R_m	Opór magnetyczny	Parameter describing linear magnetic element, ratio of the magnetic voltage drop to the flux flowing Parametr opisujacy liniowy element magnetyczny $R_m = l/(\mu_r \mu_0 S)$ l - mean length of a core, S - its cross-section area. średnia droga magnetyczna, pole przekroju
Resistance ***R***	Oporność (opór)	Coefficient of proportionality between the voltage and the current of linear resistor. Unit: ohm [Ω]=[V/A]. Współczynnik proporcjonalności między prądem a napięciem na liniowym oporniku.
Resistance in phasor-domain $R(\omega)$	Rezystancja	*See Impedance in phasor-domain*
Resistor	Opornik	Element whose primary purpose is to introduce resistance, i.e. to impede current flow and voltage drop into a circuit. Resistor converts electric energy into heat. Element zamieniający energię elektryczną na ciepło.
Resonance	Rezonans	Condition in a two-terminal circuit, occurring at the resonant frequency, when the equivalent impedance $Z(j\omega)$ or admittance $Y(j\omega)$ becomes a real number (circuit becomes non-reactive). Stan pracy dwójnika, w którym impedancja zastępcza posiada tylko część rzeczywistą (urojona jest równa zero).
Resonant frequency ω_r	Częstotliwość rezonansowa	Frequency at which a two-terminal circuit becomes purely resistive. In the series or parallel *RLC* circuit, also frequency of the undamped transient response: $\omega_r = 1/\sqrt{LC}$. Częstotliwość dla której dwójnik znajduje się w rezonansie.
Right hand rule	Reguła prawej dłoni	If a current-carrying conductor is grasped in the right hand with the thumb pointing in the direction of the conventional current, the fingers will then point in the direction of the magnetic lines of flux. Jeśli objąć przewód prawą dłonią tak by kciuk wskazywał przepływ prądu, to pozostałe palce wskażą kierunek wytworzonego strumienia magnetycznego.

English	Polish	Description (opis)
Secondary coil (winding)	Uzwojenie wtórne	Coil shown on the right-hand side of the model of a transformer. Winding connected to a load. Cewka z prawej strony modelu transformatora. Jej uzwojenie połączone jest z obciążeniem.
Sensitivity of circuit variable S_X^F	Wrażliwość funkcji układowej	Sensitivity of *F* with respect to *X*: $S_X^F = (\partial F / \partial X)\big\|_{\mathbf{X}=\mathbf{X}^n}$
Separation (voltage/current substitution) principle	Zasada wyodrębnienia	Two subcircuits connected in *m* nodes can be separated by means of $m-1$ pairs of voltage or current sources connected between the arbitrarily selected reference node and each of other $m-1$ nodes. Value of the voltage source connected between two nodes is equal to the original circuit voltage. Value of the current source equals the total current entering/leaving the node from/to one of the subcircuits. Dwa obwody połączone w *m* węzłach można odseparować od siebie włączając *m*–1 par sił elektromotorycznych między kolejne węzły a *m*-ty węzeł odniesienia, o wartościach jak przed wyodrębnieniem.
Series connection	Połączenie szeregowe	Circuit of a series of elements connected so that the same current passes through each element. Obwód zbudowany z elementów połączonych tak by płynął przez nie ten sam prąd.
Series Resonant Circuit	Szeregowy obwód rezonansowy	Circuit with a series connection of a resistor, capacitor and inductor. Szeregowy obwód *RLC*.
Set of independent cutsets	Zbiór odcięć niezależnych	Cutsets around all individual nodes except the reference one, their number: $t = n-1$ Zbiór odcięć wokół wszystkich węzłów za wyjątkiem węzła odniesienia.
Set of independent loops	Zbiór oczek niezależnych	All meshes of a circuit. Their number: $l = b - n + 1$ Zbiór wszystkich oczek.
Short-circuit (sc)	Zwarcie	Condition that exists when the voltage across two terminals is zero, irrespective of the current between the two terminals. Stan pracy, przy którym zaciski dwójnika są zwarte = napięcie między nimi jest równe zero..
Short-circuit current J_s	Prąd zwarcia	Current passing an active element (practical source) in the short-circuit condition. Prąd płynący przez zaciski dwójnika w stanie zwarcia.
Siemens **S**	Siemens	*See Conductance*

