FUNDAMENTALS

of

VIBRATION STUDY

By

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With a Foreword by

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FOREWORD

APPLIED vibration study is an important or even a major influence in many branches of modern engineering practice where fluctuating stresses originated by periodically varying components of the motive forces may seriously jeopardise structural stability.

The general term " vibration " is given to the effect produced by periodic forces, and this may range from physiologically unpleasant tremors to disturbances of sufficient magnitude to cause mechanical failure of structural components.

Recognition of the ill effects that are apt to appear when vibration problems inherent in engineering structures are neglected has led to intensive studies of fatigue phenomena in materials, and other relevant subjects. This has resulted in more widespread appreciation of the importance of avoiding sharp corners and other abrupt structural discontinuities from the point of view of minimising local concentrations of stress; and the necessity for adjusting the dynamical characteristics of the system so that important zones of resonant vibration do not occur in the operating speed ranges.

The foregoing remarks are particularly true of sea, land, and air transport services, where modern requirements demand a clear understanding of attendant vibration problems if operational troubles of one kind or another are to be avoided. In this connection it should be borne well in mind that modern requirements are not necessarily fulfilled by designs which merely provide for freedom from mechanical breakdown.

Most engineers now and then experience the satisfaction of handling a machine in which the different design factors have been so carefully balanced that the resultant blend is a product of outstanding merit. Even to-day, however, such examples are comparatively rare and in many cases are confined to individual specimens from a production batch, so that the achievement appears to be largely the result of a happy accident which somehow or other provides a product which is neither too sluggish and lifeless nor too lively and over-sensitive. This suggests that an important contributory factor towards the achievement of these ideal results is accidental realisation of the correct solution of the vibration problems associated with each case. Vibration study is therefore likely to exert considerable influence on engineering development, and already several industrial concerns making products which are particularly susceptible to vibratory influences have established special vibration engineering departments.

In the past the tendency has been to disregard vibration during initial design and consequently the work of the vibration specialist has been largely confined to the correction of faults which have all too often appeared in the product after it has been put into service. Only occasionally has this procedure proved to be entirely satisfactory, because it is almost impossible to cure a really difficult case without resorting to a drastic re-design of the whole system. There is nearly always some increase of weight accompanied by a reduction of operational efficiency due either to the necessity of imposing restrictions on operating speeds or introducing energy-absorbing damping devices. In other words, failure to take the vibration problem into account during initial design nearly always results in inefficiency of one kind or another.

It is to be hoped, therefore, that in the future many more engineers will become interested in vibration study so that design work will proceed along lines which are fundamentally sound. At the same time, there are certain aspects of vibration study which must always remain the special province of the vibration engineer ; in particular, the task of accurately recording and analysing structural vibration is a matter which requires the skill and experience of trained personnel.

Although a considerable volume of literature has accumulated on this important subject, there is a lack of information on the fundamental principles underlying present-day vibration study presented in such a way as to be readily assimilated by the busy engineer. The author of the present work is therefore to be congratulated on having produced a book which should be welcomed by everyone wishing to acquire a good working knowledge of these principles. In particular, the author is to be commended on dealing in a straightforward manner with mechanical vibrations as such without resort to electrical or other analogies.

There is no doubt that the student or engineer who studies

Mr. Manley's work will find little difficulty in extending the principles so ably set forth to meet the needs of his special problems. This is largely because the work is based on the results of actual first-hand experience in the Vibration Department of a leading industrial establishment.

W. KER WILSON.

London, 1942.

AUTHOR'S PREFACE

The need for a really *introductory* book on vibration theory has for some time been felt, both by technical staff engaged in vibration research and by others whose work brings them into contact with the problems of the subject. The available textbooks are, of course, excellent and comprehensive, but experience has shown that it is difficult for beginners to extract from the somewhat lengthy standard texts the precise information they are seeking.

The present work is designed to serve as an introduction to the subject, so that the existing textbooks can subsequently be approached with some degree of confidence. In accordance with this plan, attention has been confined to the basic theory, a knowledge of which is essential to a thorough understanding of the physical phenomena ; instrumentation and testing technique fall outside the scope of the book. No previous knowledge of differential equations is assumed, this troublesome part of the theory being dealt with by the powerful and easy method of operators ; indeed, the only mathematical equipment is that which should be at the command of anyone who has gone but a little way beyond the Matriculation syllabus. A series of notes in the appendices serves as a bridge between the school work and those parts of analysis which are required in the text.

The most important advance, in recent years, in vibration theory has been the development of the concept of *effective inertia*, with its collateral "mechanical impedance" (herein termed *dynamic stiffness* for reasons stated in the text) ; these methods of attack have been utilised with conspicuous success in the treatment of torsional vibration problems, particularly in aircraft applications. An entirely new approach to the idea of effective inertia is described in the text—an approach which, besides being direct, has been "tried out" with students and is evidently easily grasped.

Particular emphasis is given to resonance, which is the really significant phenomenon for the engineer; the usual assumption concerning the practical equivalence of natural and resonant frequencies in lightly-damped systems is justified by an argument based on the method of partial inertias which is described in the chapter on effective inertia.

A few exercises are included at the end of each chapter; these are for the most part analytical in nature, and serve slightly to extend the work covered in the chapter. Some numerical examples are collected at the end of the book.

In the Bibliography are listed four standard works only. These are books which should be readily available to all who have any working connection with the subject, and between them they cover the entire field of vibration theory and practice.

The volume embodies the subject-matter of a course of lectures delivered by the author to members of the Vibration Department of the de Havilland Aircraft Company; in this and other connections the author's thanks are due to many members of this Company, particularly: to Dr. W. Ker Wilson for his interest and encouragement, and for the many helpful suggestions he has made; to Mr. R. N. Hadwin for the facilities so readily granted for the compilation of the book, especially access to his library of technical works; and to Miss M. K. B. Harwood, both for much numerical work embodied in the text and exercises, and also for ungrudging co-operation in the tedious task of checking the typescript and proofs.

R. G. MANLEY.

EDGWARE, *January 1942.*

CONTENTS

LIST OF SYMBOLS

The symbols listed below are those that occur most frequently in the text. All symbols are defined where they first occur, and in many cases the meaning is reiterated in subsequent references.

The figures in parentheses denote the pages on which the symbols are first used or defined.

a, A	amplitude of vibration (ins. or radians) . . .	(5, 13)
c	damping coefficient, i.e. force per unit velocity (lbs.ins.$^{-1}$ secs.)	(17)
c_0	critical damping coefficient $= 2\sqrt{mk}$ . . .	(20)
c_{Xy}, etc.	coupling coefficients	(46)
C	torsional stiffness (lbs.ins./radian)	(12)
D	differential operator $\equiv \frac{d}{dt}$	(3)
	$\equiv \frac{d}{dx}$	(77)
E	Young's modulus (lbs./ins.^{2})	(76)
f, P	force (lbs.)	(16, 22)
F	frequency (cycles per second)	(7)
$F_{res.}$	resonant frequency	(29)
F	shear force (lbs.)	(76)
g	acceleration due to gravity (ins./secs.^{2}) . . .	(2)
G	shear modulus (lbs./ins.^{2})	(70)
i	$\sqrt{-1}$	(3)
I	moment of inertia (lbs.ins.secs.^{2})	(10)
I	second moment of area (ins.^{4})	(70)
j	versor-operator turning associated vector through $\pi/2$ radians	(34)
J	polar moment of inertia (lbs.ins.secs.^{2}) . . .	(12)
J_e	effective inertia (lbs.ins.secs.^{2})	(53)
k	linear spring stiffness (lbs./ins.)	(2)
k	$\sqrt{\frac{\rho}{G}}$ (heavy shafts)	(72)
K	l/GIy (heavy shafts)	(73)
l	length (ins.)	(70)
m	mass (slugs = lbs.ins.$^{-1}$ secs.^{2})	(1)
m	mass per unit length of beam (slugs/ins.) . . .	(76)
M	dynamic magnifier	(27)
$M_{res.}$	dynamic magnifier at resonance	(27)

M	bending moment (lbs.ins.)	(76)
M_B	mass of beam (slugs)	(83)
M_L	loading mass (slugs)	(85)
$p/2\pi$	frequency of forced motion (C.P.S.)	(22)
$q/2\pi$	natural frequency with damping	(20)
q	shear stress (lbs./ins.^{2})	(110)
t	time (secs.)	(3)
T	torque (lbs.ins.)	(10)
u	complementary function	(24)
v	particular integral	(24)
y	ωkl (heavy shafts)	(73)
Z	dynamic stiffness (lbs.ins./radian)	(64)
z	dynamic stiffness of damped system	(68)
$\|z\|$	modulus of $z = \sqrt{\alpha^2+\beta^2}$ where $z = \alpha \pm i\beta$	(68)
γ	damping constant $= c/2m$	(18)
δ, δ_s	static deflection under gravity-load (ins.)	(2, 8)
Δ	logarithmic decrement $= \dfrac{\pi c}{mq}$	(22)
θ	angle (radians)	(10)
λ, μ	partial inertias (lbs.ins.secs.^{2})	(53)
ρ	density (slugs/ins.^{3})	(71)
ρ	radius of curvature (ins.)	(108)
ϕ, ψ	phase-angle (radians)	(5, 23)
ψ	shear strain (radians)	(110)
ω	angular velocity of rotating vector (radians/secs.)	(6)
$\omega/2\pi$	frequency (C.P.S.)	(7)

CHAPTER I

SYSTEMS HAVING ONE DEGREE OF FREEDOM

(*Undamped Motion*)

1. A simple vibrating system.

The system shown diagrammatically in Fig. 1 is an example of the simplest type of mechanical vibrating system that can exist. It consists simply of a mass m fixed to the lower end of a light coiled spring, the upper end of which is attached to a rigid support. If the mass is pulled downwards and then released, practical experience shows that it will oscillate in the vertical direction, the amplitude of oscillation diminishing in time until finally the system appears to come to rest. This decay of the motion is due to dissipation of energy by the action of damping forces, the nature and effect of which are described in Chapter II. It will be convenient to consider first the motion of the system in the absence of such forces.

The spring is therefore considered to be idealised, so that the force required to extend or compress it along its axis is proportional to the extension or compression only. Fig. 1a shows the

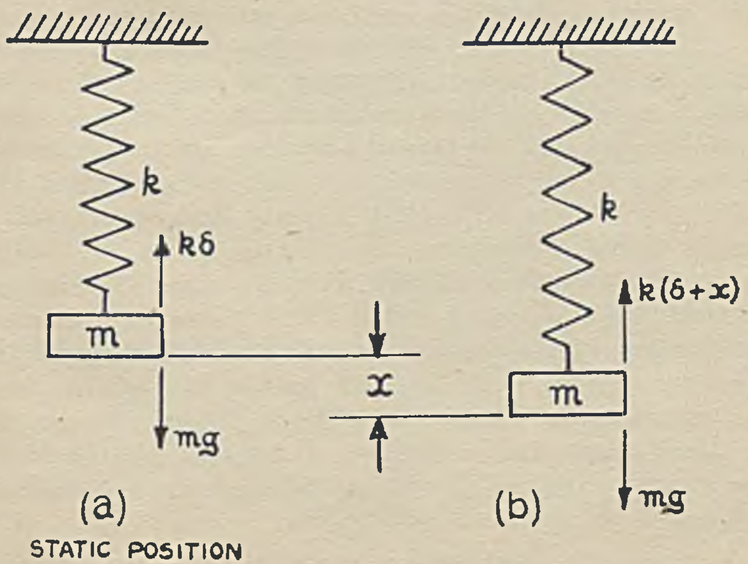


Fig. 1.—Simple spring-mass system: (a) static position; (b) at time t.

position of the system in static equilibrium, the spring being extended a distance δ beyond its natural length by the action of the gravity force mg. The spring-constant of proportionality, or *stiffness*, k, is then given by the equation:

$$k\delta = mg \qquad . \quad . \quad . \quad . \quad (1.1)$$

The mass is then pulled downwards a further distance x_0 and released with zero velocity. After a time t from the instant of release the displacement of the mass below its equilibrium position is x, as shown in Fig. 1*b*. The real forces acting on the spring are:

(i) the gravity force mg downwards, and
(ii) the spring force $k(\delta+x)$ upwards, as the reaction of the spring is proportional to its extension beyond the natural length and tends to return the spring to its natural length.

These two forces have a resultant kx upwards, as the part $k\delta$ of the spring force is balanced by the gravity force mg, as shown by (1.1). The downward acceleration of the mass is given by the equation of motion:

$$m\ddot{x} = -kx, \text{ or } \ddot{x} = -\frac{k}{m}x \qquad . \quad . \quad (1.2)$$

(The dot notation for differential coefficients with respect to time will be employed wherever possible.)

(1.2) is a linear differential equation of the second order in one dependent variable, i.e. there are no second or higher powers of any variable quantity, the highest differential coefficient involved is the second (two dots) and there is only one dependent variable (x). On account of this last property the system is said to have one degree of freedom. In order to obtain an equation giving the displacement x directly in terms of time, it is necessary to integrate the equation of motion twice, and at each indefinite integration an arbitrary constant must be inserted for generality. The general solution must therefore contain two arbitrary constants, the values of which can be chosen to make the solution correspond to particular conditions. It has been postulated that at the instant of release ($t = 0$) the displacement is x_0 and the velocity is zero. With the substitution

$$\omega^2 = \frac{k}{m} \qquad . \quad . \quad . \quad . \quad (1.3)$$

the three equations to be satisfied are :

$$\left.\begin{array}{lr} \text{equation of motion, } \ddot{x} = -\,\omega^2 x & (a) \\ \text{initial conditions, } x = x_0 \text{ at } t = 0 & (b) \\ \dot{x} = 0 \text{ at } t = 0 & (c) \end{array}\right\} \quad . \quad (1.4)$$

2. The differential operator D.

It can easily be shown by substitution that the function

$$x = x_0 \cos \omega t \quad . \quad . \quad . \quad . \quad (2.1)$$

is a solution of (1.4). Differential equations are frequently solved by guesswork—indeed, guessing is very often the only way of arriving at a solution—but there is a perfectly general method of solving equations of the type of (1.4*a*).

Write D for $\frac{d}{dt}$, so that Dx stands for $\dot{x}$ and D^2x for $\ddot{x}$. The symbol D represents the *operation* of differentiating with respect to time, and is called the *differential operator*. (1.4*a*) can be written, with this notation, as

$$D^2x = -\,\omega^2 x \quad . \quad . \quad . \quad . \quad (2.2)$$

Proceeding in an experimental manner, the common factor x is "dropped" by analogy with ordinary algebraic equations, leaving

$$D^2 = -\,\omega^2 \quad . \quad . \quad . \quad . \quad (2.3)$$

(2.3) is to be interpreted as stating that the operation of double differentiation with respect to time, when performed on the appropriate function, is equivalent to the operation of multiplying the same function by $-\,\omega^2$. It remains to determine the appropriate function. Performance of the square-root operation on (2.3) leads to :

$$D = \pm\, i\omega, \text{ i.e. } \frac{dx}{dt} = \pm\, i\omega x \quad . \quad . \quad (2.4)$$

where $$i = \sqrt{-1}$$

Separating the variables x and t, and taking first the positive sign,

$$\frac{dx}{x} = i\omega . dt$$

$$\int \frac{dx}{x} = i\omega . \int dt$$

whence

$$\log_e x = i\omega t + \text{ a constant}$$
$$= i\omega t + \log_e C_1 \text{ (say).}$$

This last result can be written as:

$$x = C_1 e^{i\omega t}.$$

Direct substitution shows that this function does in fact satisfy (1.4*a*). Taking the negative sign in (2.4) leads to another solution:

$$x = C_2 e^{-i\omega t}$$

and substitution again shows that the combined function

$$x = C_1 e^{i\omega t} + C_2 e^{-i\omega t} \quad . \quad . \quad . \quad (2.5)$$

satisfies the equation of motion (1.4*a*), and it contains the two arbitrary constants necessary for generality. Before determining the values of these constants so that the initial conditions (1.4*b*, *c*) are satisfied, it is advisable to restate (2.5) in more suitable terms. Elementary mathematical analysis (see Appendix I, section 33) shows that

$$e^{i\theta} = \cos\theta + i.\sin\theta$$
$$e^{-i\theta} = \cos\theta - i.\sin\theta$$

Putting ωt for θ, (2.5) can therefore be written:

$$x = A.\sin\omega t + B.\cos\omega t \quad . \quad . \quad . \quad (2.6)$$

where
$$A = i(C_1 - C_2)$$
$$B = C_1 + C_2$$

and
$$\omega^2 = \frac{k}{m}.$$

The form $x = A.\sin\omega t + B.\cos\omega t$ is the general solution of the equation of motion (1.4*a*). From the initial conditions (1.4*b*, *c*) the values of A and B are determined as

$$A = 0, \quad B = x_0$$

and the particular solution for these initial conditions is therefore

$$\left.\begin{aligned} x &= x_0.\cos\omega t \\ \text{where} \quad \omega^2 &= \frac{k}{m} \end{aligned}\right\} \quad . \quad . \quad . \quad (2.7)$$

The solution can be obtained by more direct methods in this particular case, but the method of operators described above is a very powerful one and can easily be extended to more complicated linear differential equations; it is for this reason that the method has been described in some detail. Using exactly the same procedure it can be shown that the solution of the equation

$$Dx = ax$$

is
$$x = Ce^{at}$$

and that the solution of the equation

$$D^2x + PDx + Qx = 0,$$

where P and Q are independent of x and t, is

$$x = C_1e^{\alpha t} + C_2e^{\beta t},$$

where α, β are the roots in D of the equation $D^2 + PD + Q = 0$. (See Exercise 1 at the end of this chapter.)

3. General solution and interpretation.

The function $x = A.\sin \omega t + B.\cos \omega t$ has been shown to be the general solution of the equation of motion (1.4*a*), and it has already been noted that the constants A and B are to be determined by specific conditions of the motion. The conditions most commonly given are the displacement and velocity at the instant $t = 0$. At this instant let

$$\left.\begin{aligned} x &= x_0 \\ \dot{x} &= \dot{x}_0 \end{aligned}\right\} \text{at } t = 0 \qquad . \quad . \quad . \quad (3.1)$$

Then
$$x_0 = B$$

and
$$\dot{x}_0 = A\omega,$$

and the solution is:

$$x = \frac{\dot{x}_0}{\omega} \sin \omega t + x_0 . \cos \omega t \quad . \quad . \quad . \quad (3.2)$$

The general solution (2.6) can be put into a more convenient form by using the relation

$$\sin(\theta + \phi) = \sin \theta . \cos \phi + \cos \theta . \sin \phi.$$

Dividing (2.6) by $\sqrt{A^2 + B^2}$ gives

$$\frac{x}{\sqrt{A^2+B^2}} = \frac{A}{\sqrt{A^2+B^2}} \sin \omega t + \frac{B}{\sqrt{A^2+B^2}} \cos \omega t,$$

and if

$$\frac{A}{\sqrt{A^2+B^2}} = \cos \phi$$

$$\frac{B}{\sqrt{A^2+B^2}} = \sin \phi$$

the result is obtained:

$$\left.\begin{aligned} x &= a.\sin(\omega t + \phi) \\ \text{where} \quad a &= \sqrt{A^2+B^2} \\ \tan \phi &= \frac{B}{A} \end{aligned}\right\} \qquad . \quad . \quad . \quad (3.3)$$

In this form of the solution there are again two constants (a and ϕ) to be determined. If the conditions at the instant $t = 0$ are $x = x_0$, $\dot{x} = \dot{x}_0$, then

$$x_0 = a.\sin\phi$$
$$\dot{x}_0 = a\omega.\cos\phi$$

whence
$$\left.\begin{aligned} \tan\phi &= \frac{\omega x_0}{\dot{x}_0} \\ a &= \sqrt{x_0{}^2 + \left(\frac{\dot{x}_0}{\omega}\right)^2} \end{aligned}\right\} \quad . \quad . \quad . \quad (3.4)$$

The angle ϕ is termed the *phase-angle* and the factor a is termed the *amplitude*, a useful interpretation of both terms being readily available. In Fig. 2 the point P moves in the counter-clockwise

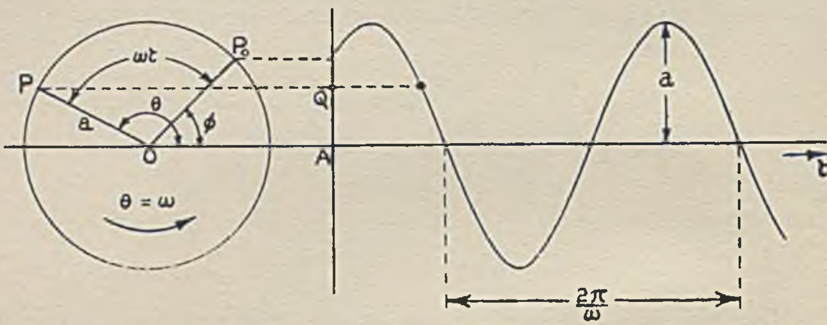


FIG. 2.—Generation of sine-wave by rotating vector.

direction on a circle, radius a and centre O, so that the line OP rotates about O with a constant angular velocity ω. If OA is a reference line, the projection AQ of the rotating vector OP on a line perpendicular to OA is $a.\sin\theta$, where θ is the angle POA. If P is at P_0 when $t = 0$, and the angle P_0OA is ϕ, then $AQ = a.\sin(\omega t + \phi)$. Thus AQ represents x, and the right-hand part of the diagram shows the graph of the displacement x plotted against time t.