English	Polish	Description (opis)
Signal variable in time $f(t) = f$	Sygnał zmienny w czasie	Real valued function of time; waveform that conveys information, denoted by a small letter. Funkcja czasu, oznaczana mała literą.
Standing wave ratio ***S***	Współczynnik fali stojącej	Ratio of the maximum to the minimum rms voltages along a line Stosunek maksymalnej amplitudy do minimalnej amplitudy fali stojącej. $S = U_{x\max} / U_{x\min} = (1+\lvert N\rvert)/(1-\lvert N\rvert)$
Standing waves	Fale stojące	Plots of $\lvert U_x(j\omega)\rvert$ and $\lvert I_x(j\omega)\rvert$ with their maxima and minima occurring at fixed locations along the line.
Star or wye (Y) connection	Skojarzenie w gwiazdę	Individual phase windings/loads are joined in a common junction point. Wye (Y) connection in case of a three-phase system. Wszystkie fazy mają wspólny zacisk.
Steady-state analysis	Analiza w stanie ustalonym	Analysis of a circuit behavior resulting after have been on for a long time. Analiza w stanie, w którym wszystkie przebiegi osiągają wartości stałe, niezmienne w czasie.
Step-down transformer	Transformator obniżający	Transformer of the turns ratio less than one. Transformator o przekładni mniejszej od 1.
Step-up transformer	Transformator podwyższający	Transformer of the turns ratio greater than one. Transformator o przekładni większej od 1.
Stray capacitance	Pojemność między węzłem a masą	Unwanted capacitance between a circuit node (element terminal) and ground. Pasożytnicza pojemność między zaciskiem elementu a masą układu.
Stray or parasitic inductance	Indukcyjność doprowadzeń	Unwanted inductance of element connections. Pasożytnicza indukcyjność doprowadzeń elementu.
Stray or parasitic or shunting capacitance	Pojemność bocznikująca	Unwanted capacitance that exists between element terminals or between a terminal and ground. Pasożytnicza pojemność między zaciskami elementu.
Superposition	Superpozycja	When a number of inputs are applied simultaneously to a linear circuit, the response is the sum of responses due to each input acting alone. W liniowym układzie o wielu pobudzeniach odpowiedź można wyznaczyć sumując odpowiedzi na każde z pobudzeń z osobna, przy pozostałych wyzerowanych. *See Linearity*

English	Polish	Description (opis)
Superposition principle	Zasada superpozycji	For a linear circuit containing independent sources, the voltage across (or the current through) any element may be obtained by adding algebraically all the individual voltages (or currents) caused by each independent source acting alone with all other sources deactivated. Remark: Power can't be found by superposing power losses. W obwodzie liniowym, w którym działa wiele źródeł niezależnych, dowolny prąd lub napięcie wyznaczyć można sumując składowe wywołane przez każde ze źródeł z osobna, przy pozostałych wyzerowanych. Dla mocy zasada superpozycji nie obowiązuje. *See Deactivation of independent source*
Susceptance $B(\omega)$	Susceptancja	*see Admittance in phasor-domain*
System	System	Interconnection of electrical elements and circuits to achieve a desired objective. Połączenie elementów i obwodów dla uzyskania pożądanego celu.
Tesla T	Tesla	*See Magnetic flux density*
Thevenin equivalent	Schemat zastępczy Thevenina	Independent voltage source E_o or $E_o(j\omega)$ in series with a resistance R_t or impedance $Z_t(j\omega)$ Schemat zastępczy powstały z szeregowego połączenia SEM biegu luzem z opornością wewnętrzną dwójnika.
Thevenin's theorem	Twierdzenie Thevenina	For any linear active two-terminal subcircuit its linear equivalent circuit can be found. This circuit consists of the series connection of a voltage source and total (equivalent) resistance (impedance): the voltage source is the open-circuit voltage of the subcircuit the resistance (impedance) is the resistance (impedance) at the terminals when all the independent sources are deactivated. Liniowy dwójnik aktywny zastąpić można schematem zastępczym Thevenina: szeregowym połączeniem SEM biegu luzem i oporności widzianej z zacisków po wyzerowaniu źródeł. *See Deactivation of independent source*