A fundamental property of the sine function is that if the argument is increased by 2π radians the value of the function is unaltered, i.e.

$$\sin(\theta + 2\lambda\pi) = \sin\theta,$$

where λ is any integer; hence after successive intervals of time $\frac{2\pi}{\omega}$ the motion repeats itself exactly.

Displacement $x = a.\sin(\omega t+\phi)$
Velocity $\dot{x} = a\omega.\cos(\omega t+\phi)$
Acceleration $\ddot{x} = -a\omega^2.\sin(\omega t+\phi)$

$$\text{and } \begin{pmatrix}\sin\\\cos\end{pmatrix}\left[\omega\left(t+\frac{2\pi}{\omega}\right)+\phi\right] = \begin{pmatrix}\sin\\\cos\end{pmatrix}(\omega t+\phi).$$

The time-interval $\frac{2\pi}{\omega}$ is termed the *period* of vibration, and the reciprocal of the period is the *frequency*, or number of *cycles* (i.e. complete repetitions of the motion) in unit time. As the unit of time in every system of measurements is the second, the frequency is given as a number of cycles per second, commonly abbreviated to C.P.S. In practical work it is more convenient to state the frequency in cycles per minute, usually written C.P.M. The whole motion of the mass in Fig. 1*b*, under the initial conditions specified, can be described concisely thus: the displacement of the mass below the static position is a sine-function of time, the maximum displacements being $\pm x_0$ and the frequency of vibration being

$$F = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \text{ C.P.S.} = \frac{60}{2\pi}\sqrt{\frac{k}{m}} \text{ C.P.M.} \quad . \quad (3.5)$$

This frequency of free vibration of the system, with no damping or external applied forces, is termed the *natural frequency* of the system.

4. Units and general remarks.

In making use of the frequency formula

$$F = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \text{ C.P.S.},$$

due consideration must be given to the units of measurement. In scientific units the spring constant k is in poundals per foot deflection, or in dynes per centimetre, and the mass m is in pounds or grams. In the engineers' system of units k is in pounds per inch and m is in *slugs*, one slug being that mass to which unit force (one pound weight) imparts unit acceleration (one inch per second per second). Thus one slug equals g pounds, where g is the gravitational constant in *inch* units = 386 inches per second per second; so that one slug equals 386 lbs. (see Appendix II, section 37). Any of these three systems may be used so long as care is taken to ensure that all the quantities involved are given in consistent units.

Checking the units of the frequency formula (3.5),

$$k \equiv \text{lbs.ins.}^{-1}$$

$$m \equiv \frac{\text{lbs.}}{g} \equiv \text{lbs.ins.}^{-1}\text{secs.}^{2}$$

and thus
$$F \equiv \sqrt{\frac{\text{lbs.ins.}^{-1}}{\text{lbs.ins.}^{-1}\text{secs.}^{2}}} \equiv \text{secs.}^{-1}.$$

That this result is correct is evident from the fact that the frequency is the reciprocal of a time-period.

As a numerical example, suppose that the mass m is 200 lbs., and that the static deflection of the spring under the weight of 200 lbs. is 0·5 inches. Then

$$k = 2 \times 200 \text{ lbs./in.}$$

$$m = \frac{200}{386} \text{ slugs,}$$

and
$$F = \frac{1}{6{\cdot}28}.\sqrt{772} = 4{\cdot}42 \text{ c.p.s.}$$

$$= 265 \text{ c.p.m.}$$

This example suggests an alternative frequency formula in terms of the gravity deflection. If δ_s is the static deflection of the spring under the gravity-load of the mass, then

$$k = \frac{mg}{\delta_s}$$

and
$$F = \frac{1}{2\pi}\sqrt{\frac{g}{\delta_s}} \text{ c.p.s.} \quad . \quad . \quad . \quad (4.1)$$

This formula is sometimes quoted as $F = \frac{188}{\sqrt{\delta_s}}$ where F is the frequency in cycles per *minute* and δ_s is the static deflection in *inches*.

The amplitude and frequency are usually of far greater practical importance than the phase-angle. The two forms

$$A.\sin \omega t + B.\cos \omega t$$

and
$$a.\sin (\omega t + \phi)$$

both contain two constants which have to be determined by initial conditions, but in the first form both A and B must be found before the amplitude can be calculated; the form

$$x = a.\sin(\omega t + \phi)$$

is therefore normally to be preferred. There are two conventions with regard to the sign of the amplitude and the range of the phase-angle: (i) if the amplitude is considered as essentially positive, the phase-angle may have any value $0 \leqslant \phi < 2\pi$; (ii) if the amplitude is allowed to take a negative value, the phase-angle is restricted to the range $0 \leqslant \phi < \pi$, for

$$a.\sin(\omega t + \alpha + \pi) = -a.\sin(\omega t + \alpha).$$

The choice between these alternatives depends upon the nature of the problem.

It is well to note here what assumptions have been made in the preceding analysis. First, the effect of damping forces has been neglected; these forces are taken into account in Chapter II. Secondly, the spring is assumed to have a symmetrical linear characteristic, i.e. the forces required to extend and to compress it by equal distances are equal and proportional to the distance. This assumption is justified for small vibrations in very many practical applications; the analysis of unsymmetrical and non-linear spring characteristics is beyond the scope of this present work, but an excellent discussion is given by Den Hartog (see reference 1 in the Bibliography at the back of the book). The physical conditions postulated for the system analysed above do not arise in practice, but the type of resultant motion (i.e. sinusoidal) does occur. It is shown in Chapter VI that any cyclical variation can be expressed as the sum of a number of sinusoidal variations of different frequencies, and for this reason the properties of sinusoidal motion are very important.

5. Simple harmonic motion, and equivalent systems.

The type of motion in which the restoring force is proportional to the displacement from a mean position, and which therefore is expressed by an equation similar to (3.3), is termed *simple harmonic motion*. From the equation of motion (1.2) it is found that the acceleration for a unit displacement is numerically equal to k/m; the frequency equation (3.5) can therefore be written:

$$\begin{matrix}\text{frequency} \\ \text{(c.p.s.)}\end{matrix} = \frac{1}{2\pi}\sqrt{-\begin{matrix}\text{(acceleration for unit} \\ \text{displacement)}\end{matrix}} \quad . \quad (5.1)$$

and this property is typical of simple harmonic motion. If it is known that the restoring force on a body is proportional to the displacement from a fixed position, formula (5.1) can be applied at once to determine the frequency of vibration. The system of Fig. 3*a*, which consists of a mass m on the end of a light cantilever spring, is fundamentally the same as that of Fig. 1, for transverse vibrations; the spring-constant of the cantilever can be calculated from the physical constants (see

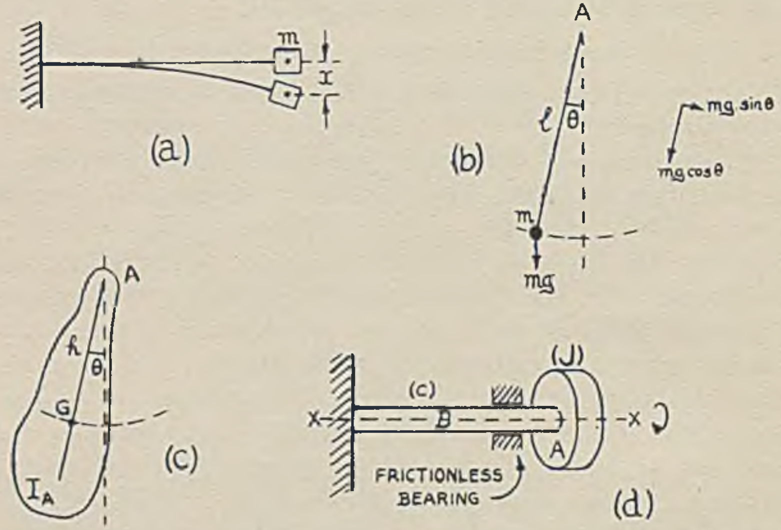


Fig. 3.—Systems equivalent to Fig. 1: (*a*) light cantilever; (*b*) simple pendulum; (*c*) compound pendulum; (*d*) torsional pendulum.

Appendix II, section 38). Two other examples of systems to which the formula (5.1) can be applied are (i) the pendulum, Figs. 3*b*, *c* and (ii) the torsional pendulum, Fig. 3*d*.

Rotatory systems.

It is shown in Appendix II, section 37, that if a rigid body rotates about a fixed axis the torque T and angular acceleration $\ddot{\theta}$ about that axis are linearly related by the equation

$$T = I\ddot{\theta},$$

where I is the moment of inertia of the body about that axis; and if the body is considered as being composed of infinitesimal particles typified by the particle of mass δm at a radius r from the axis, $I = \Sigma r^2 . \delta m$, the summation being extended over the whole body. There is a general analogy between rectilinear and angular motion, the corresponding quantities being listed in Table I together with the British scientific and engineers' units.

TABLE I

	Linear	Angular
Action Brit. Scientific . . . Engineers'	force poundals lbs.	torque poundals-ft. lbs.ins.
Response Brit. Scientific . . . Engineers'	acceleration ft./sec.^{2} ins./sec.^{2}	angular acceleration } radians/sec.^{2}
Inertia Brit. Scientific . . . Engineers'	mass lbs. lbs.ins.$^{-1}$sec.^{2}	moment of inertia lbs.ft.^{2} lbs.ins.sec.^{2}

Fig. 3*b* illustrates a simple pendulum, consisting of a particle of mass m hung on a light inextensible string of length l. If the string is deflected in a vertical plane through the point of support A, and the deflection from the vertical position is θ radians, the restoring torque about A is that due to the weight mg resolved normal to the string, i.e. $mgl.\sin\theta$. As this is a restoring torque tending to *decrease* θ, the equation of motion analogous to (1.2) is

$$mgl.\sin\theta = -I\ddot{\theta} = -ml^2\ddot{\theta}$$

For small displacements $\sin\theta = \theta$, so that the equation of motion becomes

$$mgl\theta = -ml^2\ddot{\theta}$$

or

$$\ddot{\theta} = -\frac{g}{l}\theta \quad . \quad . \quad . \quad . \quad (5.2)$$

As g and l are constant the motion is simple harmonic and the frequency is given by

$$F = \frac{\omega}{2\pi} \text{ C.P.S., where } \omega^2 = \frac{g}{l} \quad . \quad . \quad (5.3)$$

Similarly, in the compound pendulum illustrated in Fig. 3*c*, if m is the mass of the pendulum, I_A its moment of inertia about an axis through the support A and normal to the plane of motion, and h is the distance between the point of support and the centre of gravity of the pendulum, then

$$mgh\theta = -I_A\ddot{\theta}, \text{ or } \ddot{\theta} = -\frac{mgh}{I_A}\theta \quad . \quad . \quad (5.4)$$

for small values of θ.

Hence the frequency of vibration is given by

$$F = \frac{\omega}{2\pi} \text{ c.p.s., where } \omega^2 = \frac{mgh}{I_A} \quad . \quad . \quad (5.5)$$

Torsional systems.

Fig. 3*d* illustrates a torsional pendulum. A rigid flywheel A is attached to one end of a light shaft B, the other end of which is clamped. The gravity load of the weight of A is taken on a bearing which is supposed frictionless, and the flywheel A is free to rotate against the torsional stiffness of the shaft. Let this torsional stiffness be C, so that the torque that must be applied at A to twist it through an angle θ is $C\theta$, and let the moment of inertia of A about the axis $X - X$ be J. Then the equation of motion analogous to (1.2) is

$$J\ddot{\theta} = -C\theta$$

or

$$\ddot{\theta} = -\frac{C}{J}\theta \quad . \quad . \quad . \quad . \quad (5.6)$$

Hence the frequency of vibration is given by

$$F = \frac{\omega}{2\pi} \text{ c.p.s., where } \omega^2 = \frac{C}{J} \quad . \quad . \quad (5.7)$$

6. Two-mass systems with one degree of vibrational freedom.

The torsional system depicted diagrammatically in Fig. 4 is an example of a two-mass system having only one degree of

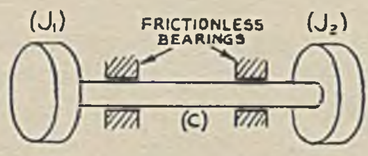


Fig. 4.—Two-mass system with one natural frequency.

freedom so far as vibrations are concerned. Let the angular displacements of the inertias J_1 and J_2 from the static position be θ_1, θ_2 in the same sense, and let the torsional stiffness of the shaft be C. Then, as the torque T transmitted by the shaft is proportional to the twist between the ends,

$$\left.\begin{aligned} J_1\ddot{\theta}_1 &= -C(\theta_1-\theta_2) \\ J_2\ddot{\theta}_2 &= -C(\theta_2-\theta_1) \end{aligned}\right\} \quad . \quad . \quad . \quad (6.1)$$

These equations can be written in the operator form :

$$\left.\begin{aligned} (J_1D^2+C)\theta_1 &= C\theta_2 \qquad (a) \\ (J_2D^2+C)\theta_2 &= C\theta_1 \qquad (b) \end{aligned}\right\} \quad . \quad . \quad (6.2)$$

Equations (6.2) are simultaneous operational equations which may tentatively be treated as algebraic equations, so long as the result obtained in this way is checked in the original equations. The method of treatment is as follows : operate on both sides of (6.2a) with the operator (J_2D^2+C) which occurs on the left-hand side of (6.2b). Thus :

$$(J_2D^2+C)(J_1D^2+C)\theta_1 = C(J_2D^2+C)\theta_2$$

The right-hand side of this last equation is, by reason of the equation (6.2b), equal to $C^2\theta_1$. Hence,

$$(J_2D^2+C)(J_1D^2+C)\theta_1 = C^2\theta_1$$

Dropping the variable θ_1 and expanding the brackets,

$$J_1J_2D^4+(J_1+J_2)CD^2 = 0$$

i.e.

$$D^2 = 0 \text{ or } -\frac{C(J_1+J_2)}{J_1J_2} \quad . \quad . \quad . \quad (6.3)$$

Comparing the solutions (6.3) with equation (2.3) it is evident that the two frequencies of vibration are

$$F = \frac{\omega}{2\pi} \text{ c.p.s., where } \omega^2 = 0 \text{ or } \frac{C(J_1+J_2)}{J_1J_2} \quad . \quad (6.4)$$

The zero frequency solution corresponds to a steady rotation $\theta_1 = \theta_2 = At+B$ (for which clearly $D^2 = 0$), while the finite solution gives the frequency of free vibration of the system. That there is only one degree of vibrational freedom is shown by the fact that the amplitudes and phase-angles of the motion of the two inertias are related by (6.1). Let the motion of the two inertias be

$$\theta_1 = A_1.\sin(\omega t+\phi_1)$$

$$\theta_2 = A_2.\sin(\omega t+\phi_2)$$

then

$$\ddot{\theta}_1 = -\omega^2A_1.\sin(\omega t+\phi_1)$$

and

$$\ddot{\theta}_2 = -\omega^2A_2.\sin(\omega t+\phi_2).$$

But from (6.1) $J_1\ddot{\theta}_1 = -J_2\ddot{\theta}_2$

hence

$$J_1A_1.\sin(\omega t+\phi_1) = -J_2A_2\sin(\omega t+\phi_2) \quad . \quad (6.5)$$

If the amplitudes A_1 and A_2 are regarded as positive (see the note on the sign convention in section 4) then $\sin(\omega t+\phi_2) = -1$ whenever $\sin(\omega t+\phi_1) = 1$. Hence ϕ_1 and ϕ_2 are separated by an odd integral multiple of π, for if

$$\omega t+\phi_1 = 2K_1\pi+\pi/2, \text{ then } \sin(\omega t+\phi_1) = 1$$

and $\omega t+\phi_2 = (2K_2+1)\pi+\pi/2$, K_1 and K_2 being integral. Thus

$$\phi_2-\phi_1 = [2(K_2 - K_1)+1]\pi.$$

Furthermore, (6.5) gives the result $J_1A_1 = J_2A_2$. If the motion of J_1 is $A_1.\sin(\omega t+\phi)$, then that of J_2 is

$$\theta_2 = \frac{J_1}{J_2}A_1.\sin(\omega t+\phi+\pi) \quad . \quad . \quad . \quad (6.6)$$

as $$\begin{matrix}(\sin)\\(\cos)\end{matrix}[2(K_2 - K_1)+1]\pi = \begin{matrix}(\sin)\\(\cos)\end{matrix}\pi.$$

The vibrational motion of the two masses is therefore such that they move in opposite directions, and it is important to note that there are only two arbitrary constants A_1 and ϕ in the formula (6.6). The initial conditions of the motion can be given as the displacements and velocities of the two inertias at the instant $t = 0$, and it is left as an exercise for the reader to include appropriate terms in the solution to express the steady rotation of the system, and generally to determine the four necessary arbitrary constants to make the solution fit the general initial conditions. (See Exercise 2 at the end of this chapter.)

The equivalent linear system would consist of two masses connected by a spring and supported on a smooth table, the motion being horizontal along the line joining the two masses. The simplified system of Fig. 1 is the result of making one such mass infinite and rotating the system from a horizontal to a vertical position.

EXERCISES I

1. Find the general solution of the equation

$$\ddot{x}+P\dot{x}+Qx = 0,$$

where P and Q are independent of x and t.

(See end of section 2.)

2. Determine the motion of the two flywheels in Fig. 4 if the initial displacements and velocities are Θ_1, Θ_2, $\dot{\Theta}_1$, $\dot{\Theta}_2$; and find the conditions (i) for steady motion only, with no vibration, and (ii) for vibrational motion only, with no steady rotation.

(Method: show that the steady rotation formula $\theta_1 = \theta_2 = At+B$ is a solution of the equations of motion, and determine these two constants and the two constants contained in the formula for vibratory motion, section 6, in terms of the initial conditions. Note that in the steady rotation there is no relative motion between the flywheels.)

3. Find the general solution, in terms of exponential functions, of the fourth-order equation

$$\frac{d^4y}{dx^4} = k^4y.$$

(This equation occurs in the theory of beam vibration, see Chapter V, section 25. Proceed as in section 2; the general solution contains four arbitrary constants.)

CHAPTER II

DAMPING AND FORCED VIBRATION

(One Degree of Freedom—continued)

7. Dissipation of energy.

It has already been noted that the amplitude of free vibration of a practical system diminishes in time so that the motion decays to zero, and that during this process there is a loss of energy. It is now necessary to determine the origin of this energy loss. If a system is constructed to resemble that in Fig. 1, and it is set in vibratory motion by an initial displacement, the motion is observed to be approximately sinusoidal during successive small intervals of time; in fact, it differs from sinusoidal motion only in the diminution of amplitude. In order to discover the cause of the loss of energy, the motion will be assumed sinusoidal and the work done during one cycle of the motion by an impressed force will be calculated.

Let the displacement be given by

$$x = X.\sin \omega t \quad . \quad . \quad . \quad . \quad (7.1)$$

and let the force f be a linear function of the displacement, velocity and acceleration, say

$$f = A + Px + Q\dot{x} + R\ddot{x} \quad . \quad . \quad . \quad (7.2)$$

This force is applied to the mass, and the work done by the force is given by

$$W = \int f.dx$$

But $\dfrac{dx}{dt} = \dot{x}$, and so $dx = \dot{x}.dt$. Thus

$$W = \int f\dot{x}.dt.$$

If attention is confined to one cycle of the motion the limits of integration differ by the period of vibration. The frequency being $\omega/2\pi$, the period is $2\pi/\omega$, and the work done by the force during one cycle is

$$W = \int_0^{\frac{2\pi}{\omega}} f\dot{x}.dt$$

Thus $$W = \int_0^{\frac{2\pi}{\omega}} (A+Px+Q\dot{x}+R\ddot{x})\dot{x}\,.\,dt.$$

Giving x its sinusoidal form (7.1) the integral becomes

$$W = \int_0^{\frac{2\pi}{\omega}} [A+(P-R\omega^2)X.\sin\omega t+Q\omega X.\cos\omega t]\omega X.\cos\omega t.dt$$

$$= \int_0^{\frac{2\pi}{\omega}} A\omega X.\cos\omega t.dt + \int_0^{\frac{2\pi}{\omega}} \tfrac{1}{2}\omega X^2(P-R\omega^2)\sin 2\omega t.dt$$

$$+ \int_0^{\frac{2\pi}{\omega}} \tfrac{1}{2}X^2\omega^2 Q(1+\cos 2\omega t)dt$$

$$= \left[\tfrac{1}{2}\omega^2X^2Qt\right]_0^{\frac{2\pi}{\omega}} = \omega X^2Q\pi \qquad . \quad . \quad . \quad . \quad (7.3)$$

It is evident that the only part of the force that has done work over the whole cycle is the part $Q\dot{x} = QX\omega\cos\omega t$; the other parts of the force each contribute nothing to the work done over the cycle. This result suggests that an explanation of the dissipation of energy during the motion of the practical system may be that work is being done against a *damping* force proportional to the velocity of the mass. The theoretical results obtained by analysing the motion on such an assumption are in good agreement with practical results.

8. Damped motion of single-mass system.

The theoretical system of Fig. 1 is now extended, as shown in Fig. 5, by the addition of a " dashpot," the function of which is to exert a force $c\dot{x}$ proportional to the velocity of the mass and opposed to the direction of motion. The symbolic spring and dashpot do not necessarily represent separate parts of the system, but rather different properties of the same part, i.e. the practical spring. The symbolic spring represents that property of the spring whereby it opposes motion of the mass with a force proportional to the displacement, and is merely an energy-storing device incapable of dissipation; the constant of proportionality, k, is sometimes referred to as the *non-dissipative stiffness*. The symbolic dashpot represents that property of the spring (and surrounding air) whereby motion is opposed with a force $c\dot{x}$ proportional to the velocity, and is merely a dissipative device incapable of storing energy; the constant c is called the *damping coefficient* or *resistance*.