English	Polish	Description (opis)
Three-phase source	Generator trójfazowy	Three voltage sources of the same frequency and magnitude, and the phase shift or 120° between any two of them, connected in the form of Y or Δ. Trzy źródła napięcia sinusoidalnego o tej samej częstotliwości, amplitudzie i przesunięciem 120º między każdą parą.
Time constant ***T***	Stała czasowa	Parameter of exponentially decaying or rising response. After one time constant the response drops to $\approx 38\%$ of its initial value or rises to $\approx 62\%$ of its end value, Parametr krzywej wykładniczej. Po upływie jednej stałej czasowej krzywa zanika do 38% wartości początkowej lub narasta do 62% wartości końcowej. for *RL* circuit: $T = L / R_t$, for *RC* circuit: $T = R_t C$.
Tolerance margins of circuit parameter X^-, X^+	Granice obszaru tolerancji	Margins specifying the allowed (by the design) variation of parameter *X* from its nominal value X^n : Marginesy odchyłek parametru (górnej i dolnej) od wartości nominalnej, dopuszczalnych tolerancjami projektowymi: $X^- = X^n - \Delta X \quad X^+ = X^n + \Delta X$.
Tolerance region (box)	Obszar tolerancji	Parallelepiped in the parameter space $\Re^P$ with planes parallel with the coordinate axes, designated by the tolerance margins of all circuit parameters. Równoległościan w przestrzeni parametrów wyznaczony tolerancjami projektowymi.
Total flux or flux linkage Φ_t	Strumień całkowity	The total flux linked by the *z* turns of the coil, Iloczyn strumienia i liczby zwojów, $\Phi_t = z\Phi$. *See Magnetic flux*
Transfer function in *s*-domain $K(s)$	Transmitancja operatorowa	Ratio of the response (output signal) of a circuit to an excitation (input signal) expressed as a function of *s* (initial conditions are assumed to be zero). Stosunek operatorowej postaci sygnału wejściowego do sygnału wejściowego.
Transfer function (gain) for DC signals ***K***	Transmitancja	Ratio of the response (output signal) of a circuit to an excitation (input signal) Stosunek odpowiedzi układu do pobudzenia DC.
Transfer function in frequency domain $K(j\omega)$	Częstotliwościowa funkcja przejścia	*See Frequency response*