Apart from air-resistance, the energy is dissipated by way of internal friction in the spring; in the system of Fig. 5 the dashpot represents both the resistance of the spring and that of the air surrounding the system. The force exerted by the dash-

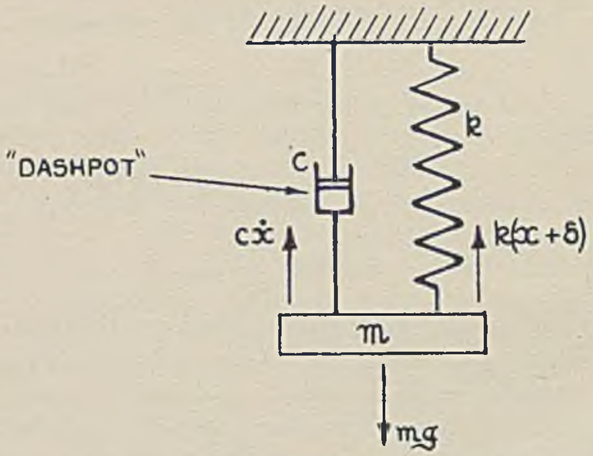


Fig. 5.—System of Fig. 1 modified by inclusion of a dashpot.

pot on the mass is $-c\dot{x}$, and the equation of motion is therefore

$$m\ddot{x} = -kx - c\dot{x} \quad . \quad . \quad . \quad (8.1)$$

As before, let $\omega^2 = k/m$, and further put $2\gamma = c/m$. (8.1) then becomes

$$\ddot{x}+2\gamma\dot{x}+\omega^2 x = 0,$$

and the operational equation obtained by using the operator D and dropping the variable x is

$$D^2+2\gamma D+\omega^2 = 0 \quad . \quad . \quad . \quad (8.2)$$

Proceeding on the lines of the previous work (Section 2) this equation is solved for D, thus:

$$D = \binom{\alpha}{\beta} = -\gamma \pm \sqrt{\gamma^2 - \omega^2}$$

and in this solution α and β take the place of $\pm i\omega$ in (2.4). The form of (2.5) suggests the solution

$$x = C_1 e^{\alpha t}+C_2 e^{\beta t}$$

i.e.

$$x=C_1 \exp(-\gamma+\sqrt{\gamma^2-\omega^2})t+C_2 \exp(-\gamma-\sqrt{\gamma^2-\omega^2})t \quad . \quad (8.3)$$

where the notation $\exp\theta$ is used in place of e^θ for convenience. Direct substitution shows that the function (8.3) does in fact satisfy the equation of motion (8.1), and as it contains the necessary two arbitrary constants it is the required general solution.

The physical significance of the solution (8.3) depends upon the nature of the expression $\sqrt{\gamma^2 - \omega^2}$. Three cases need be considered, according to whether γ is greater than, equal to, or less than ω.

(i) If $\gamma > \omega$, both roots α and β are real and negative, for $\sqrt{\gamma^2 - \omega^2}$ is then real and numerically less than γ; the graph of the displacement plotted against time is therefore the sum of two decreasing exponential curves (see Appendix I, section 31).

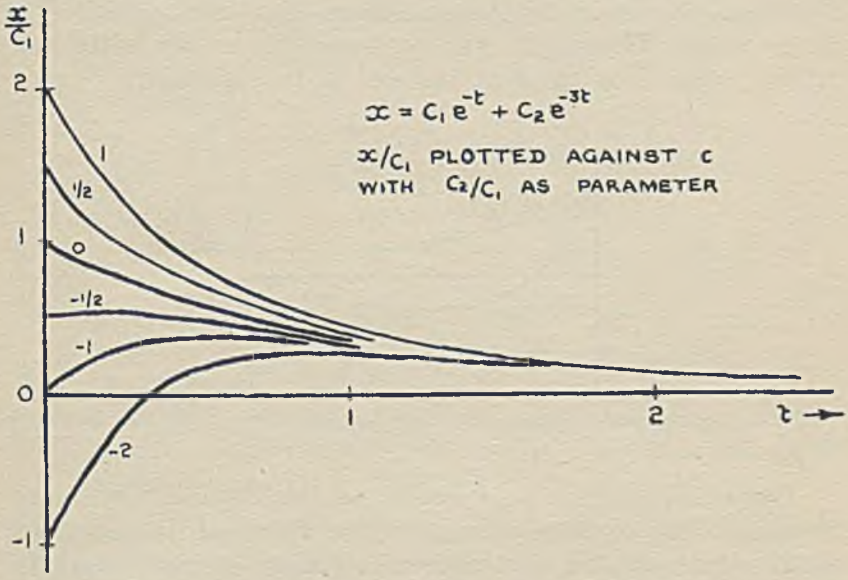


FIG. 6.—Displacement : time curve for damped motion (more than critical damping).

The shape of the graph depends upon the values of C_1 and C_2, which in turn depend on the initial conditions of the motion. Fig. 6 illustrates some typical shapes; in this diagram the function x/C_1 is plotted against time for the particular values $\alpha = -1$, $\beta = -3$, and for various values of the ratio C_2/C_1.

(ii) If $\gamma = \omega$, both roots α and β are equal to γ. (8.3) becomes

$$x = (C_1 + C_2)e^{-\gamma t},$$

but this function involves only one arbitrary constant $(C_1 + C_2)$ and cannot be the most general solution. To find the general solution the following procedure is adopted :

Let $x = z.\exp(-\gamma t)$, where z is to be determined. Then

$$Dx = -\gamma e^{-\gamma t}.z + e^{-\gamma t}.Dz$$

and $$D^2x = \gamma^2 e^{-\gamma t}.z - 2\gamma e^{-\gamma t}.Dz + e^{-\gamma t}.D^2z.$$

Substitution of these values in (8.2) leads to the result

$$(\omega^2 - \gamma^2)e^{-\gamma t}.z + e^{-\gamma t}.D^2z = 0$$

i.e. $D^2z = 0$, as $\gamma = \omega$.

Double integration with respect to time gives :

$$z = C_3t + C_4,$$

and $$x = (C_3t + C_4)e^{-\gamma t} \quad . \quad . \quad . \quad (8.4)$$

The displacement-graph plotted against time may have either of the forms indicated in Fig. 7, depending on the initial conditions. (See Exercise 1 at the end of this chapter.)

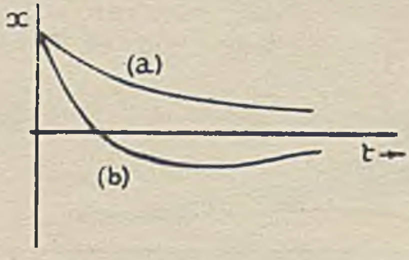


FIG. 7.—Displacement : time curve for damped motion (critical damping).

The condition $\gamma = \omega$ is known as the condition of *critical damping*; the corresponding critical value of the damping coefficient ($c = c_0$) is a convenient standard of comparison for damping coefficients and is easily found :

$$\gamma = \omega = \sqrt{\frac{k}{m}}$$

and $$c = c_0 = 2m\gamma = 2m\omega$$

whence $$c_0 = 2\sqrt{mk} \quad . \quad . \quad . \quad . \quad . \quad (8.5)$$

The physical significance of the critical-damping condition is that in this condition the system is brought practically to rest in a shorter time than if the damping is either greater or less than the critical value (see Exercise 4 at the end of the chapter).

(iii) If $\gamma < \omega$, let $q = \sqrt{\omega^2 - \gamma^2}$, so that q is real.

Then $$x = e^{-\gamma t}(C_1e^{iqt} + C_2e^{-iqt}),$$

and as in Section 2 of Chapter I this expression may be rewritten in trigonometric form :

$$\left.\begin{aligned} x &= e^{-\gamma t}(A.\sin qt+B.\cos qt) \\ \text{where} \quad A &= i(C_1-C_2) \\ B &= C_1+C_2 \\ \text{or alternatively,} \quad & \\ x &= e^{-\gamma t}.a.\sin(qt+\phi) \\ \text{where} \quad a &= \sqrt{A^2+B^2} \\ \tan\phi &= B/A \end{aligned}\right\} \quad . \quad . \quad . \quad (8.6)$$

The graph of the displacement plotted against time has the form indicated in Fig. 8, the curve being a damped sine-wave

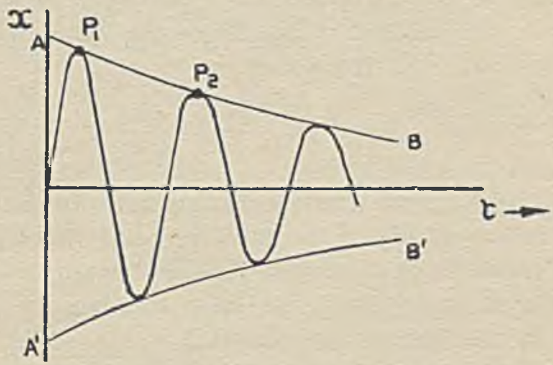


FIG. 8.—Displacement : time curve for damped motion (less than critica damping).

lying between a decreasing exponential curve AB and its mirror-image $A'B'$ in the time-axis.

Case (iii), when the damping coefficient c is less than the critical value c_0, is the most important case in practice. A useful property of the curve (Fig. 8) is the manner in which the wave decreases in amplitude. The points P_1, P_2 are successive maxima at times t_1 and t_2, so that $t_2 = t_1+2\pi/q$; the amplitudes x_1 and x_2 at these times are very nearly equal to the values of the exponential function $a.\exp(-\gamma t)$ at P_1 and P_2, i.e.

$$\left.\begin{aligned} x_1 &= a.e^{-\gamma t_1} \\ x_2 &= a.e^{-\gamma t_2} \end{aligned}\right\}\text{approximately.}$$

Let the ratio x_2/x_1 be μ. Then $\mu = \exp[-\gamma(t_2-t_1)]$

i.e. $\quad \mu = \exp(-2\gamma\pi/q) = \exp(-\pi c/mq)$

This ratio is independent of the initial amplitude and the time

t_1, and is thus the same for all pairs of consecutive maxima. If $\Delta = -\log_e \mu$

then
$$\Delta = \frac{\pi c}{mq} \quad . \quad . \quad . \quad . \quad (8.7)$$

This quantity is termed the *logarithmic decrement*, and provides a ready means of determining the damping coefficient c of a system, as all the quantities in (8.7) except c can be evaluated experimentally from the physical constants of the system and a displacement curve or *vibration record*.

There is no real cycle of the motion, as the amplitude of vibration is constantly decreasing, but the maxima and minima of the displacement curve (Fig. 8) each recur after successive equal intervals of time $2\pi/q$, and the quantity $q/2\pi$ is termed the " frequency " of vibration. The larger the value of ($\gamma = c/2m$) is made, the lower the frequency

$$\frac{q}{2\pi} = \frac{1}{2\pi}\sqrt{\omega^2 - \gamma^2}$$

becomes.

As a practical result it is interesting to note that after ω/γ " cycles " the amplitude of the motion is only 0·19 per cent. of its original value, so that in most practical examples the motion is effectively damped-out after this number of cycles (see Exercise 5 at the end of the chapter).

9. Forced motion.

So far the analysis has been concerned with free vibrations, i.e. vibrations maintained solely by the energy stored in the system at the commencement of the motion. If the vibration

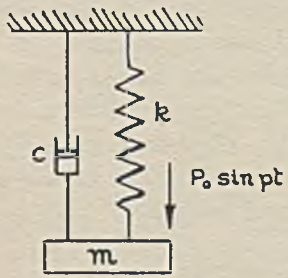


FIG. 9.—System of Fig. 5 modified by inclusion of an applied force.

of a damped system is to be maintained at constant amplitude there must be an energy input. Fig. 9 represents the system of Fig. 5 acted on by a harmonic force $P = P_0 \sin pt$. The equation of motion is

$$m\ddot{x} = P_0 \sin pt - kx - c\dot{x}$$

i.e. $$\left.\begin{aligned} \ddot{x}+2\gamma\dot{x}+\omega^2 x &= \frac{P_0}{m}\sin pt \\ 2\gamma = c/m,\ \omega^2 &= k/m \end{aligned}\right\} \quad . \quad . \quad (9.1)$$

where

In operator form this becomes

$$[D^2+2\gamma D+\omega^2]x = \frac{P_0}{m}\sin pt.$$

Operating on both sides of the equation with $[D^2 - 2\gamma D+\omega^2]$

$$[(D^2+\omega^2)^2 - 4\gamma^2 D^2]x = \frac{P_0}{m}[D^2 - 2\gamma D+\omega^2]\sin pt$$

and this last equation can tentatively be written as :

$$x = \frac{P_0}{m}.\frac{[D^2 - 2\gamma D+\omega^2]}{[(D^2+\omega^2)^2 - 4\gamma^2 D^2]}\sin pt$$

or $$x = \frac{P_0}{m}[D^2 - 2\gamma D+\omega^2][(D^2+\omega^2)^2 - 4\gamma^2 D^2)^{-1}\sin pt \quad (9.2)$$

If the second bracket, with its negative index, is now expanded by the Binomial Theorem, it is seen that every term in the expansion contains powers of D^2. But

$$D^2 \sin pt = -p^2 \sin pt$$
$$D^4 \sin pt = \quad p^4 \sin pt, \text{ etc.,}$$

so that wherever the operator D^2 appears in (9.2) the multiplier $-p^2$ can be substituted. (It is to be noted that the validity of this procedure is to be checked by ascertaining whether the derived solution satisfies the original equation.) 9.2 now becomes

$$x = \frac{(P_0/m)[\omega^2 - p^2 - 2\gamma D]\sin pt}{(\omega^2 - p^2)^2+4\gamma^2 p^2}$$

and performing the operation indicated by the square bracket the result is obtained :

$$x = \frac{P_0/m}{(\omega^2 - p^2)^2+4\gamma^2 p^2}[(\omega^2 - p^2)\sin pt - 2\gamma p.\cos pt]$$

which can be written as :

$$\left.\begin{aligned} x &= b.\sin(pt - \psi) \\ b &= \frac{P_0/m}{[(\omega^2 - p^2)^2+4\gamma^2 p^2]^{\frac{1}{2}}} \\ \tan\psi &= \frac{2\gamma p}{\omega^2 - p^2}. \end{aligned}\right\} \quad . \quad . \quad (9.3)$$

Using the equations $2\gamma = c/m$, and $c_0 = 2\sqrt{mk}$, (9.3) can be written in an alternative form which is sometimes more useful:

$$\left.\begin{aligned} b &= \frac{P_0}{m\omega^2\left[\left(1-\frac{p^2}{\omega^2}\right)^2+4\left(\frac{c}{c_0}\right)^2\left(\frac{p}{\omega}\right)^2\right]^{\frac{1}{2}}} \\ \text{and} \qquad \tan\psi &= \frac{2.\frac{c}{c_0}.\frac{p}{\omega}}{1-\frac{p^2}{\omega^2}} \end{aligned}\right\} \quad . \quad (9.4)$$

This solution contains no arbitrary constants and cannot therefore be the general solution. It is termed the *particular integral* of the equation (9.1). To determine the general solution the procedure is as follows:

Let $x = u+v$, where v stands for the solution (9.3). Substituting in the operator form of (9.1),

$$[D^2+2\gamma D+\omega^2](u+v) = \frac{P_0}{m}\sin pt$$

i.e. $$[D^2+2\gamma D+\omega^2]u+[D^2+2\gamma D+\omega^2]v = \frac{P_0}{m}\sin pt$$

But v satisfies (9.1) and hence

$$[D^2+2\gamma D+\omega^2]v = \frac{P_0}{m}\sin pt$$

thus $$[D^2+2\gamma D+\omega^2]u = 0.$$

This last equation is exactly similar to that discussed in Section 8, and if $\gamma < \omega$ the solution is given by (8.6), i.e.

$$u = e^{-\gamma t}.a.\sin(qt+\phi) \quad . \quad . \quad . \quad (9.5)$$

The function u contains two arbitrary constants (a and ϕ) and is itself the solution of the equation obtained by putting zero in place of the right-hand side of (9.1). Such a function is termed a *complementary function*, and it is a general result in the theory of differential equations that the complete solution of an equation such as (9.1) is the sum of the complementary function and a particular integral determined as in (9.3). (See Exercise 2 at the end of this chapter.)

The complete solution x is given by

$$x = u+v = b.\sin(pt-\psi)+e^{-\gamma t}.a.\sin(qt+\phi) \quad . \quad (9.6)$$

The term due to the complementary function u is a decreasing time-function, as in Fig. 8; after a time this part of the motion is practically damped out, and effectively the displacement function consists merely of the particular integral v. Such a motion is termed "steady-state forced vibration," and is the only part of the complete motion that is at all important.

10. Resonance.

The frequency of the steady-state motion is the same as that of the impressed force, namely $p/2\pi$. It is of interest to investigate the relation between the amplitude of steady-state motion and the frequency. From (9.3) the amplitude is given by

$$b = \frac{P_0/m}{[(\omega^2 - p^2)^2 + 4\gamma^2 p^2]^{\frac{1}{2}}}$$

If P is independent of the frequency, b has a stationary value whenever

$$B = (\omega^2 - p^2)^2 + 4\gamma^2 p^2$$

has a stationary value (maximum or minimum). Differentiating B with respect to p, which is proportional to the frequency of the impressed force,

$$\begin{aligned}\frac{dB}{dp} &= 8\gamma^2 p + 2(\omega^2 - p^2)(-2p)\\ &= 4[2\gamma^2 p + p(p^2 - \omega^2)].\end{aligned}$$

Hence b has a stationary value when $p = 0$

or $$2\gamma^2 + p^2 - \omega^2 = 0$$

i.e. $$p = \sqrt{\omega^2 - 2\gamma^2} \quad . \quad . \quad . \quad (10.1)$$

It can be shown that when $p = 0$, b is a minimum, and for the value (10.1) b is a maximum; thus the amplitude of the steady-state motion has a maximum at the frequency

$$\frac{p}{2\pi} = \frac{1}{2\pi}\sqrt{\omega^2 - 2\gamma^2}$$

and the graph of displacement-amplitude plotted against frequency has the general form indicated in Fig. 10.

The condition of maximum displacement-amplitude for a given impressed force is termed *resonance*, and the corresponding frequency is termed the *resonant frequency.* The practical importance of resonance cannot be emphasised too strongly. Indeed, the treatment of vibration problems in engineering is almost entirely a matter of detecting resonant frequencies and avoiding resonant conditions, for at resonance the vibration-amplitude of the system is maximum and the resulting stresses in the system similarly have maximum values. The normal procedure in dealing with practical vibration problems is to determine the source of vibration (i.e. the origin of the input force) and the resonant frequencies, either theoretically or practically, and then to arrange matters so that the machine does not run in a resonant condition. There are usually four ways in which this may be done, any or all of which may be possible in a particular case: (i) eliminating the source of vibration, (ii) absorbing the input

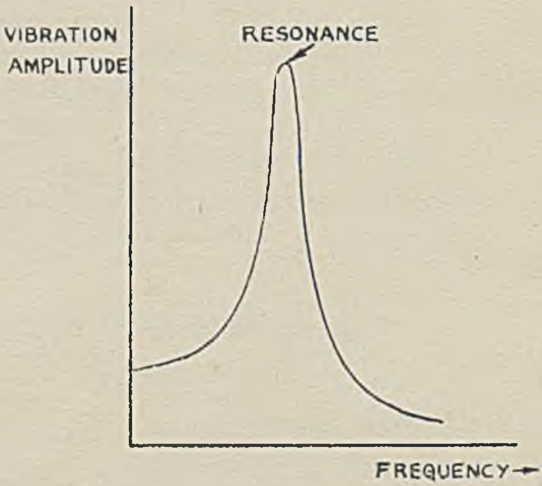


Fig. 10.—Resonance curve for system of Fig. 9.

energy in some way, (iii) altering the speed-range of the machine to avoid resonance, and (iv) removing the resonance from the operating-range of speed by suitable alteration of the dynamic constants of the system. The precise technique involved is a matter of engineering detail and as such is outside the scope of this work, but one type of energy-absorber is discussed in Chapter III, section 15.

A measure of the intensity of the resonance effect is afforded by the quantity known as the *dynamic magnifier*. The dynamic magnifier at a point in a vibrating system is defined for a particular frequency as the ratio of two deflections: (i) the dynamic amplitude at that frequency (as given, for example, by b in (9.3)) and (ii) the deflection caused by the application of a static force equal in magnitude to the amplitude of the forcing input. In the system of Fig. 9 the dynamic forcing input is $P_0 \sin pt$, and the static deflection under the action of a force P_0 is P_0/k at the mass. The dynamic magnifier at the mass is thus given by

$$M = \frac{\omega^2}{[(\omega^2 - p^2)^2 + 4\gamma^2 p^2]^{\frac{1}{2}}} \quad . \quad . \quad . (10.2a)$$

as $\omega^2 = k/m$

or

$$M = \frac{1}{\sqrt{\left(1 - \frac{p^2}{\omega^2}\right)^2 + 4\frac{c^2}{c_0^2} \cdot \frac{p^2}{\omega^2}}} \quad . \quad (10.2b)$$

using the relations $2\gamma = c/m$, $c_0 = 2\sqrt{mk}$.

Using the equation (10.1) it can easily be shown that at resonance this expression becomes :

$$M_{res.} = \frac{1}{2\frac{c}{c_0}\sqrt{1 - \left(\frac{c}{c_0}\right)^2}} \quad . \quad . \quad (10.3)$$

where $c_0 = 2\sqrt{mk}$ as before.

The greater the ratio of the damping coefficient c to its critical value c_0, the less becomes the value of the dynamic magnifier at resonance. Fig. 11*a* represents the dynamic magnifier curves for the one-mass system for two values of the damping coefficient : 10% and 20% of the critical value. The magnification of amplitude due to the resonant condition is seen to be very much greater for small damping forces than it is for large damping forces. If there is no damping, the dynamic magnifier at resonance has an infinite value, but this is of course a purely theoretical consideration ; the curve for zero damping is drawn dotted in Fig. 11*a*, and it is evident that for a lightly damped system the magnifier is not very different from that for zero damping, except in the neighbourhood of the resonant frequency.

The general shape of the dynamic magnifier curve Fig. 11*a* is typical for single-mass systems with a forcing input of the form $P = P_0 \sin pt$, where P_0 is constant ; this type of forcing is commonly termed " constant excitation." A type of excitation that is of very frequent occurrence in engineering is that due to centrifugal action on unbalanced rotors ; the reactions at the bearings supporting the shaft are then of the form $P = Qp^2 \sin pt$, as the centrifugal force acting on the unbalanced rotor is proportional to the square of the rotational speed. The dynamic magnifier curve for this type of forcing, commonly termed "inertia excitation," has the form indicated in Fig. 11*b*, from which it can be seen that, for inertia excitation, the magnifier has zero value at zero frequency and approaches unit value at

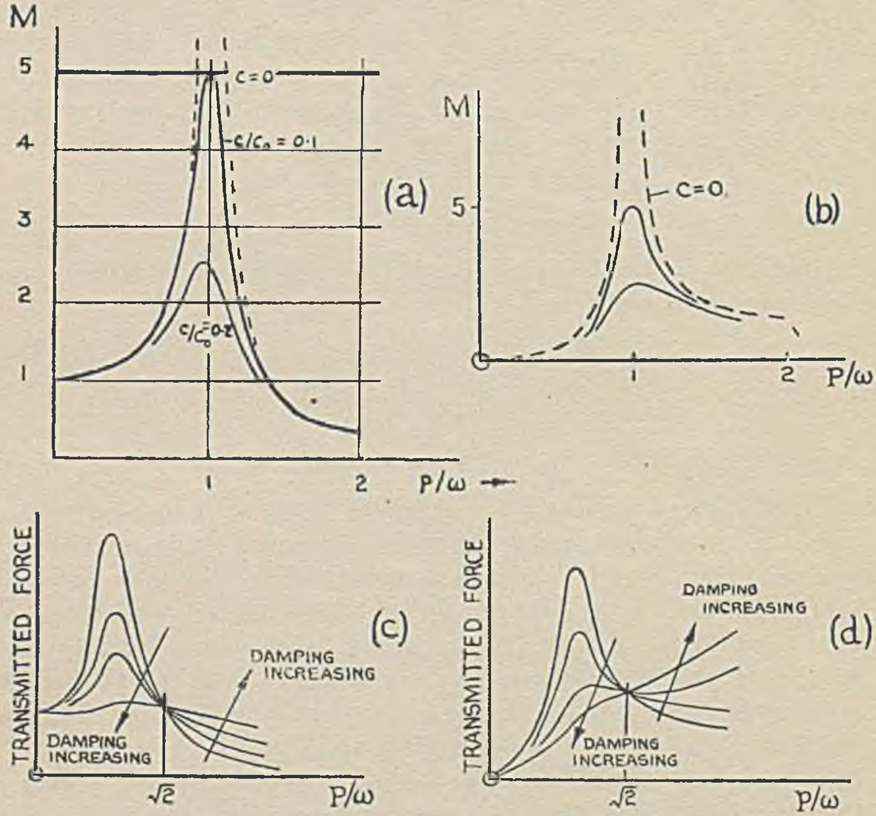


Fig. 11.—Dynamic magnifier at the mass in Fig. 9: (*a*) constant excitation; (*b*) inertia excitation. Force transmitted to support: (*c*) constant excitation; (*d*) inertia excitation.

very high frequencies. (*Note.*—In this case the relevant static deflection to be used in the definition of the dynamic magnifier —see the paragraph immediately preceding equations 10.2—is the deflection Q/k under the action of a static force Q, not Qp^2 which has zero value at zero frequency.)

Although the free vibration, given by the complementary function (9.5), is damped-out after a short time owing to the decreasing exponential factor, its amplitude may be quite considerable at the beginning of the motion, the amplitude of the total motion (9.6) being thereby increased temporarily to an extent which can be troublesome in practice; this effect is particularly significant in cases where the exciting force is started and stopped frequently, when the "starting transient" (i.e. the complementary function) may seriously increase the risk of fractures being caused by excessive vibration amplitudes. Many instruments are critically-damped to avoid this vicious condition. (See Exercise 6 at the end of the chapter.)

11. Resonance and natural frequency.

In the present section the resonance characteristics of a single-mass system under the action of a constant excitation force are discussed. Analogous results are easily obtainable for the case of inertia excitation, the most important practical characteristic of which is concerned with the problem of insulation, see Section 12, Example I.

The resonant frequency of the single-mass system of Fig. 9 is, from (10.1),

$$F_{res.} = \frac{p}{2\pi} = \frac{1}{2\pi}\sqrt{\omega^2 - 2\gamma^2} \qquad . \qquad . \; (11.1)$$

and this expression can be rewritten in terms of the critical damping coefficient ($c_0 = 2\sqrt{mk}$) as

$$F_{res.} = \frac{\omega}{2\pi}\sqrt{1 - 2\left(\frac{c}{c_0}\right)^2} \; . \qquad . \qquad . \; (11.2)$$

It can be seen from (11.2) that if the damping force is small so that the square of the ratio c/c_0 can be neglected in comparison with unity, the resonant frequency is equal to $\omega/2\pi$. This frequency is the natural frequency for zero damping (Chapter I), and if the system is lightly damped no great error is involved in neglecting the damping force altogether and making use of the natural frequency of the resulting system instead of the true resonant frequency.

It is shown in Chapter IV, section 18, that this result can be extended to multi-mass systems, for such systems vibrating at *any* frequency can be considered as built-up of partial single-degree-of-freedom systems, plus an auxiliary mass known as the *effective inertia* of the complete system; and the resonant frequency of each such partial system is approximately equal to the natural frequency for zero damping, so long as the damping forces are small. Some examples of the error involved are listed in Table II for various values of the ratio c/c_0.

TABLE II

Damping coeff. / Critical value c/c_0	Natural frequency for zero damping / Resonant frequency ω/p	Error %
0	1·00	0
0·1	1·01	1
0·2	1·04	4
0·3	1·10	10
0·4	1·21	21
0·5	1·41	41

The error quoted is that involved in using the natural frequency $\omega/2\pi$ instead of the resonant frequency $p/2\pi$.

The practical importance of this result is that it is very much easier to neglect damping forces altogether, and calculate the *natural* frequencies of the resulting modified system, than it is to calculate the *resonant* frequencies; Table II shows that the error involved in this procedure is negligible in the case of lightly damped systems.

One further useful property of the resonant condition of lightly damped systems is the value of the phase-angle ψ in the expression (9.3) or (9.4). When $p = \omega$, i.e. at the natural frequency for zero damping, the angle ψ has the value $\pi/2$, as the

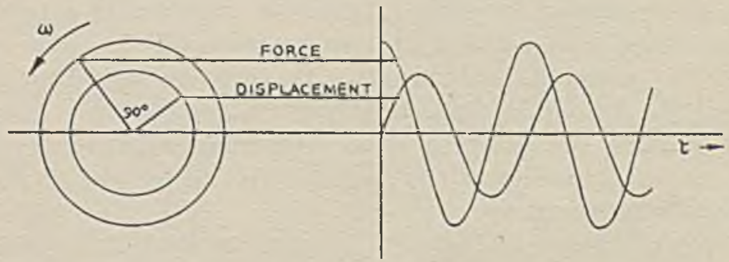


FIG. 12.—90° phase-angle between force and displacement waves at resonance.

denominator of the expression for the tangent is then zero, and the tangent has an infinite value. Thus in a lightly damped system the phase-angle at resonance is approximately $\pi/2$. This angle is the angle by which the displacement-wave $b.\sin(pt - \psi)$ lags behind the force-wave $P_0 \sin pt$, as illustrated in Fig. 12. Fig. 13 shows representative curves of the phase-angle for various values of the ratio c/c_0, plotted against the non-dimensional ratio

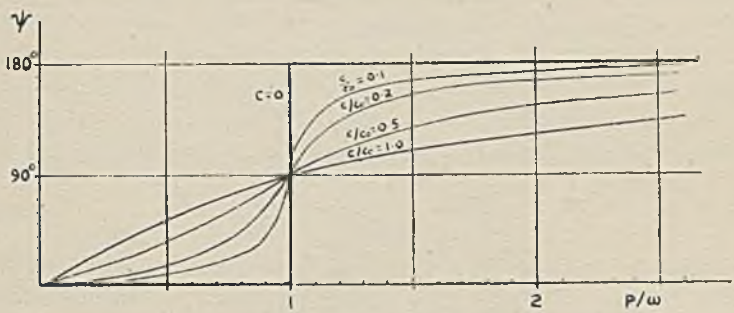


FIG. 13.—Variation of phase-angle with frequency.

p/ω which is proportional to the frequency $p/2\pi$. For very small values of the ratio c/c_0 the curve approximates to the discontinuous line for $c = 0$, the characteristic of which is that for excitation frequencies below the natural frequency the displacement is in phase with the force, whereas for frequencies above the natural frequency the two waves are 180° out of phase so that the rotating vectors in Fig. 12 are separated by 180° or π radians.

12. Practical examples.

In order to illustrate the use of the foregoing analysis, two practical examples are discussed in this section. The first involves the principles of vibration *insulation*, whereby the force transmitted from a vibrating machine to its support is reduced by means of an elastic mounting ; the second is the case of a seismic vibrograph for recording vibration-amplitudes, and in the treatment of this problem the complex-number method of calculation is introduced.

I. Insulation.

A machine, which may be considered as a rigid mass of 1,930 lbs., is acted on by an alternating vertical force, whose amplitude is 400 lbs., at a frequency of 3,000 C.P.M. The damping forces in the system can be made to have 20% of the critical value. If the alternating load is taken equally by four spring feet of identical design, what must the stiffness of each mounting be in order that the total alternating force transmitted to the base is only 100 lbs. maximum ?

In essentials the system is that of Fig. 9 inverted, the four identical spring mounts being considered as combined together. In Fig. 9 the total alternating reaction on the support is $kx+c\dot{x}=f_r$ and if $x = b.\sin(pt-\psi)$ the force is

$$f_r = b[k.\sin(pt-\psi)+cp.\cos(pt-\psi)]$$
$$= b\sqrt{k^2+c^2p^2}.\sin(pt+\varepsilon)$$

where the phase-angle ε is not important. The maximum value of the reaction force is

$$f_{max.} = b\sqrt{k^2+c^2p^2}$$

which can be written in a form similar to that used in (9.4) as :

$$f_{max.} = bm\omega^2\sqrt{1+4\lambda\mu} = 2bm\omega^2\sqrt{\tfrac{1}{4}+\lambda\mu}$$

where $$\lambda = \left(\frac{p}{\omega}\right)^2$$

and $$\mu = \left(\frac{c}{c_0}\right)^2 = (0{\cdot}2)^2 = 0{\cdot}04$$

i.e. $$f_{max.} = 2bm\omega^2\sqrt{0{\cdot}25+0{\cdot}04\lambda}.$$

Substituting for b from (9.4),

$$f_{max.} = \frac{2P_0\sqrt{0{\cdot}25+0{\cdot}04\lambda}}{[(1-\lambda)^2+0{\cdot}16\lambda]^{\frac{1}{2}}}$$

and when $P = 400$ lbs., $f_{max.}$ must be 100 lbs.

Hence $$8\sqrt{0{\cdot}25+0{\cdot}04\lambda} = \sqrt{(1-\lambda)^2+0{\cdot}16\lambda}$$

Solving for λ,

$$\left(\frac{p}{\omega}\right)^2 = \lambda = 6{\cdot}66$$

i.e. $$\frac{p}{\omega} = 2{\cdot}58$$

But it is given that $p/2\pi = 3{,}000/60 = 50$ C.P.S.

Hence $\omega = 50 \times 2\pi \div 2{\cdot}58 = 122$

Also $\omega = \sqrt{\frac{k}{m}}$, and $m = 1{,}930 \div 386 = 5$ slugs.

Hence $k = 5 \times 122^2 = 7{\cdot}45 \times 10^4$ lbs./in.

The total stiffness of the four similar mounts is thus $7{\cdot}45 \times 10^4$ lbs./in., and that of each mount is one-quarter of this value, i.e. $1{\cdot}86 \times 10^4$ lbs./in.

This example shows the property of partial insulation possessed by a spring mounting; by choosing the mounting so that the working frequency is 2·58 times the natural frequency, the force transmitted to the base is reduced to a quarter of the applied force.

In connection with this insulation effect, it is a matter of some practical importance to determine the precise result of increasing the damping forces in the systems. By the method given above the general formula for the transmitted force due to an applied force $P = P_0 \sin pt$ is obtained as:

$$F_{max.} = \frac{P_0\sqrt{1+4\lambda\mu}}{\sqrt{(1-\lambda)^2+4\lambda\mu}} \quad . \quad . \quad . \quad (12.1)$$

By differentiating this expression with respect to μ, keeping λ constant, it is found that for λ less than 2, i.e. for a forcing frequency less than $\sqrt{2}$ times the undamped natural frequency, an increase in the damping force causes a decrease in the transmitted force, whereas for λ greater than 2, i.e. for a forcing frequency greater than $\sqrt{2}$ times the undamped natural frequency, an increase in the damping force causes an increase in the transmitted force. Representative curves are shown in Fig. 11*c*.

If Qp^2 is substituted for P_0 in (12.1), so that the expression corresponds to the case of inertia-excitation, the increase in transmitted force due to an increase in the damping force, when the forcing frequency is greater than $\sqrt{2}$ times the undamped natural frequency, is more violent than for the case of constant excitation; representative curves for this type of excitation are shown in Fig. 11*d*.

The choice between great or small values of the damping coefficient in any particular practical case depends to a large extent on whether there is any likelihood of the system being vibrated accidentally in a resonant condition. For high frequencies of excitation, large damping forces are a disadvantage, but in the neighbourhood of resonance the effect of decreasing the damping forces may easily be disastrous, owing to the much greater effect of magnification at resonance.

II. Seismic vibrograph.

Fig. 14 illustrates the seismic element of a simple type of vibrograph. The base A is attached to the structure whose vibration-amplitude it is required to measure, and means are included for recording the amplitude of the relative displacement between A and the mass m. It is required to draw the "response curve" of the instrument, i.e. a curve showing how the ratio

$$\frac{\text{amplitude of relative motion}}{\text{amplitude of motion of base}}$$

varies with the frequency. The natural frequency of the instrument is 500 cycles per minute, and the damping can be made to be between 0·1 and 0·7 of the critical value.

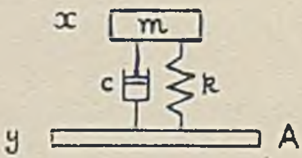


FIG. 14.—Seismic element of simple vibrograph (recording-gear not shown).

Let the displacements of the base and the mass be x and y respectively, x being a sinusoidal time-function of frequency $p/2\pi$ cycles per second. The alternating forces acting on the mass are $k(x-y)$ and $c(\dot{x}-\dot{y})$, and the equation of motion is

$$k(x-y)+c(\dot{x}-\dot{y}) = m\ddot{y}$$

i.e.

$$m\ddot{y}+c\dot{y}+ky = c\dot{x}+kx \quad . \quad . \quad . \quad \text{(i)}$$

Let the amplitude of x be X, so that

$$x = X.\sin pt.$$

It is evident that the right-hand side of the equation (i) represents a sinusoidal force of frequency $p/2\pi$, and the displacement y can be obtained by a method similar to that adopted in Section 9. It is convenient here, however, to introduce an alternative method of solution involving complex-numbers.

It has been shown in Section 3 that a sine-function of frequency F can be regarded as the projection of a vector rotating with angular velocity $2\pi F$ (see Fig. 2). If the displacement $x = X.\sin pt$, the velocity $\dot{x} = Xp.\cos pt$, or

$$\dot{x} = Xp.\sin\left(pt+\frac{\pi}{2}\right),$$

and the corresponding vector is phased $\pi/2$ radians ahead of the displacement vector. Elementary vector-analysis (see Appendix I, section 35) utilises the symbol j to represent a *versor-operator* which turns the associated vector through an angle $\pi/2$ radians in the positive sense. If the vector resulting from this rotation is operated on again by j it is turned through a further angle $\pi/2$ radians, so that it is finally phased π radians ahead of the original vector. As

$$\sin(\theta+\pi) = -\sin\theta,$$

the operation j^2 is equivalent to changing the sign, i.e.

$$j^2 \equiv -1$$

and in general the operator j has the algebraic properties of $i = \sqrt{-1}$.

The form of equation (i) shows that the displacement y is a sinusoidal time-function of frequency $p/2\pi$. Thus the velocity $\dot{y}$ can be written jpy and the acceleration $\ddot{y} = -p^2y$. (As a check on the method, it will be observed that in Section 2 it was found that the velocity $\dot{x}$ of a displacement $x = X.\sin\omega t$ is

$$\dot{x} = Dx = i\omega x \quad \text{(see 2.4)}$$

and in the present section the operator j is being used in place of the number i.)

Rewriting the equation of motion in terms of j,

$$(k-mp^2+jcp)y = (k+jcp)x.$$

Hence
$$\frac{y}{x} = \frac{k+jcp}{k-mp^2+jcp}$$

and
$$\frac{y-x}{x} = \frac{mp^2}{k-mp^2+jcp} = \frac{mp^2(k-mp^2-jcp)}{(k-mp^2)^2+c^2p^2}.$$

Substituting the sinusoidal form for x,

$$\frac{y-x}{X} = \frac{mp^2[(k-mp^2)\sin pt - cp.\cos pt]}{(k-mp^2)^2+c^2p^2}$$

$$= \frac{mp^2}{[(k-mp^2)^2+c^2p^2]^{\frac{1}{2}}}\sin(pt-\phi)$$

where
$$\tan\phi = \frac{cp}{k-mp^2}.$$

The amplitude of the relative motion is therefore

$$\text{rel. amp.} = \frac{mp^2X}{[(k-mp^2)^2+c^2p^2]^{\frac{1}{2}}}$$

or, making use of the substitutions $\omega^2 = k/m$, $c_0 = 2\sqrt{mk}$, $\lambda = (p/\omega)^2$ and $\mu = (c/c_0)^2$,

$$\text{rel. amp.} = \frac{\lambda X}{[(1-\lambda)^2+4\lambda\mu]^{\frac{1}{2}}}$$

Fig. 15 shows curves of the relative amplitude plotted against frequency (it is given that the natural frequency is 500 C.P.M.) for values of the damping coefficient from 0·1 to 0·7 of the critical value. For frequencies above twice the natural frequency, i.e. above 1,000 C.P.M., the response of the instrument is within $2\frac{1}{2}\%$ of the motion of the base if the damping has 70% of the critical value. At such frequencies the mass m is practically stationary in the gravitational field of the earth, and the recorded relative motion is therefore almost exactly the same as the absolute motion of the base in the gravitational field.

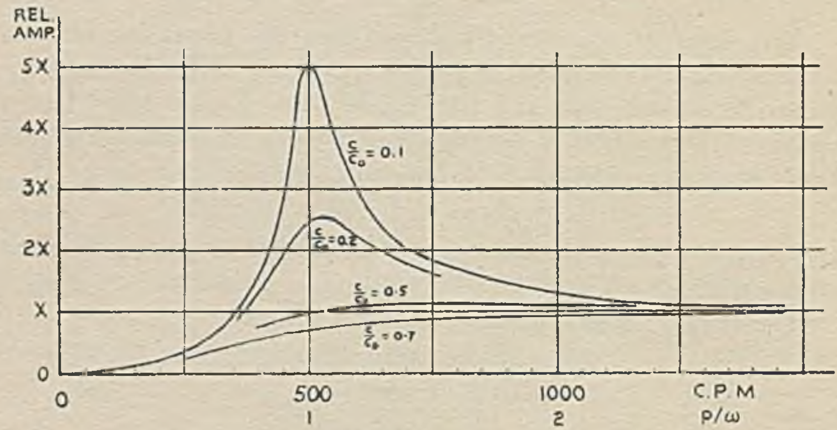


FIG. 15.—Response-curve for Fig. 14.

Although it is no part of this present work to enter into engineering details, it is relevant to observe here that large damping forces can be introduced into the system deliberately by means of an electro-magnetic device, the damping effect of which can be varied over a wide range by adjustment of the electric current flowing through the coil of an electromagnet.

EXERCISES II

1. Determine the condition distinguishing between the two forms of displacement curve illustrated in Fig. 7 and derived from equation (8.4), and show that the curve cannot cross the time-axis more than once.

(Method : the type of motion depends upon the initial displacement and velocity ; solve to find the time at which $x = 0$ in terms of these initial values.)

2. Extend the method of Section 9 to prove the general theorem :

If $[\psi(D)]x = [a_0 + a_1D + a_2D^2 + \dots + a_nD^n]x$, where the a's are constants, and $x = v$ is a particular integral of the equation

$$[\psi(D)]x = f(t)$$

then $x = u+v$ is the general solution of this equation, where $x = u$ is the general solution of the equation

$$[\psi(D)]x = 0.$$

Show also that this result is not true for non-linear equations such as

$$\dot{x}^2 = ax + bt.$$

3. Show that if the force input in Fig. 9 is of the form $P = Qp^2 \sin pt$, where Q is a constant, the resonant frequency formula analogous to (10.1) is

$$P = \frac{\omega^2}{[\omega^2 - 2\gamma^2]^{\frac{1}{2}}}$$

Draw a curve showing the relation between the displacement amplitude and the frequency of the applied force. Show that for both types of motion under the action of forces of the form $P = P_0 \sin pt$ and $P = Qp^2 \sin pt$ the displacement time-graph has no maximum if the damping coefficient has a value greater than $1/\sqrt{2}$ times the critical value.