English	Polish	Description (opis)
Transformation	Transformacja	Conversion of a set of equations from one domain to another, e.g. from the *t*-domain to the *s*-domain. Konwersja równań z dziedziny czasu w dziedzinę operatorową
Transformer	Transformator	Magnetic circuit with two or more multi-turn coils wound on a common core. Obwód magnetyczny o dwóch uzwojeniach.
Transient analysis (state)	Analiza stanów przejściowych (nieustalonych)	Analysis of a circuit behavior for a period of time immediately after independent source or sources have been turned on or turned off, at $t=0$. In stable circuits, transient state vanishes after $5T_{\max}$, where $T_{\max}$ is the maximum time constant. Analiza zachowania obwodu od momentu powstania zaburzenia do momentu ustalenia się odpowiedzi. W obwodzie stabilnym, czas osiągnięcia stanu ustalonego wyznacza pięciokrotność największej stałej czasowej odpowiedzi.
Transmission line	Linia długa	Two-wire line connecting the input circuit with the output circuit. Linia dwuprzewodowa łącząca źródło z obciążeniem.
Traveling waves	Fale wędrujące	Initial and reflected waves traveling from the line input to its output – forward waves and reflected waves traveling in the opposite direction – backward waves. Fala pierwotna i fale odbite, od początku i od końca linii. The reflected waves interference can be disregarded if time parameter(s) of a transmitted signal is(are) much greater than the line propagation time τ. Efekt nakładania sie fal odbitych na falę pierwotną można pominąć jeśli czas charakterystyczny trsansmitowanego sygnału jest znacznie wiekszy od czasu propagacji τ.
Turns ratio ***n***	Przekładnia	Ratio $n = z_2 / z_1$, where z_2 and z_1 are turns in secondary and primary coil of an ideal transformer. Stosunek liczby zwojów uzwojenia wtórnego do liczby zwojów uzwojenia pierwotnego.
Two-port	Czwórnik	Four terminal element identified by two distinct pairs of terminals - ports. Element o dwóch wrotach.
Two-terminal element or circuit	Dwójnik	Element/circuit connected at a pair of terminals, described by a single $I-U$ relationship. Element/obwód dwuzaciskowy. *See Element law...*
Undamped response	Odpowiedź nie tłumiona	Transient response in *LC* (resistiveless) circuit. Odpowied obwodu *LC* (bezoporowego).

English	Polish	Description (opis)
Undamped natural frequency	Częstotliwość drgań własnych	*See Resonant frequency*
Underdamped response	Odpowiedź słabo tłumiona	Periodic response of the 2nd order circuit – condition that exists when two poles of the response are complex conjugates. Odpowiedź periodyczna obwodu słabo tłumionego, ma miejsce gdy w odpowiedzi występują dwa bieguny zespolone sprzężone.
Unit Impulse (Dirac delta) function $\delta(t)$	Funkcja impulsowa	Infinitely short pulse of infinitely large magnitude - its value is zero for $t \neq 0$, infinity at $t = 0$ and its area is equal to 1. Unit: [1/s]. Nieskończenie krótki impuls o nieskończonej amplitudzie i polu jednostkowym.
Unit Ramp function $r(t)$	Funkcja liniowo narastająca	$r(t) = t\mathbf{1}(t)$, an integral of the unit step function. całka skoku jednostkowego.
Unit Step function $\mathbf{1}(t)$	Skok jednostkowy	Dimensionless function of time that is zero for $t < 0$ and unity for $t \geq 0$. Bezwymiarowa funkcja przyjmująca wartość 1 dla czasów większych od 0, wartość 0 dla pozostałych czasów.
Unity coupling	Sprzężenie idealne	Coupling with $k = 1$. Sprzężenie ze wspóczynnikiem jednostkowym. *See Mutual inductance*
Var	Var	*See Reactive power*
Voltage (potential difference) U or u	Napięcie (różnica potencjałów)	Work required to move a unit charge $Q = 1$ [C] from one point A to another B, Praca niezbędna do przemieszczenia ładunku jednostkowego z punktu A do punktu B, $U_{\mathrm{AB}} = W_{\mathrm{AB}}\big\vert_{Q=1}$. Unit: volt [V].
Voltage divider	Dzielnik napięcia	Circuit of a series of resistors that divides the input voltage U by the ratio of the R_i to the total series resistance Obwód zbudowany z n oporników połączonych szeregowo, dzielący napięcie jak poniżej: $R_t = \sum_{i=1}^{n} R_i$, $U_i = UR_i / R_t$.
Weber Wb	Weber	*See Magnetic flux*

English	Polish	Description (opis)
Worst Case	Najgorszy przypadek	Case when deviation of circuit variable, caused by the design tolerances, reaches its maximum $\Delta F_{\max}$. Przypadek największej odchyłki funkcji układowej od wartości nominalnej, spowodowanej tolerancjami projektowymi parametrów obwodu.
Zeros of transfer function q_k	Zera transmitancji	Roots of the numerator polynomial of the transfer function $K(s)$. Pierwiastki wielomianu licznika operatorowej funkcji przejścia. *See Transfer function in s-domain*

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