(This type of motion occurs when the exciting force is the result of centrifugal action on an unbalanced rotor.)

4. Show that for damped free motion of a single-mass system the mass is effectively brought to rest in a shorter time if the damping is critical than if the damping is either greater or less than critical.

(Consider a particular system in which the damping has the values 1·1, 1·0, and 0·9 times the critical value, and the initial displacement and velocity are x_0 and zero respectively. Find the times taken for the displacement (or in the case of 90% critical damping, the exponential "envelope" AB in Fig. 8) to fall to 10% of the initial value.)

5. Show that when the damping is very much less than critical, the amplitude of the exponential envelope AB in Fig. 8 is only 0·19% of its initial value after a time $2\pi/\gamma$, i.e. after ω/γ "cycles"; and show further that after a time $1/\gamma$ the amplitude is $1/e$ or 36·8% of its initial value.

6. The mass in Fig. 9 is held rigidly while a harmonic force $P = P_0 \sin pt$ is applied; it is released at an instant when $\sin pt = 1$; if $p = \omega/10$, and $\gamma = \omega/5$, draw a curve of displacement against time.

(Let the amplitude of the forced motion be b as in (9.4); at the instant of release the displacement due to the forcing must have this maximum value, so that the displacement due to the free motion, at a frequency $\omega/2\pi$, must be $-b$; similarly the velocity of the free motion is zero. The complementary function (9.5) therefore has an initial amplitude b and a phase angle $3\pi/2$. Sufficient is now known to enable the required curve to be drawn. It will be seen that shortly after the instant of release the displacement is instantaneously nearly double the amplitude of the steady-state motion.)

CHAPTER III

UNDAMPED MOTION WITH TWO DEGREES OF FREEDOM

13. Two-mass systems.

In this chapter two different types of system with two degrees of freedom are discussed; the first is typified by the system of Fig. 16, in which each of the two masses has one degree of freedom—namely, motion in the vertical direction—and Fig. 21 illustrates the second type, in which one mass has two degrees of freedom as denoted by the arrow diagram in this figure. The two types of system are fundamentally different and are treated in order.

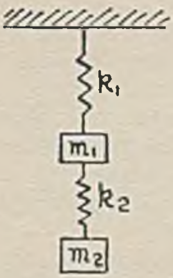


Fig. 16.—Two-mass system with two natural frequencies.

It has been shown in Section 11 that the resonant frequency of a lightly-damped system with a single degree of freedom is practically equal to the natural frequency calculated on the assumption of zero damping. This result will be extended to cover multi-mass systems in Chapter III, section 18, and present attention is confined to the determination of the natural frequencies of the two-mass system in Fig. 16. Let the displacements of the two masses below the position of static equilibrium be x_1 and x_2. The extension of the lower spring is then $x_2 - x_1$, and the equations of motion for the two masses are:

$$m_1\ddot{x}_1 = -k_1x_1 + k_2(x_2 - x_1)$$

i.e.

$$\left.\begin{aligned} m_1\ddot{x}_1 &= -(k_1+k_2)x_1 + k_2x_2 \\ m_2\ddot{x}_2 &= k_2x_1 - k_2x_2 \end{aligned}\right\} \quad . \quad . \quad (13.1)$$

and

The spring stiffnesses are constant and the motion of the system can therefore be assumed to be sinusoidal; the validity of the assumption is, as usual, to be checked by investigating whether the solution obtained does in fact satisfy the equations of motion. Without loss of essential generality the phase-angle for the displacement of the mass m_1 can be taken as zero, and the displacements can be written:

$$\left.\begin{aligned} x_1 &= a_1 \sin \omega_1 t \\ x_2 &= a_2 \sin (\omega_2 t + \phi) \end{aligned}\right\} \quad . \quad . \quad . \quad (13.2)$$

Substitution of these values in (13.1) gives the equations :

$$\left.\begin{aligned}(-m_1\omega_1^2+k_1+k_2)a_1\sin\omega_1 t &= k_2a_2\cos\phi\sin\omega_2 t\\ &\quad +k_2a_2\sin\phi\cos\omega_2 t \quad . \quad (a)\\ (-m_2\omega_2^2+k_2)a_2\cos\phi\sin\omega_2 t &\\ +(-m_2\omega_2^2+k_2)a_2\sin\phi\cos\omega_2 t &= k_2a_1\sin\omega_1 t \quad (b)\end{aligned}\right\}(13.3)$$

These equations are true for all instants of time ; put $t = 0$, then (13.3a) becomes

$$k_2a_2\sin\phi = 0.$$

Now, k_2 is not zero by hypothesis, and hence either a_2 or $\sin\phi$ is zero. If a_2 is zero, so is a_1 from (13.3b), but if both a_1 and a_2 are zero there is no motion. Thus $\sin\phi = 0$, and equations (13.3) become

$$\left.\begin{aligned}(-m_1\omega_1^2+k_1+k_2)a_1\sin\omega_1 t &= k_2a_2\cos\phi\sin\omega_2 t \quad (a)\\ (-m_2\omega_2^2+k_2)a_2\cos\phi\sin\omega_2 t &= k_2a_1\sin\omega_1 t \quad (b)\end{aligned}\right\}(13.4)$$

$$(\text{where } \cos\phi = \pm 1).$$

The ratio a_1/a_2 obtained from (13.4a) is

$$\frac{a_1}{a_2} = \frac{k_2\cos\phi}{-m_1\omega^2+k_1+k_2}.\frac{\sin\omega_2 t}{\sin\omega_1 t}.$$

This fraction being merely the ratio of two lengths, it must be independent of the time ; thus the right-hand factor must reduce to a constant, i.e.

$$\sin\omega_2 t = c.\sin\omega_1 t,$$

where c is as yet undetermined. If possible, let $c < 1$, and put $t = \pi/2\omega_2$, then

$$1 = c.\sin\frac{\pi\omega_1}{2\omega_2}$$

i.e. $\sin\dfrac{\pi\omega_1}{2\omega_2} > 1$, which is impossible.

Thus c cannot be less than unity ; similarly, it can be shown that c cannot be greater than unity ; hence $c = 1$, and $\sin\omega_2 t = \sin\omega_1 t$. For this last result to be true at all times, $\omega_2 = \omega_1$. Let $\omega = \omega_1 = \omega_2$, then the ratio a_1/a_2 becomes :

$$\frac{a_1}{a_2} = \frac{k_2\cos\phi}{-m_1\omega^2+k_1+k_2} \quad . \quad . \quad (13.5a)$$

This same ratio can be evaluated from (13.4b) as

$$\frac{a_1}{a_2} = \frac{(-m_2\omega^2+k_2)\cos\phi}{k_2} \quad . \quad . \quad (13.5b)$$

By equating the two expressions for the ratio, the so-called "frequency equation" is obtained:

$$\omega^4 - \left[\frac{k_2}{m_2}+\frac{k_1+k_2}{m_1}\right]\omega^2+\frac{k_2^2}{m_1m_2} = 0 \quad . \quad (13.6)$$

As this equation is a quadratic in ω^2 there are two natural frequencies of the system, $\omega_L/2\pi$ and $\omega_H/2\pi$ c.p.s. where

$$2\omega_L^2 = \frac{k_2}{m_2}+\frac{k_1+k_2}{m_1} - \left[\left(\frac{k_2}{m_2}+\frac{k_1+k_2}{m_1}\right)^2 - \frac{4k_2^2}{m_1m_2}\right]^{\frac{1}{2}}$$

$$2\omega_H^2 = \frac{k_2}{m_2}+\frac{k_1+k_2}{m_1} + \left[\left(\frac{k_2}{m_2}+\frac{k_1+k_2}{m_1}\right)^2 - \frac{4k_2^2}{m_1m_2}\right]^{\frac{1}{2}}$$

Rearranging the expression in square brackets,

$$2\omega_L^2 = \frac{k_2}{m_2}+\frac{k_1+k_2}{m_1} - \left[\left(\frac{k_2}{m_2}-\frac{k_1+k_2}{m_1}\right)^2 + \frac{4k_1k_2}{m_1m_2}\right]^{\frac{1}{2}}$$

$$= \frac{k_2}{m_2}+\frac{k_1+k_2}{m_1} - \left[\frac{k_2}{m_2}-\frac{k_1+k_2}{m_1}+C\right]$$

where $C > 0$

$$= 2\frac{k_1+k_2}{m_1} - C.$$

Hence

$$\omega_L^2 < \frac{k_1+k_2}{m_1} \quad . \quad . \quad . \quad (13.7a)$$

Similarly it can easily be shown that

$$\omega_H^2 > \frac{k_2}{m_2} \quad . \quad . \quad . \quad (13.7b)$$

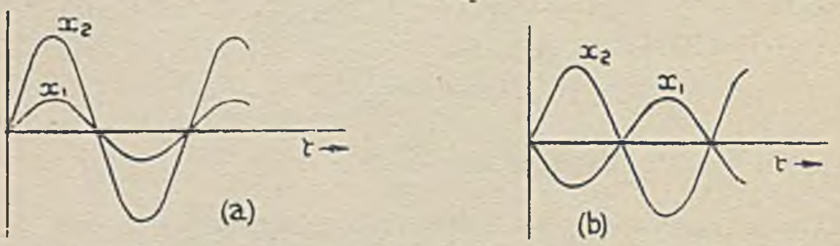


Fig. 17.—Displacement : time curves for the two masses of Fig. 16 : (*a*) lower natural frequency ; (*b*) higher natural frequency.

It is convenient here to regard the amplitudes as being essentially positive (see note in Section 4); the results (13.7) when substituted in (13.5) then give:

$$\left.\begin{aligned}\cos\phi &= 1, \text{ i.e. } \phi = 0 \text{ if } \omega = \omega_L \\ \cos\phi &= -1, \text{ i.e. } \phi = \pi \text{ if } \omega = \omega_H\end{aligned}\right\} \quad . \quad (13.8)$$

(It has already been shown that $\cos\phi = \pm 1$.)

Two very important results have now been obtained:

(i) both masses vibrate with the same frequency, and

(ii) there are two natural frequencies, at the lower of which the two masses move in phase ($\phi = 0$) as in Fig. 17*a*, whereas at the higher frequency the two masses always move in opposite directions ($\phi = \pi$) as in Fig. 17*b*.

14. Deformation diagrams and nodes.

The configuration of the system when vibrating is known as the "deformation shape," and is illustrated by a "deformation diagram" similar to Fig. 18, in which the amplitudes of the motion of the two masses are plotted at right-angles to the axis of the system. As the springs are supposed to be composed of

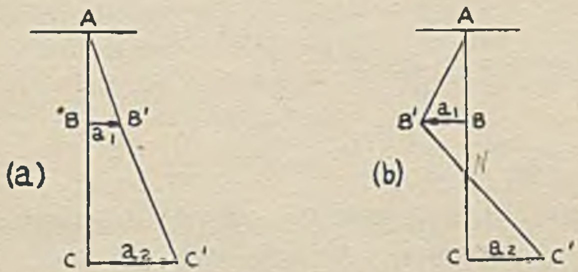


FIG. 18.—Deformation diagrams for the two masses of Fig. 16: (*a*) lower natural frequency; (*b*) higher natural frequency.

uniform material uniformly distributed, the deformation shapes for the springs are straight lines AB' and $B'C'$, where BB' and CC' are the plotted displacements of the two masses and A is the suspension point. The deformation diagrams are quite different for the two natural frequencies.

(i) Lower natural frequency ($\omega = \omega_L$).

From (13.5*b*), when $\omega = \omega_L$

$$\frac{a_1}{a_2} = 1 - \frac{m_2\omega_L{}^2}{k_2} < 1.$$

Moreover, a_1 and a_2 are positive; hence $a_1 < a_2$ and the deformation diagram has the form indicated in Fig. 18*a*.

(ii) Higher natural frequency ($\omega = \omega_H$).

The masses move in opposite directions, and there is a point N on BC such that the displacement at N is zero at all times, Fig. 18*b*. Such a point is termed a *node*.

At both natural frequencies the displacement of the suspension point A is zero, since the support is taken to be rigid and so does not move. The point A is therefore a node in both cases.

The shape of the deformation diagram, specified by the position of the nodes, is termed the *mode.* The two modes of vibration of the two-mass system of Fig. 16 are therefore :

(i) a one-node mode, the node being at A, and

(ii) a two-node mode, one node being at A and the other between the two masses.

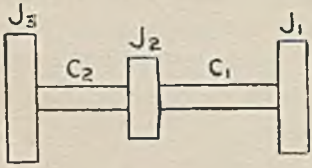


FIG. 19.—Three-mass system with two natural frequencies.

In Chapter I, section 6, it was shown that the torsional two-mass system of Fig. 4 has only one natural frequency ; similarly it can be shown that the three-mass system of Fig. 19 has only two natural frequencies. It is left as an exercise for the reader to show that the corresponding modes of vibration are :

(i) a one-node mode, and

(ii) a two-node mode.

(See Exercise 1 at the end of the chapter.)

It must be emphasised that the same result would hold for a linear system consisting of three masses, adjacent pairs being connected by springs, the motion being in a horizontal line on the surface of a smooth table ; the torsional case is more practicable, however, and is therefore to be preferred as an illustration. It will be seen from the results obtained that the number of natural frequencies is, in all cases of systems so far examined, the same as the number of springs, whether linear or torsional. This is true of any multi-mass system in which the masses and springs occur alternately (see Chapter IV, section 17).

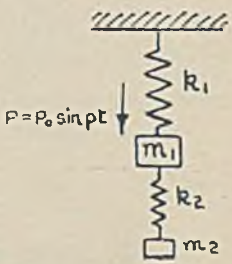


FIG. 20.—Undamped vibration absorber (linear).

15. Undamped vibration absorber.

The system illustrated in Fig. 20 is a simple type of *dynamic absorber* layout. The mass m_1 is forced by the input $P = P_0 \sin pt$, and it is desired to reduce the amplitude of vibration of m_1 to zero at a particular value of the forcing frequency $p/2\pi$. The auxiliary spring-mass system k_2, m_2 effects this by absorbing the input energy at its own natural frequency. By precisely the same method as used in Section 13 it can be shown that the two masses move with the same frequency $p/2\pi$ as the exciting force, and further that they move either in phase or π radians out of phase with each other. Allowing the amplitudes to take negative values, the displacements of the two masses can therefore be expressed as :

$$\left.\begin{aligned} x_1 &= a_1 \sin pt \\ x_2 &= a_2 \sin pt \end{aligned}\right\} \qquad (15.1)$$

The equations of motion are :

$$\left.\begin{aligned} m_1\ddot{x}_1 + (k_1+k_2)x_1 - k_2x_2 &= P_0 \sin pt \\ m_2\ddot{x}_2 - k_2x_1 + k_2x_2 &= 0 \end{aligned}\right\} \qquad (15.2)$$

Substitution of the sinusoidal expressions (15.1) in the equations of motion, and elimination of a_2 between the two resulting equations, lead to an expression for the amplitude a_1 of motion of the mass m_1 :

$$a_1 = \frac{P_0(k_2 - m_2p^2)}{(k_2 - m_2p^2)(k_1+k_2 - m_1p^2) - k_2^2} \qquad (15.3)$$

If the forcing frequency $p/2\pi$ is such that $m_2p^2 = k_2$, the numerator of the fraction (15.3) is zero but the denominator is not ; thus at the natural frequency of the auxiliary system, considered as a separate system similar to Fig. 1, the amplitude of motion of the mass m_1 is reduced to zero. The auxiliary system is thus absorbing all the vibratory energy in the system, and is termed a *dynamic absorber*. The name "dynamic damper" is frequently used but in error, as the property utilised is that of absorption, not of dissipation by way of damping.

The arrangement is of great utility in cases where it is required to reduce the amplitude of motion of a system without altering the suspension arrangements. In practice the damping forces must be taken into account ; Den Hartog deals with the subject of the damped absorber in some detail (see reference 2 in the Bibliography). The dynamic absorber in its torsional form is of still greater utility.

Two fundamentally different types of absorber are being applied to an ever-increasing number of troublesome torsional systems. One type is merely a torsional version of the linear absorber, the spring attachment being in some cases a patented rubber mounting ; this variety of absorber is effective at a particular frequency in the same way as is the linear analogue. The other type is more interesting and at the same time more useful; it consists essentially of a pendulum hinged to a convenient part of the system—in the case of internal combustion engines, to a crankweb—and the dynamic conditions of the system are such that the pendulum absorbs energy, not at a certain constant frequency, but at a frequency which is a constant multiple of the rotational speed of the engine. Thus, for example, the absorber may be effective for frequencies $2\frac{1}{2}\times$ engine R.P.M., and it is then termed a "$2\frac{1}{2}\times$" absorber. By suitable design of absorbers two or three harmonics ($2\frac{1}{2}$, $3\frac{1}{2}$, and $6\times$ engine R.P.M., for example) may be absorbed simultaneously. A very full discussion of this important subject is given by Dr. Ker Wilson (see reference 3 in the Bibliography).

16. Coupled vibrations.

Fig. 21 illustrates a system of the second type mentioned at

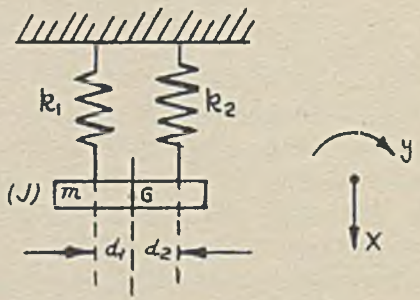


FIG. 21.—One-mass system with two natural frequencies.

the beginning of this chapter—one in which a single mass has two degrees of freedom. The centre of gravity of the mass is constrained to move vertically, and the springs react against this motion with restoring forces in the vertical direction; and the mass rotates about its centre of gravity, also against the spring restraints. If G is the centre of gravity of the mass, the two degrees of freedom are therefore:

(i) vertical motion of G, and
(ii) rotation of the mass about G.

When both linear and angular displacements occur in the same problem it is convenient to denote them by capital and small letters respectively ; in this system let the two displacements be X and y as shown in the diagram. The arrow-diagram on the right of the figure shows the positive directions of motion. In general both displacements occur at the same time, and for small amplitudes the spring forces produced by the motion are proportional to the displacements. Suppose first that the mass moves vertically with no rotation, the displacement being X downwards. The spring k_1 exerts a restoring force k_1X upwards, and this force has a moment k_1d_1X about G in the clockwise direction ; similarly the spring k_2 exerts a restoring force k_2X upwards, and this force has a moment $-k_2d_2X$ clockwise about G. Thus the total restoring *force* on G is $(k_1+k_2)X$ upwards, and the total moment about G in the positive sense is $(k_1d_1-k_2d_2)X$. This moment tends to produce a rotation round G, and the two types of motion are therefore *coupled*. It is to be noted that the displacements are assumed small, so that the moment-arms d_1 and d_2 do not change appreciably during the motion.

Now suppose the mass to rotate through a small angle y without linear motion of the centre of gravity G ; the rotation being about G, the displacements at the points of attachment of the springs are $-d_1y$ and d_2y downwards, as y is positive when clockwise. The forces exerted by the springs are therefore k_1d_1y and $-k_2d_2y$ downwards, and these forces have moments $-k_1d_1^2y$ and $-k_2d_2^2y$ clockwise about G. The total spring force on G is thus $(k_1d_1-k_2d_2)y$ downwards, and the total moment about G is $-(k_1d_1^2+k_2d_2^2)y$ clockwise.

If now the mass is supposed to move generally—i.e. to have both the displacements X and y—the forces and moments induced are the sum of those induced by the two motions occurring separately. Let the moment of inertia of the mass about G be J, with respect to the rotation y, then the equations of motion are :

$$\left.\begin{aligned} m\ddot{X} &= -(k_1+k_2)X+(k_1d_1-k_2d_2)y \\ J\ddot{y} &= (k_1d_1-k_2d_2)X-(k_1d_1^2+k_2d_2^2)y \end{aligned}\right\} \quad . \quad (16.1)$$

In the usual manner (see Section 13) it can be shown that the displacements can be expressed as :

$$X = A.\sin \omega t$$
$$y = a.\sin (\omega t+\phi)$$

where $\phi = 0$ or π.

It is convenient here to allow the constant a to take negative values, and to write accordingly :

$$\left.\begin{aligned} X &= A.\sin \omega t \\ y &= a.\sin \omega t \end{aligned}\right\} \quad . \quad . \quad . \quad (16.2)$$

where A is positive but a may be either positive or negative. Substitution of these values in (16.1) leads to the equations :

$$[m\omega^2 - (k_1+k_2)]A + [k_1d_1 - k_2d_2]a = 0$$
$$[J\omega^2 - (k_1d_1{}^2+k_2d_2{}^2)]a + [k_1d_1 - k_2d_2]A = 0$$

which can be written as :

$$\left.\begin{aligned} (m\omega^2 - c_{XX})A + c_{Xy}a &= 0 \\ c_{yX}A + (J\omega^2 - c_{yy})a &= 0 \end{aligned}\right\} \quad . \quad . \quad (16.3)$$

where :

$$\left.\begin{aligned} c_{XX} &= k_1+k_2 \\ c_{yy} &= k_1d_1{}^2+k_2d_2{}^2 \\ c_{Xy} = c_{yX} &= k_1d_1 - k_2d_2 \end{aligned}\right\} \quad . \quad . \quad (16.4)$$

The substitutions indicated in (16.4) are very generally useful, for $-c_{XX}$ is the force parallel to the X-axis produced by a unit displacement along that axis, $-c_{Xy}$ is the force parallel to the X-axis produced by a unit angular displacement about the y-axis, etc. The equality of the two constants c_{Xy} and c_{yX} is not merely coincidental, but is an example of a well-known reciprocation theorem due to Maxwell.

By obtaining the ratio A/a in each of the equations (16.3) the " frequency equation " is obtained :

$$\omega^4 - \left(\frac{c_{XX}}{m}+\frac{c_{yy}}{J}\right)\omega^2 + \frac{c_{XX}c_{yy} - c_{Xy}c_{yX}}{mJ} = 0$$

or, as $c_{Xy} = c_{yX}$,

$$\omega^4 - \left(\frac{c_{XX}}{m}+\frac{c_{yy}}{J}\right)\omega^2 + \frac{c_{XX}c_{yy} - c_{Xy}{}^2}{mJ} = 0 \quad . \quad (16.5)$$

(16.5) is very significant when expressed in terms of the " uncoupled " frequencies of the system. The constants $c_{Xy} = c_{yX}$ are a measure of the degree of coupling between the two modes of vibration, i.e. between X and y. If these equal constants are zero, which occurs when

$$k_1d_1 = k_2d_2 \quad . \quad . \quad . \quad . \quad (16.6)$$

the two degrees of freedom are uncoupled and each can take place without involving the other ; for a displacement X then induces zero moment about G, and an angular displacement y induces zero force parallel to the X-axis. (16.3) then become:

$$(m\omega^2 - c_{XX})A = 0$$
$$(J\omega^2 - c_{yy})a = 0$$

and if the corresponding values of ω are denoted by ω_X, ω_y,

$$\left.\begin{aligned}\omega_X^2 &= \frac{c_{XX}}{m}\\ \omega_y^2 &= \frac{c_{yy}}{J}\end{aligned}\right\} \quad . \quad . \quad . \quad (16.7)$$

From (16.7) the "uncoupled" frequencies $\omega_X/2\pi$ and $\omega_y/2\pi$ can be calculated. (16.5) can be written as:

$$\omega^4 - (\omega_X^2+\omega_y^2)\omega^2+(\omega_X^2\omega_y^2 - B) = 0 \quad . \quad (16.8)$$

where $$B = \frac{c_{Xy}^2}{mJ} > 0.$$

Denoting the roots of (16·8) by ω_L^2 and ω_H^2, where ω_L^2 is the smaller root, the elementary theory of equations shows that

$$\left.\begin{aligned}\omega_L^2+\omega_H^2 &= \omega_X^2+\omega_y^2 & (a)\\ \omega_L^2\omega_H^2 &= \omega_X^2\omega_y^2 - B < \omega_X^2\omega_y^2 & (b)\end{aligned}\right\} \quad . \quad (16.9)$$

By using the identity $(m+n)^2 = (m - n)^2+4mn$ it is easy to deduce from (16·9) that

$$\omega_H^2 - \omega_L^2 > \omega_X^2 - \omega_y^2 \text{ numerically} \quad . \quad (16.10)$$

(16.9a) shows that the mean of the two coupled values of ω^2 is equal to the mean of the two uncoupled values, while (16.10) shows that the difference between the coupled values is greater than the difference between the uncoupled values. The lower

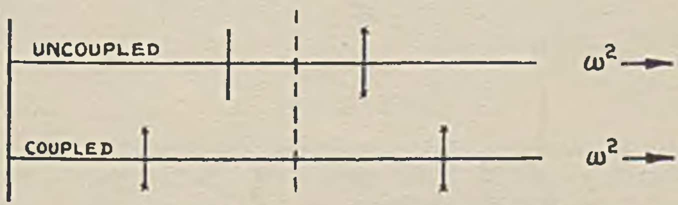


Fig. 22.—Frequency spectrum for Fig. 21.

coupled value is therefore lower than the lower uncoupled value, and the higher coupled value is higher than the higher uncoupled value, as shown in Fig. 22. Such a diagram is termed loosely a "frequency spectrum."

If the mass in Fig. 21 is free to move in a horizontal direction against spring restraints, the system then has three degrees of freedom and the equation corresponding to (16.8) is a cubic in ω^2. The general case of a mass free to move in space under spring restraints, as shown in Fig. 23, has six degrees of free-

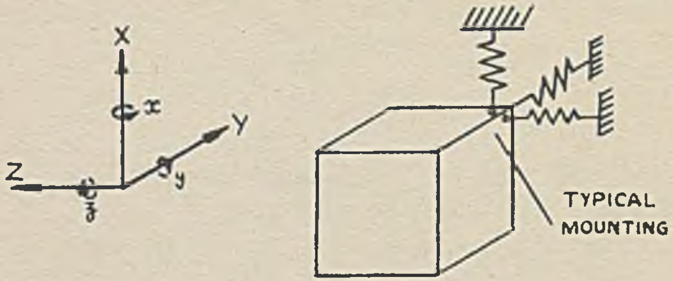


FIG. 23.—One-mass system with six degrees of freedom.

dom, namely, three linear and three angular displacements. The resulting "frequency equation" is of the sixth degree in ω^2, but symmetry in the system can reduce the equations of motion to separate groups by rendering some of the coupling coefficients zero. This occurs in the orthodox type of in-line aero-engine mounting; in such a system the vertical motion, for example, is uncoupled from rolling motion (i.e. rotation about a fore-and-aft axis) while usually remaining coupled to pitching motion (i.e. rotation about an axis parallel to the wings of the aircraft).* The more symmetrical the system is, the simpler the solution becomes; in the case of a solid of revolution such as an airscrew spinner, with spring mountings symmetrically disposed about the axis of revolution, the general solution reduces to two quadratic equations similar to (16.8) and two "uncoupled" equations similar to (16.7).

EXERCISES III

1. By the methods of Sections 6 and 13 show that there are only two natural frequencies of the three-mass torsional system of Fig. 19, and that the corresponding modes are:

(i) a one-node mode, and

(ii) a two-node mode.

2. Derive an expression for the amplitude of motion of the lower mass (m_2) in Fig. 20, and draw curves showing how the amplitudes of motion of both masses vary with the forcing frequency $p/2\pi$.

* See Appendix III, section 41.

(It will be found that these amplitudes have infinite values when the forcing frequency equals the natural frequencies of the system. If small damping forces are included in the system the infinite amplitudes are reduced to the practical resonance "peaks.")

3.—Find the condition that one of the coupled frequencies of the system shown in Fig. 24 should be zero, the centre of gravity of the mass being constrained to move in the direction X.

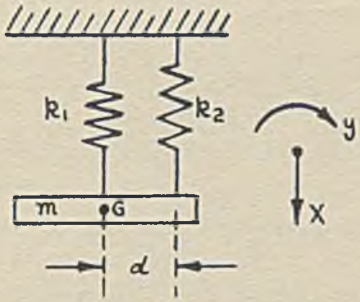


Fig. 24.—Ex. III, 3.

(Obtain a frequency equation analogous to (16.5) and find the condition that ω^2 is a root, i.e. that the term independent of ω^2 is zero.)

4. Derive the frequency equation for the system of Fig. 23 in terms of appropriate coupling-coefficients similar to (16.4).

(See Appendix III, section 41.)

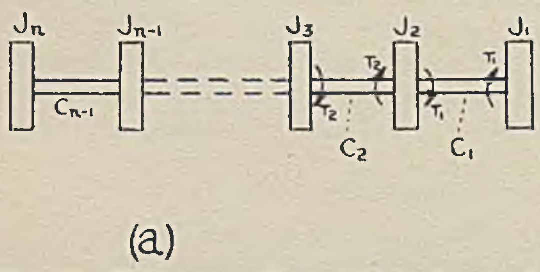
CHAPTER IV

MANY DEGREES OF FREEDOM

Effective Inertia and Dynamic Stiffness

17. Multi-mass systems.

The purely analytical methods employed in Chapters I–III prove to be cumbersome when applied to systems having more than two degrees of freedom. Particularly is this so in the calculation of resonant frequencies by differentiation of amplitude-against-frequency curves, but fortunately it is found that for lightly-damped systems the damping forces can be disregarded and the natural frequencies of the resulting system calculated instead of the resonances. This useful fact has been proved for systems with one degree of freedom in Chapter II, section 11, and will be proved for multi-mass systems in Section 18 ; and when dealing with lightly-damped systems attention will be confined to determining the natural frequencies and deformation shapes. The analytical method can be used to show that a torsional system as that of Fig. 25*a* has one less natural frequency than the number of inertias ; if



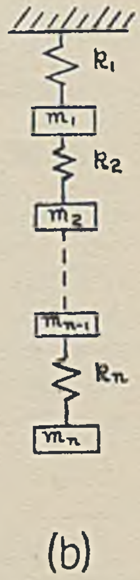


Fig. 25.—Multi-mass systems : (*a*) $n-1$ natural frequencies ; (*b*) n natural frequencies.

the last inertia J_n is removed and the system clamped at this point, thus replacing the finite inertia by the effectively infinite inertia of the earth, to which the clamp is rigidly attached, there will still be $n-1$ natural frequencies. These results are easily derived, as follows :

Let the angular displacements of the inertias from the static position be $\theta_1, \theta_2, \theta_3 \ldots \theta_n$. There are n equations of motion :

$$\left.\begin{aligned} J_1\ddot{\theta}_1 &= -C_1(\theta_1-\theta_2) \\ J_2\ddot{\theta}_2 &= -C_2(\theta_2-\theta_3)+C_1(\theta_1-\theta_2) \\ &\text{etc. etc.} \\ J_{n-1}\ddot{\theta}_{n-1} &= -C_{n-1}(\theta_{n-1}-\theta_n)+C_{n-2}(\theta_{n-2}-\theta_{n-1}) \\ J_n\ddot{\theta}_n &= +C_{n-1}(\theta_{n-1}-\theta_n) \end{aligned}\right\} \quad (17.1)$$

By extension of the method of Section 6 it can easily be shown that all the inertias have sinusoidal motion at the same frequency, and that all the phase-angles differ from each other by zero or a multiple of π radians, and that the displacements can therefore be put in the form

$$\left.\begin{aligned} \theta_1 &= A_1.\sin\omega t \\ \theta_2 &= A_2.\sin\omega t \\ &\text{etc. etc.} \\ \theta_n &= A_n.\sin\omega t \end{aligned}\right\} \quad . \quad . \quad . \quad (17.2)$$

where the constants A are allowed to take negative values (see note on the sign conventions, Chapter I, section 4). The equations of motion can then be written as :

$$\left.\begin{aligned} (C_1-J_1\omega^2)A_1 &= C_1A_2 \\ (C_1+C_2-J_2\omega^2)A_2 &= C_1A_1+C_2A_3 \\ &\text{etc. etc.} \\ (C_{n-1}-J_n\omega^2)A_n &= C_{n-1}A_{n-1} \end{aligned}\right\} \quad . \quad . \quad (17.3)$$

Elimination of the amplitude ratios A_2/A_1, A_3/A_1, etc., leads to an equation of degree n in ω^2, of which $\omega^2 = 0$ is a solution corresponding to $A_1 = A_2 = \ldots = A_n$. As in Section 6 this solution refers to steady rotation, and there are thus $n-1$ natural frequencies of vibration. It has not been shown that these natural frequencies are all distinct, but that they are so will be evident in the course of the present chapter. An important point to be noted is that the number of natural frequencies is equal to the number of springs, in systems similar to those of Fig. 25, wherein the inertias and springs occur alternately ; whether the system is a linear or a torsional one is, of course, immaterial.

18. Effective inertia.

The multi-mass systems that occur most frequently in practice are torsional systems, and the discussion which follows is therefore concerned primarily with torsional vibration; the equivalent results for linear systems can be obtained by means of the linear-torsional analogues listed in Table I (Chapter I, section 5).

When a torsional system such as that of Fig. 25*a* is vibrating at a natural frequency in the absence of damping forces, the motion is maintained by the energy stored in the system initially. If the displacement at a point P in the system is

$$\theta = A . \sin \omega t$$

the acceleration is $$\ddot{\theta} = - A\omega^2 \sin \omega t,$$

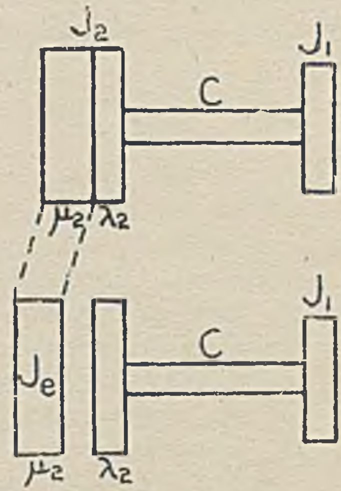


FIG. 26.—Partial inertias.

and there is no applied torque required to maintain this acceleration. The *effective inertia* of the system towards vibrational motion at a natural frequency is thus zero, for

$$\text{inertia} = \frac{\text{torque}}{\text{angular acceleration}}.$$

If it is possible to determine the effective inertia of the system as a function of frequency, the natural frequencies can be found by investigating for what frequencies the effective inertia is zero.

Consider the two-mass system shown in Fig. 26. The natural frequency is

$$F = \frac{1}{2\pi}\sqrt{\frac{C(J_1+J_2)}{J_1 J_2}} \text{ (see Section 6).}$$

Suppose that the inertia J_2 is divided into two parts, λ_2 and μ_2, the *partial inertia* λ_2 being such that the natural frequency of the sub-system $J_1 - (C) - \lambda_2$ is $\omega/2\pi$. Thus

$$\omega = \sqrt{\frac{C(J_1+\lambda_2)}{J_1\lambda_2}}$$

$$\left.\begin{aligned} &\text{i.e.} \quad \lambda_2 = \frac{CJ_1}{\omega^2 J_1 - C} \quad &(a) \\ &\text{and} \quad \mu_2 = J_2 - \lambda_2 \quad &(b) \end{aligned}\right\} \quad . \quad . \quad (18.1)$$

μ_2 is then the effective inertia J_e of the system at the frequency $\omega/2\pi$, for the remainder of the system is vibrating at its natural frequency and thus has zero effective inertia. The effective inertia J_e of the whole system is thus given by :

$$J_e = J_2 - \lambda_2 = J_2 - \frac{CJ_1}{\omega^2 J_1 - C} \quad . \quad . \quad (18.2)$$

Putting $J_e = 0$ naturally leads back to the formula (6.4). The method is, however, capable of immediate extension to multi-mass systems. In the system of Fig. 25*a* let

$$\left.\begin{aligned} J_2 &= \lambda_2+\mu_2 \\ J_3 &= \lambda_3+\mu_3 \\ &\text{etc. etc.} \\ J_n &= \lambda_n+\mu_n \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (18.3)$$

As before, let the sub-system $J_1 - (C_1) - \lambda_2$ have a natural frequency $\omega/2\pi$; then as in (18.1),

$$\mu_2 = J_2 - \frac{C_1 J_1}{\omega^2 J_1 - C} \quad . \quad . \quad . \quad (18.4a)$$

Now determine λ_3 so that the sub-system $\mu_2 - (C_2) - \lambda_3$ also has a natural frequency $\omega/2\pi$. Thus

$$\mu_3 = J_3 - \lambda_3 = J_3 - \frac{C_2\mu_2}{\omega^2\mu_2 - C_2} \quad . \quad (18.4b)$$

In this manner the whole system is split up into sub-systems, for each of which $\omega/2\pi$ is the natural frequency. In the next stage after (18.4*b*),

$$\mu_4 = J_4 - \frac{C_3\mu_3}{\omega^2\mu_3 - C_3} \quad . \quad . \quad . \quad (18.4c)$$

and finally, for the last inertia J_n,

$$\mu_n = J_n - \frac{C_{n-1}\mu_{n-1}}{\omega^2\mu_{n-1} - C_{n-1}} \quad . \quad . \quad . \quad (18.4d)$$

The whole system can now be considered as consisting of two parts :

(i) the system $J_1 - J_2 - J_3 \ldots J_{n-1} - \lambda_n$
(ii) the remaining inertia μ_n.

Part (i) is a system vibrating at a natural frequency, and has therefore zero effective inertia ; thus the effective inertia of the whole system at the point J_n is μ_n.

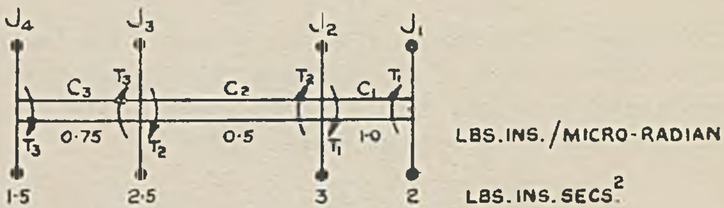


Fig. 27.—Four-mass torsional system.

Apart from providing a method of determining the natural frequencies of multi-mass systems, this conception of partial-inertias and sub-systems enables two very important general conclusions to be drawn :

(i) All the masses move so that their displacements can be expressed as in (17.2), for it has already been established (Section 6 in Chapter I) that the two partial inertias comprising each sub-system move in this manner, and the two partial inertias comprising each inertia J_1, J_2, J_3, etc., are rigidly connected. Thus any pair of adjacent inertias move either in phase or π radians out of phase with each other.

(ii) The resonant frequencies of a lightly-damped multi-mass system are very nearly equal to the natural frequencies calculated on the assumption of zero damping ; for this result has also been proved for a single-mass, single-degree-of-freedom system, and therefore by implication for a two-mass, single-degree-of-freedom system such as that of Fig. 4 with light damping forces included ; but the sub-systems are of the same type as that of Fig. 4, hence the general result.

(A more rigorous demonstration of this second result is to be found in reference 4 in the Bibliography at the back of the book.)

As an example of the application of the method, consider the four-mass system of Fig. 27. The stiffness values are given in the practical unit : lbs.ins. per microradian (i.e. per millionth of a radian) and must be multiplied by a million before being used in the formulae (18.4).

At the frequency 13,500 C.P.M., i.e. 225 C.P.S.,

$$\omega = 2\pi \times 225 = 1{,}414 \text{ and } \omega^2 = 2 \times 10^6.$$

At this frequency,

$$\mu_2 = J_2 - \frac{C_1 J_1}{\omega^2 J_1 - C_1} = 2 \cdot 333$$

$$\mu_3 = J_3 - \frac{C_2 \mu_2}{\omega^2 \mu_2 - C_2} = 2 \cdot 220$$

and

$$J_e = \mu_4 = J_4 - \frac{C_3 \mu_3}{\omega^2 \mu_3 - C_3} = 1 \cdot 049.$$

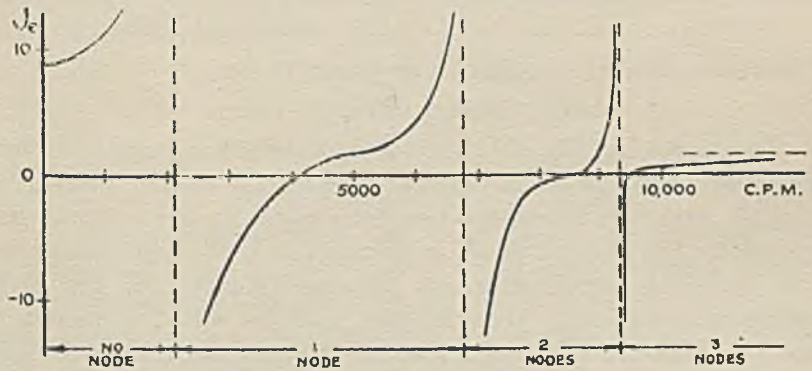


FIG. 28.—Effective inertia at J_4 in Fig. 27.

At the frequency 6,750 C.P.M., i.e. 112·5 C.P.S.,

$$(\omega = 2\pi \times 112 \cdot 5 = 707 \text{ and } \omega^2 = 0 \cdot 5 \times 10^6)$$

$$\mu_2 = \pm \infty, \qquad \mu_3 = 1 \cdot 5, \qquad J_e = \mu_4 = \pm \infty.$$

At the frequency 9,550 C.P.M., i.e. 159 C.P.S.,

$$(\omega = 2\pi \times 159 = 1{,}000 \text{ and } \omega^2 = 1 \cdot 0 \times 10^6)$$

$$\mu_2 = 1, \qquad \mu_3 = 1 \cdot 5, \qquad J_e = \mu_4 = 0.$$

Thus 9,550 C.P.M. is a natural frequency of the system. The effective inertias at J_4 of the system are calculated in a similar manner for various frequencies so as to determine the other natural frequencies. Values are given in Table III, and the curve of effective inertia plotted against frequency in Fig. 28.

TABLE III

Frequency (C.P.M.)	$\omega^2 \div 10^6$	J_e
0	0	9
955	0·01	10·6
1,910	0·04	30·1
3,020	0·1	− 6·83
4,100	0·1838	0
5,230	0·3	1·97
6,040	0·4	4·26
6,750	0·5	∞
7,400	0·6	− 4·89
8,000	0·7	− 0·84
8,620	0·8162	0
9,160	0·9216	0·98
9,360	0·9604	− 1·03
9,550	1·0	0
10,500	1·21	0·58
etc.	etc.	

The three natural frequencies are seen to be:

4,100, 8,620, and 9,550 C.P.M.

The curve in Fig. 28 is typical of effective-inertia curves. The zeros correspond to natural frequencies of the system of Fig. 27, and the discontinuities correspond to the natural frequencies when the inertia J_4 is clamped so that its displacement and hence its acceleration are necessarily zero, for the ratio torque/acceleration is then infinite, and the effective inertia is equal to this ratio. Between adjacent zeros there must be one discontinuity, and between each pair of adjacent discontinuities there must be one zero, so that zeros and discontinuities occur alternately as the frequency is increased; for the discontinuities correspond to the natural frequencies when an additional constraint is applied (the clamping of J_4) and it is a general dynamic theorem that the natural frequencies of such a system must lie between the natural frequencies of the original system. (See E. T. Whittaker, *Analytical Dynamics*, Cambridge, 3rd Edition, pp. 191, 192; and see also Biot, reference 5 in the Bibliography, from whom the reference to Whittaker is taken.)

19. Swinging forms.

The deformation curve of a torsional system undergoing vibration is called the *swinging form*, and it is frequently very important to know the shape of this curve, i.e. what are the relative amplitudes of motion of the various inertias. If the swinging form of a system at a particular frequency is known, then from experimental observations of the amplitude at one point in the system the amplitudes at all other points can be calculated.

In the two-mass system of Fig. 26, let the displacements of the two inertias J_1 and J_2 be

$$\theta_1 = A_1 \sin \omega t$$

and

$$\theta_2 = A_2 \sin \omega t.$$

The velocities are

$$\dot{\theta}_1 = A_1 \omega \cos \omega t$$

and

$$\dot{\theta}_2 = A_2 \omega \cos \omega t.$$

At a natural frequency of the system there is no applied force required to maintain the motion in the absence of damping forces, so that the rate of change of the total angular momentum $(J_1\dot{\theta}_1 + J_2\dot{\theta}_2)$ must be zero, by Newton's second Law of Motion (see Appendix II, section 37).

Thus

$$J_1\ddot{\theta}_1 + J_2\ddot{\theta}_2 = 0, \text{ i.e.}$$

$$J_1 A_1 + J_2 A_2 = 0,$$

or

$$\frac{A_2}{A_1} = -\frac{J_1}{J_2} \quad . \quad . \quad (19.1)$$

Now, the partial inertia λ_2 has been chosen so that the sub-system $J_1 - (C) - \lambda_2$ has $\omega/2\pi$ as its natural frequency (18.1a), and therefore (19.1) holds true at any frequency $\omega/2\pi$ if λ_2 is substituted for J_2. Similarly, in the system of Fig. 25a, the following amplitude ratios are obtained:

$$\left.\begin{aligned} \frac{A_2}{A_1} &= -\frac{J_1}{\lambda_2} \\ \frac{A_3}{A_2} &= -\frac{\mu_2}{\lambda_3} \\ &\text{etc. etc.} \\ \frac{A_n}{A_{n-1}} &= -\frac{\mu_{n-1}}{\lambda_n} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (19.2)$$

An arbitrary value of unity is usually assigned to the amplitude A_1, and the equations (19.2) then give the corresponding values of the amplitudes of motion of the other inertias. As an example, Table IV gives the amplitudes of the four inertias of the system of Fig. 27 at the three natural frequencies.

TABLE IV

Frequency (C.P.M.)	A_1	A_2	A_3	A_4
4,100	1	0·6325	– 0·7999	– 1·2648
8,620	1	– 0·6325	– 0·7999	1·2648
9,550	1	– 1	1	– 1

The corresponding swinging forms are illustrated in Fig. 29 ; the three modes are :

(i) one node between J_2 and J_3 ;

(ii) two nodes, between J_1 and J_2 and between J_3 and J_4 ; and (iii) three nodes, one in each spring.

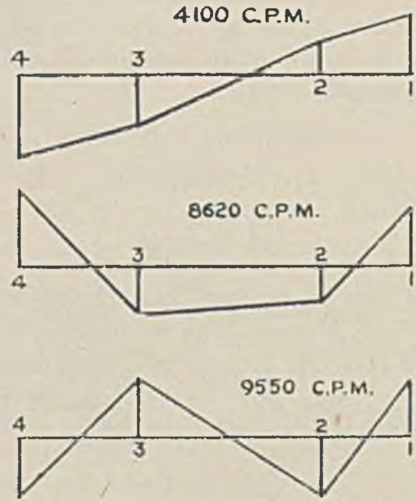


FIG. 29.—Swinging forms for Fig. 27 (at natural frequencies).

The method apparently breaks down when any of the partial inertias are infinite, as for example at the frequency 6,750 C.P.M. At this frequency

$$\lambda_2 = \infty \qquad \mu_2 = -\infty$$
$$\lambda_3 = 1 \qquad \mu_3 = 1{\cdot}5$$

Thus $$\frac{A_2}{A_1} = -\frac{J_1}{\lambda_2} = 0, \text{ and } A_2 = 0$$

and $$\frac{A_3}{A_2} = -\frac{\mu_2}{\lambda_3} = \infty, \text{ whence } A_3 = \infty \times 0$$

and is indeterminate. The amplitude can however be determined in such cases by a consideration of the torques in the shafts to either side ; in this example let T_1 and T_2 be the torques in the shafts C_1 and C_2. These torques are proportional to the twists in their respective shafts, and $A_2 = 0$. Hence

$$T_1 = C_1(A_2 - A_1) = -C_1A_1$$

and $$T_2 = C_2(A_3 - A_2) = C_2A_3.$$

Furthermore, as the amplitude A_2 is zero, the acceleration of J_2 is zero, and hence the resultant torque on J_2 is likewise zero ; i.e.

$$T_2 - T_1 = C_2A_3 + C_1A_1 = 0,$$

and hence $$\frac{A_3}{A_1} = -\frac{C_1}{C_2} = -2.$$

Thus if $A_1 = 1$, $A_3 = -2$. In a similar manner the swinging form can be obtained at any frequency whereat there are nodes at any of the inertias.

If it is desired to know merely how many nodes occur in the swinging form at a particular frequency, it is only necessary to count the number of discontinuities in the effective-inertia curve between that frequency and zero frequency. At these discontinuities the effective inertia is infinite and thus $A_4 = 0$. As the frequency is increased through such a value, an additional node is introduced into the swinging form. Thus for the system of Fig. 27, the effective inertia curve for which is shown in Fig. 28, if F is the frequency,

for $0 < F < 2{,}100$ C.P.M., no nodes ;
$2{,}100 < F < 6{,}750$ C.P.M., one node ;
$6{,}750 < F < 9{,}320$ C.P.M., two nodes ;
$9{,}320$ C.P.M. $< F$, three nodes.

20. Rigidly-coupled systems.

The system illustrated in Fig. 30 consists of two definite parts A and B, rigidly coupled at P ; part A is a system in which all the inertia and torsional stiffness values are known, and part B is a system in which some or all of these values are for some reason unknown. The effective inertia of system B at the coupling-point P can be determined experimentally by applying at this point a sinusoidal torque $T.\sin \omega t$ and measuring the amplitude A of the resulting motion at P ; for if the displacement is $\theta = A.\sin \omega t$, the acceleration is $\ddot{\theta} = -A\omega^2 \sin \omega t$, and the torque required to produce this acceleration is

$$J_e\ddot{\theta} = -J_eA\omega^2 \sin \omega t = T.\sin \omega t.$$

The effective inertia J_e is thus given by :

$$J_e = -\frac{T}{A\omega^2}. \qquad (20.1)$$

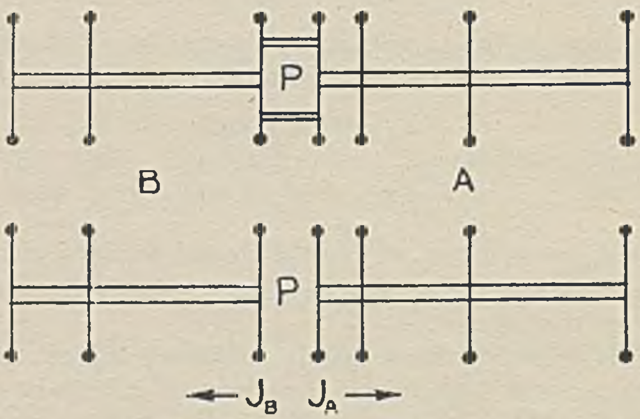


FIG. 30.—Rigidly-coupled systems.

and in this formula the quantities T and A are determined experimentally for various values of the frequency $\omega/2\pi$. It is therefore possible to obtain experimentally a curve of J_e plotted against frequency.

Let the effective inertias of the parts A and B be J_A and J_B at P. Part A consists of a system vibrating at a natural frequency together with a residual inertia J_A at P; part B consists of a system vibrating at a natural frequency together with a residual inertia J_B at P. The total effective inertia of the whole system at P is therefore J_A+J_B, and the natural frequencies of the complete system are determined by the condition

$$J_A+J_B = 0 \qquad (20.2)$$

Suppose that part A is the system of Fig. 27, P being at J_4 in this diagram, so that Fig. 28 is the curve of effective inertia J_e at P; and suppose further that the effective inertia curve of part B obtained experimentally is that shown in Fig. 31*a*. The function J_A+J_B can be plotted by adding together the corresponding ordinates in Figs. 28 and 31*a*, but the frequencies at which $J_A+J_B=0$ are more conveniently determined by plotting J_A and $-J_B$ on the same axes, as in Fig. 31*b*. The two curves intersect where $J_A=-J_B$, i.e. $J_A+J_B=0$. The natural frequencies of the complete system are thus: 2,960,

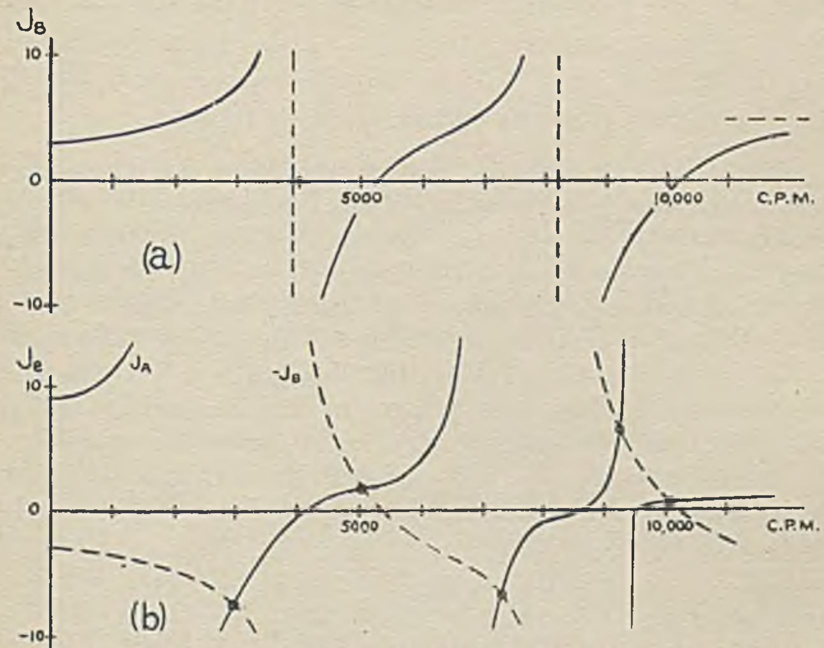


FIG. 31.—Determination of natural frequencies of Fig. 30 (J_A from Fig. 28).

5,000, 7,300, 9,200 and 10,050 C.P.M. As the two parts of the system are rigidly connected at P there are five spring connections and hence five natural frequencies. The node distributions at these frequencies are listed in Table V.

TABLE V

Frequency (C.P.M.)	Nodes in part A	Nodes in part B	Total
2,960	1	0	1
5,000	1	1	2
7,300	2	1	3
9,200	2	2	4
10,500	3	2	5

This method of calculating the natural frequencies of coupled systems is of great use even when all the dynamic constants (inertias and stiffnesses) are known, particularly when it is desired to find the effect of making alterations to one part only of the system, as it is unnecessary by this method to recalculate over the whole system; in such cases it is usual to split the system as near as possible to the point where the alterations are to be made.

21. Frequency (Torque-Summation) Tables.

The treatment, given in Sections 18 and 19, of the concept of effective inertia is a convenient one for the purpose of introducing the method, and also for enabling the two important general conclusions (Section 18) to be deduced. By far the most convenient practical method of calculating the effective inertia of a system, however, is afforded by the Frequency Tabulation, or Torque-Summation Table. By this method the effective inertia, swinging form, and variation in vibration-torque through the system are obtained by means of a systematic tabulation.

Referring to Fig. 25*a*, the torque $T_1 \sin \omega t$ in the shaft C_1 is proportional to the difference between the displacements of the inertias J_2 and J_1. Thus

$$T_1 \sin \omega t = C_1(A_2 - A_1) \sin \omega t$$

or

$$T_1 = C_1(A_2 - A_1).$$

This torque produces the acceleration $\ddot{\theta}_1 = -A_1\omega^2 \sin \omega t$, hence

$$T_1 = -J_1\omega^2 A_1 = C_1(A_2 - A_1).$$

Let the difference between A_1 and A_2 be $\delta_1 A$, so that

$$A_2 = A_1 - \delta_1 A$$

then

$$\delta_1 A = \frac{J_1\omega^2}{C_1} A_1 \quad . \quad . \quad . \quad . \quad (21.1)$$

Similarly the torque $T_2 \sin \omega t$ in the shaft C_2 is given by:

$$T_2 = C_2(A_3 - A_2)$$

and the acceleration of the inertia J_2 is effected by the resultant torque acting on J_2, i.e.

$$-J_2\omega^2 A_2 = T_2 - T_1$$

hence

$$T_2 = -J_1\omega^2 A_1 - J_2\omega^2 A_2 = C_2(A_3 - A_2).$$

Let $A_2 - A_3 = \delta_2 A$,

then

$$\delta_2 A = \frac{J_1\omega^2 A_1 + J_2\omega^2 A_2}{C_2} \quad . \quad . \quad . \quad (21.2)$$

Proceeding in this manner, the amplitude A_n of the last inertia J_n is calculated in terms of A_1; similarly the total torque which must be applied to J_n to maintain the motion is obtained as the sum

$$-(J_1\omega^2A_1+J_2\omega^2A_2+J_3\omega^2A_3+\ .\ .\ .\ +J_n\omega^2A_n).$$

Denoting this torque by $-\Sigma T_n$, the effective inertia J_e of the system at J_n is given by putting $-\Sigma T_n$ for T in (20.1), i.e.

$$J_e = \frac{\Sigma T_n}{A_n\omega^2} \quad . \quad . \quad . \quad . \quad (21.3)$$

As the amplitude A_1 is a constant multiplier throughout the calculations it is usually given unit value for convenience.

As an example the frequency tables for the system of Fig. 27 at the frequencies 4,100 and 7,990 C.P.M. ($\omega^2 = 0{\cdot}1838 \times 10^6$ and $0{\cdot}7 \times 10^6$ respectively) are given in Tables VI and VII.

TABLE VI

Frequency table for Fig. 27 at 4,100 C.P.M.

J	$J\omega^2 \div 10^6$	A	$J\omega^2A \div 10^6$	$\Sigma J\omega^2A \div 10^6$	$C \div 10^6$	δA
2	0·3675	1	0·3675	0·3675	1	0·3675
3	0·5513	0·6325	0·3487	0·7162	0·5	1·4324
2·5	0·4594	− 0·7999	− 0·3675	0·3487	0·75	0·4649
1·5	0·2757	− 1·2648	− 0·3487	0	—	—

$$J_e\omega^2 = \frac{\Sigma T_n}{A_n} = 0,\ J_e = 0 \text{ (natural frequency)}$$

TABLE VII

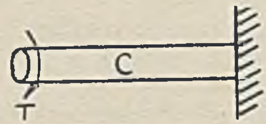
Frequency table for Fig. 27 at 7,990 C.P.M.

J	$J\omega^2 \div 10^6$	A	$J\omega^2A \div 10^6$	$\Sigma J\omega^2A \div 10^6$	$C \div 10^6$	δA
2	1·4	1	1·4	1·4	1	1·4
3	2·1	− 0·4	− 0·84	0·56	0·5	1·12
2·5	1·75	− 1·52	− 2·66	− 2·10	0·75	− 2·8
1·5	1·05	1·28	1·344	− 0·756		

$$J_e\omega^2 = \frac{\Sigma T_n}{A_n} = \frac{-0{\cdot}756}{1{\cdot}28} = -0{\cdot}591,\ J_e = -0{\cdot}845.$$

22. Dynamic stiffness.

A quantity having a close relation to the effective inertia of a system is the *dynamic stiffness.* The torsional stiffness of a shaft is defined as the ratio of the torque transmitted by the shaft to the twist induced by that torque. If one end of the shaft is clamped as in Fig. 32 and a torque T is applied at the



ANGULAR DISPLACEMENT $A = T/C$

FIG. 32.—Static stiffness of shaft.

free end, the angular displacement at the free end is T/C, where C is the torsional stiffness. If this displacement is A, then

$$C = \frac{T}{A} \quad . \quad . \quad . \quad . \quad (22.1)$$

Now consider a torque $T.\sin\omega t$ to be applied at a point P in a vibrating system, and let the resulting displacement at P be $\theta = A.\sin\omega t$. The ratio torque/displacement is defined as the "dynamic stiffness" of the system at the point P, and is denoted by Z. Thus

$$Z = \frac{T}{A} \quad . \quad . \quad . \quad . \quad (22.2)$$

The name "mechanical impedance" is sometimes given to this quantity, by a loose analogy with electrical quantities. Electrical impedance is the ratio of e.m.f. to current, and the analogous mechanical ratio is that of force (or torque) to *velocity*, not to displacement; while it is quite justifiable to choose any name for the quantity defined, it is perhaps unfortunate that the word "impedance" should be used to denote two ratios of different types. Biot (reference 5 in the Bibliography) uses the term "dynamic modulus," but this choice again is unfortunate as the word "modulus" has certain other specific meanings in mathematics, and it is indeed necessary to refer to the "modulus of the dynamic modulus" in the case of damped systems—see (22.8) and reference 4 in the Bibliography. The quantity Z defined by (22.2) is, however, the precise dynamic analogue to static stiffness, and the name "dynamic stiffness" has been adopted in this book for this reason.

Comparison of the equations (20.1) and (22.2) shows that the dynamic stiffness Z and the effective inertia J_e are related by the formula :

$$Z = - J_e\omega^2 \qquad (22.3)$$

and a curve of dynamic stiffness can easily be derived from a curve of effective inertia, or vice versa, by multiplying or dividing all the ordinates by the corresponding values of ω^2 and changing the sign. The condition for the natural frequencies of two systems rigidly coupled together is therefore

$$Z_1+Z_2 = 0 \qquad (22.4)$$

where Z_1 and Z_2 are the dynamic stiffnesses of the two systems 1 and 2 at the coupling point. (Note : the distinguishing subscripts have deliberately been changed from letters to numbers, as both notations are useful on different occasions.)

Both the effective inertia and the dynamic stiffness are measures of the response of a system to an applied sinusoidal torque ; the dynamic stiffness is frequently to be preferred when it is desired to know the response at one point in a system in terms of the response at another point, and more particularly when two systems are coupled through flexible shafts or through gearing. For these purposes three formulae relating to transference of the reference point are very useful.

(i) Transference through an inertia.

In Fig. 33*a* let the dynamic stiffnesses of the systems *to the right of* the points A and B immediately each side of the inertia J be Z_A and Z_B ; the corresponding torques and angular displacements are :

$$\begin{pmatrix} T_A \\ T_B \end{pmatrix} \sin \omega t \qquad \text{and} \qquad \begin{pmatrix} A_A \\ A_B \end{pmatrix} \sin \omega t.$$

As the inertia J is rigid, $A_A = A_B = A$ (say) and the difference between the torques T_B and T_A maintains the acceleration $\ddot{\theta} = - A\omega^2 \sin \omega t$ of the inertia. Thus

$$T_B = T_A - J\omega^2 A$$

and

$$\frac{T_B}{A_B} = \frac{T_A}{A_A} - J\omega^2$$

whence by (22.2)

$$Z_B = Z_A - J\omega^2 \qquad (22.5a)$$

(ii) Transference through a shaft.

Using the same notation for the conditions of the systems to the right of each end of the shaft in Fig. 33*b*, $T_A = T_B = T$ (say), as the shaft is supposed to be light. Furthermore, the amplitudes are connected by the relation :

$$C(A_B - A_A) = T$$

Hence
$$A_B = A_A + \frac{T}{C}$$

and
$$\frac{A_B}{T_B} = \frac{A_A}{T_A} + \frac{1}{C}$$

whence
$$\frac{1}{Z_B} = \frac{1}{Z_A} + \frac{1}{C} \qquad . \quad . \quad . \quad (22 \cdot 5b)$$

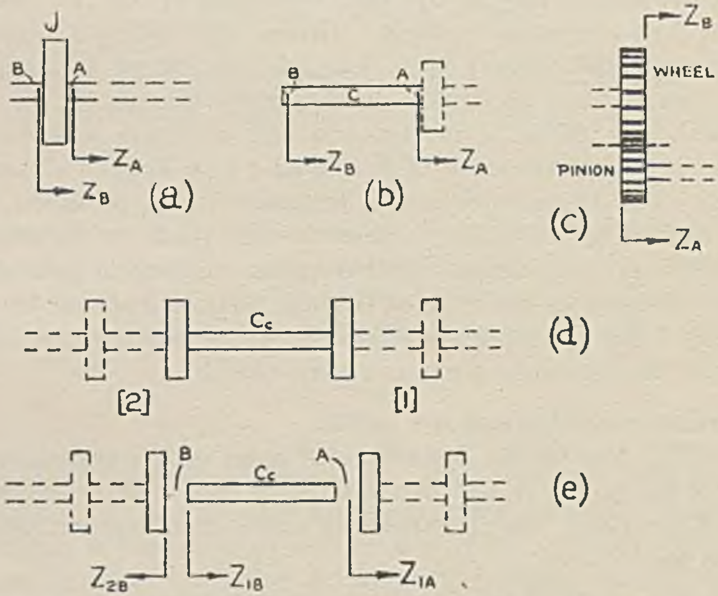


FIG. 33.—Transference of reference point for dynamic stiffness : (*a*) through a mass ; (*b*) through a shaft ; (*c*) through gearing ; (*d*) and (*e*) application to flexibly-coupled systems.

(iii) Transference through gearing.

Let Z_A in Fig. 33*c* be the dynamic stiffness of the system to the right of, and including, the gear-pinion, and let Z_B refer to the system to the right of, and excluding, the gear wheel. If the number of teeth on the pinion is n times the number of teeth on the gear-wheel, the gear-ratio is n:1. If the displacement of the pinion is θ_A, the displacement of the wheel is $n\theta_A$; whereas if the torque at the pinion is T_A, the torque at the wheel is T_A/n. From the equation (22.2) the relation between Z_B and Z_A follows:

$$Z_B = \frac{1}{n^2} Z_A \quad . \quad . \quad . \quad (22.5c)$$

The same method (of multiplying displacements and dividing torques by the gear-ratio) can be used in frequency tables.

Successive use of the two equations (22.5*a*, *b*) alternately enables the dynamic stiffness at the inertia J_4 in Fig. 27 to be determined as a function of the frequency, for the dynamic stiffness of the (zero) system to the right of J_1 is zero, and the point of reference can be shifted from right to left until the last inertia J_4 is included. This affords a third method of calculation of the effective inertia of the system.

The application of the method of dynamic stiffnesses to the problem of flexibly-coupled systems is illustrated in Figs. 33*d*, *e*. Two systems, 1 and 2, are coupled through the shaft C_C. The dynamic stiffnesses are as shown in the exploded view, Fig. 33*e*. The quantities Z_{1A} and Z_{2B} are known as functions of the frequency and it is required to find the natural frequencies of the coupled system. The stiffness Z_{1B} is given in terms of Z_{1A} by 22.5*b*), i.e.

$$\frac{1}{Z_{1B}} = \frac{1}{Z_{1A}} + \frac{1}{C_C}$$

and the condition for natural frequencies is

$$Z_{1B} + Z_{2B} = 0$$

i.e.

$$-\frac{1}{Z_{2B}} = \frac{1}{Z_{1A}} + \frac{1}{C_C} \quad . \quad . \quad . \quad (22.6)$$

The method of dynamic stiffnesses is capable of extension to damped systems, and provides the easiest way of calculating the resonant frequencies of systems with large damping forces. The full details of the method are given in reference 4 in the Bibliography, but a brief outline is not out of place here. It has already been shown (Chapter II, section 9) that in the presence of damping forces the displacement in forced motion has a non-zero phase-angle with respect to the applied force. Thus if a torque $T.\sin \omega t$ is applied at a point P in a damped system, the resultant displacement is of the form

$$\begin{aligned} \theta &= A.\sin(\omega t + \phi) \\ &= A.\cos\phi \sin\omega t + A.\sin\phi\cos\omega t. \end{aligned}$$

In Appendix I, section 35, it is shown that the cosine function can be expressed as :

$$\cos \omega t = j.\sin \omega t,$$

j being a versor operator turning the associated vector through an angle $\pi/2$ in the positive sense. (This conception has already been utilised in the discussion of the damped seismic vibrograph, Chapter II, section 12, example II.) Thus the displacement θ can be written :

$$\theta = (A.\cos \phi + Aj.\sin \phi) \sin \omega t.$$

The dynamic stiffness at P is defined, as before, as the ratio of the torque to the displacement at P, but to emphasise the complex nature of the quantity the small letter z is used. Thus

$$z = \frac{T}{A(\cos \phi + j.\sin \phi)}.$$

Multiplying top and bottom of this fraction by $(\cos \phi - j.\sin \phi)$,

$$z = \frac{T}{A}(\cos \phi - j.\sin \phi), \text{ as } \cos^2 \phi - j^2 \sin^2 \phi = 1.$$

If $\alpha = \dfrac{T}{A} \cos \phi$ and $\beta = \dfrac{T}{A} \sin \phi$,

then $$z = \alpha - j\beta \quad . \quad . \quad . \quad . \quad (22.7)$$

Formulae similar to (22.5) can be evolved for transferring the reference point through a system including dashpots which produce forces proportional to velocities. The " modulus " of the dynamic stiffness is defined as

$$|z| = \sqrt{\alpha^2 + \beta^2} \quad . \quad . \quad . \quad (22.8)$$

which, by the definitions of α and β, can be written as

$$|z| = \frac{T}{A} \quad . \quad . \quad . \quad . \quad (22.9)$$

and clearly the amplitude of motion A is a maximum when the modulus of the dynamic stiffness is a minimum. In order to determine the resonant frequencies of a damped system it is necessary therefore to obtain a curve of the modulus of the dynamic stiffness plotted against frequency ; the resonances correspond to the minima in this curve.

Note.—The present chapter has been concerned primarily with torsional vibration, as the multi-mass systems occurring most frequently in engineering are torsional ones, and all the results can be converted for use in linear systems by means of the table of analogous quantities given in Chapter I. It is no part of this present work to consider how the inertias and stiffnesses of, say, an internal combustion engine are calculated; in this work it is assumed that such data are available and to hand. Full details of the methods of calculation of these dynamic constants are given by Dr. Ker Wilson—see reference 6 in the Bibliography at the back of the book.

EXERCISES IV

(Numerical examples on the calculation of effective inertia and dynamic stiffness of systems are rather tedious unless a calculating machine is available. Such examples have, therefore, been omitted from this set of exercises, except for No. 1, the solution of which does not require more than slide-rule accuracy and for which the majority of the work has already been done in the text.)

1. Calculate the dynamic stiffness of the system of Fig. 27 from the figures of effective inertia given in Table III, section 18.

Suppose that two systems similar to that of Fig. 27 are coupled by means of a shaft joining the two inertias J_4. If the stiffness of this shaft is 1,000,000 lbs.ins. per radian, determine the natural frequencies of the complete system by means of (22.6).

2. Obtain the natural frequencies of the system of Fig. 16 by calculating the effective inertia at m_2 by the method of Section 18, and also by calculating the dynamic stiffness at m_2 by the method of Section 22.

3. Obtain the resonant frequency of the system of Fig. 9 by deriving an expression for the modulus of the dynamic stiffness at the mass (see Section 22).

CHAPTER V

CONTINUOUS SYSTEMS

(Heavy Shafts and Beams)

23. Torsional vibration of a heavy circular shaft.

THE torsional systems discussed in Chapter IV are composed of rigid inertias connected by light shafts, i.e. the flexibility of the inertias and the polar inertias of the shafts have been neglected. Many torsional systems encountered in practice can be considered as built-up in this manner; in internal combustion engines, for example, torsional vibration of the crankshaft produces vibration of the reciprocating parts (pistons, connecting-rods, etc.) and the polar inertia of the crankshaft itself is very small compared with the inertia at each cylinder due to the motion of these reciprocating parts. In the present section the torsional vibration of a heavy uniform circular shaft will be considered.

Fig. 34 shows longitudinal and transverse sections of such

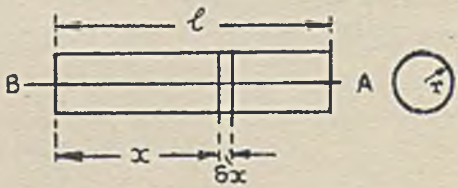


FIG. 34.—Uniform heavy circular shaft.

a shaft, the length being l and the radius r. Torsional vibrations take place about the longitudinal axis AB, and it is assumed that during the motion plane cross-sections of the shaft remain plane and undistorted. Consider a thin slice normal to the axis AB at a distance x from one end B, and of thickness δx. Let the angular displacement of the cross-section at x be θ from the static position, and let the angular displacement of the cross-section at $x+\delta x$ be $\theta+\delta\theta$. The twist over the length δx of the slice is then $\delta\theta$, and δx and $\delta\theta$ are connected by the formula

$$\frac{\delta\theta}{\delta x}=\frac{T}{GI}. \qquad (23.1a)$$

where T is the torque transmitted by the slice, G is the shear modulus of the material, and I is the polar moment of area of the cross-section, defined thus :

$$I = \Sigma r^2 \delta a$$

where δa is an elemental area of cross-section at a distance r from the axis AB, and the summation is extended over the whole cross-section. (See Appendix II, section 39.) The polar inertia of the slice about AB is

$$\delta J = I\rho\delta x \quad . \quad . \quad . \quad (23.1b)$$

where ρ is the density of the material in appropriate units ; in the engineers' system of units, δx is in inches, I in inches^4, ρ in slugs per cubic inch and δJ in lb.in.sec.^{2} (see Appendix II, section 37). The difference δT between the torques T and $T+\delta T$ at either end of the slice maintains the acceleration $\ddot{\theta}$. Thus

$$\delta T = \ddot{\theta}\delta J \quad . \quad . \quad . \quad . \quad (23.1c)$$

From (23.1b, c)

$$\frac{\partial T}{\partial x} = \rho I \ddot{\theta},$$

the partial notation being used as there are two variables x and t. Differentiation of (23.1a) gives :

$$\frac{\partial T}{\partial x} = GI\frac{\partial^2 \theta}{\partial x^2}$$

and hence

$$\ddot{\theta} = \frac{G}{\rho}\frac{\partial^2 \theta}{\partial x^2} \quad . \quad . \quad . \quad . \quad (23.2)$$

Equation (23.2) is a partial differential equation, and θ is evidently a function of both x and t. Assume that θ is such that it can be written

$$\theta = X.f(t) \quad . \quad . \quad . \quad . \quad (23.3)$$

where X is a function of x only and $f(t)$ is a function of t only. Then

$$\ddot{\theta} = X\frac{\partial^2}{\partial t^2}[f(t)]$$

and

$$\frac{\partial^2 \theta}{\partial x^2} = f(t).\frac{\partial^2 X}{\partial x^2}.$$

Substitution in (23.2) gives

$$X\frac{\partial^2}{\partial t^2}[f(t)] = \frac{G}{\rho}f(t).\frac{\partial^2 X}{\partial x^2}.$$

Now, as X is independent of t, so must be $\frac{\partial^2 X}{\partial x^2}$;

Hence $$\frac{\partial^2}{\partial t^2}[f(t)] = \left[\frac{G}{\rho}\frac{\partial^2 X}{\partial x^2}\frac{1}{X}\right]f(t) \quad . \quad . \quad (23.4)$$

and the expression in square brackets is independent of t. Denote this expression by $-\omega^2$, so that

$$\frac{G}{\rho}\frac{\partial^2 X}{\partial x^2}\frac{1}{X} = -\omega^2 \quad . \quad . \quad . \quad (23.5)$$

Then $$\frac{\partial^2}{\partial t^2}[f(t)] = -\omega^2 f(t)$$

which yields the normal simple-harmonic-motion result

$$f(t) = a.\sin(\omega t+\phi) \quad . \quad . \quad . \quad (23.6a)$$

Equation (23.5) can be written

$$\left.\begin{aligned} \frac{\partial^2 X}{\partial x^2} &= -\omega^2\frac{\rho}{G}X = -\omega^2 k^2 X \\ \text{where} \qquad k &= \sqrt{\frac{\rho}{G}} \end{aligned}\right\} \quad . \quad . \quad (23.6b)$$

the solution of which is evidently of the form

$$X = b.\sin(\omega kx+\psi).$$

It is more convenient in this case to write X in the form

$$X = b_1 \sin \omega kx + b_2 \cos \omega kx \quad . \quad . \quad (23.6c)$$

From equations (23.3) and (23.6),

$$\theta = (b_1 \sin \omega kx + b_2 \cos \omega kx)a.\sin(\omega t+\phi)$$

or, writing A for $b_1 a$, and B for $b_2 a$,

$$\theta = (A.\sin \omega kx + B.\cos \omega kx)\sin(\omega t+\phi) \quad . \quad (23.7a)$$

The torque T is given by (23.1a) and (23.7a) as:

$$T = GI\omega k(-B.\sin \omega kx + A.\cos \omega kx)\sin(\omega t+\phi) \quad (23.7b)$$

From the equations (23.7) the frequency of vibration $\omega/2\pi$ (i.e. the natural frequency) can be determined in accordance with specified end-conditions.

24. Dynamic stiffness and natural frequencies of heavy shaft.

Without loss of essential generality the phase-angle ϕ can be made zero by a suitable choice of the instant $t = 0$. In the absence of damping forces, the torques and displacements at the ends A and B of the shaft can be written:

$$\begin{pmatrix} T \\ \theta \end{pmatrix} = \begin{pmatrix} T_B \\ \beta \end{pmatrix} \sin \omega t \quad \text{at } B\ (x = 0)$$

$$\begin{pmatrix} T \\ \theta \end{pmatrix} = \begin{pmatrix} T_A \\ \alpha \end{pmatrix} \sin \omega t \quad \text{at } A\ (x = l)$$

Putting $x = 0$ in (23.7) yields

$$\beta = B \quad . \quad . \quad . \quad . \quad (24.1a)$$

$$T_B = GI\omega kA \quad . \quad . \quad . \quad (24.1b)$$

and putting $x = l$, and writing y for ωkl,

$$\alpha = A.\sin y + B.\cos y \quad . \quad . \quad . \quad (24.1c)$$

$$T_A = GI(y/l)(-B.\sin y + A.\cos y) \quad . \quad (24.1d)$$

Solving these last two equations for A and B,

$$A = \alpha \sin y + KT_A \cos y \quad . \quad . \quad (24.2a)$$

$$B = -KT_A \sin y + \alpha \cos y \quad . \quad . \quad (24.2b)$$

where $$K = \frac{l}{GIy}$$

The dynamic stiffnesses Z_A and Z_B at A and B are

$$Z_A = \frac{T_A}{\alpha}, \qquad Z_B = \frac{T_B}{\beta}$$

i.e. $$Z_B = GI\omega k\frac{A}{B}$$

$$= GI\frac{y}{l}.\frac{\alpha \sin y + KT_A \cos y}{\alpha \cos y - KT_A \sin y}.$$

This last formula reduces to

$$Z_B = \frac{1}{K}\left\{-\cot y + \frac{\operatorname{cosec} y}{\cos y - KZ_A \sin y}\right\} \quad . \quad (24.3)$$

where $$K = \frac{l}{GIy}$$

$$y = \omega kl$$

and $$k = \sqrt{\frac{\rho}{G}}.$$

Equation (24.3) gives the dynamic stiffness at B in terms of that at A; it can be applied immediately to systems similar to those discussed in Chapter IV except for the inclusion of heavy shafts, the formula (24.3) being used in place of (22.5*b*).

Four special cases are worthy of particular consideration :

(i) Z at one end of the shaft, the other end being free (Fig. 35*a*).

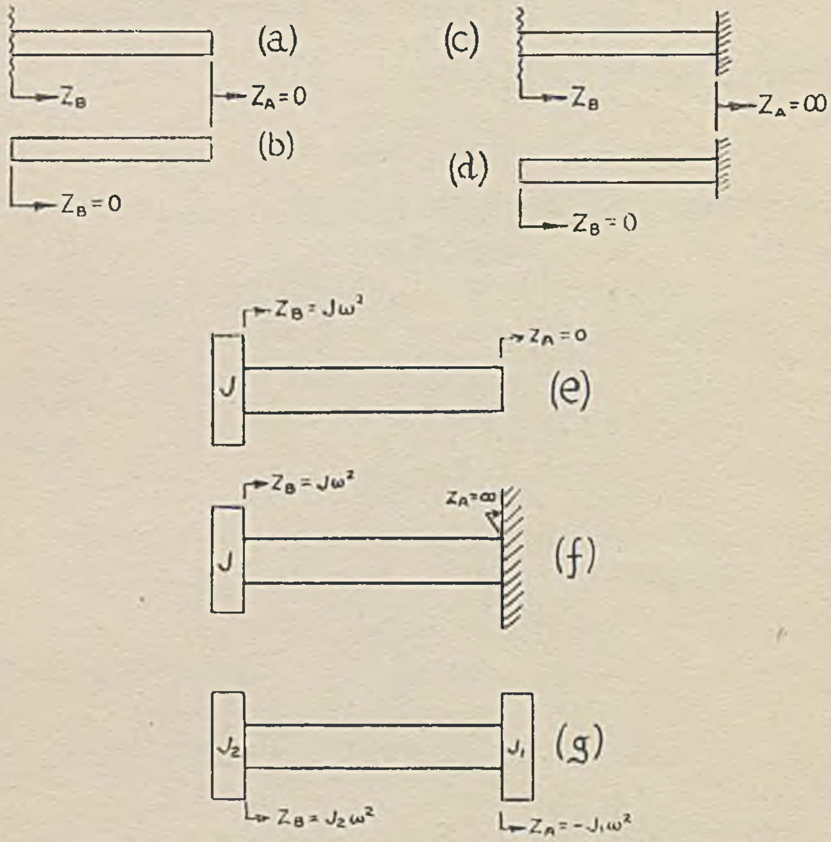


FIG. 35.—Various end conditions and special cases for Fig. 34.

In this case $Z_A = 0$,

and
$$Z_B = \frac{1}{K}\left\{-\cot y + \frac{\operatorname{cosec} y}{\cos y}\right\}.$$
$$= \frac{1}{K}\tan y$$

Hence
$$Z_B = GI\omega\sqrt{\frac{\rho}{G}}.\tan \omega l\sqrt{\frac{\rho}{G}} \quad . \quad . \quad (24.4a)$$

(ii) Natural frequencies of shaft, both ends free (Fig. 35*b*).

At the natural frequencies, $Z_B = 0$

Hence $$\omega l\sqrt{\frac{\rho}{G}} = 0,\ \pi,\ 2\pi,\ \text{etc.}$$

Apart from the zero solution, the natural frequencies are

$$\omega/2\pi$$

where $$\omega = \frac{\pi}{l}\sqrt{\frac{G}{\rho}},\ \frac{2\pi}{l}\sqrt{\frac{G}{\rho}},\ \frac{3\pi}{l}\sqrt{\frac{G}{\rho}},\ \text{etc.} \qquad . \quad (24.4b)$$

(iii) Z at one end of the shaft, the other end being clamped (Fig. 35*c*).

In this case

$$Z_A = \infty$$

and $$Z_B = \frac{1}{K}(-\cot y)\ \text{unless}\ \sin y = 0.$$

Hence $$Z_B = -GI\omega\sqrt{\frac{\rho}{G}}.\cot \omega l\sqrt{\frac{\rho}{G}} \qquad . \quad . \quad (24.4c)$$

(iv) Natural frequencies of shaft, one end clamped (Fig. 35*d*).

At the natural frequencies $Z_B = 0$

Hence $$\omega l\sqrt{\frac{\rho}{G}} = \frac{\pi}{2},\ \frac{3\pi}{2},\ \frac{5\pi}{2},\ \text{etc.}$$

or $$\omega = 0.$$

Apart from the zero solution the natural frequencies are

$$\omega/2\pi$$

where $$\omega = \frac{\pi}{2l}\sqrt{\frac{G}{\rho}},\ \frac{3\pi}{2l}\sqrt{\frac{G}{\rho}},\ \frac{5\pi}{2l}\sqrt{\frac{G}{\rho}},\ \text{etc.} \qquad . \quad (24.4d)$$

Three other special cases, illustrated in Fig. 35*e*, *f*, *g*, are of interest. The method of calculating the natural frequencies of these systems is indicated in Exercise 4 at the end of the chapter.

25. Flexural vibration of uniform beam.

Fig. 36 represents an elemental portion of a uniform heavy beam vibrating transversely. The section is at a distance x from one end of the beam and is of length δx; the total length of the beam is l. It is assumed that each point in the beam moves at right-angles to the static direction of the beam-axis

Ox. Let the displacement of the section from the static position be y. Elementary theory of elasticity (see Appendix II, section 38) gives the formulae :

$$\left.\begin{aligned} M &= \text{Bending moment} = EI\frac{\partial^2 y}{\partial x^2} \\ F &= \text{Shear force} = \frac{\partial M}{\partial x} \end{aligned}\right\} \quad . \quad . \quad (25.1)$$

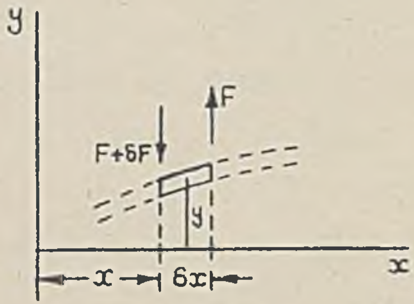


Fig. 36.—Elemental portion of vibrating beam.

where E is Young's modulus of elasticity for the material and I is the second moment of area of the cross-section defined as $\Sigma r^2 \delta a$, δa being an elemental area of cross-section at a distance r from the neutral axis of bending (see Appendix II).

Let the mass per unit length of the beam be m. The resultant shear force on the section is $-\delta F$ in the positive direction of y ; hence

$$\delta F = -m\delta x\ddot{y}$$

or
$$\frac{\partial F}{\partial x} = -m\ddot{y}$$

But from (25.1)
$$\frac{\partial F}{\partial x} = \frac{\partial^2 M}{\partial x^2} = EI\frac{\partial^4 y}{\partial x^4},$$

as the quantities E and I are constant over the whole length of the beam.

Hence
$$EI\frac{\partial^4 y}{\partial x^4} = -m\ddot{y} \quad . \quad . \quad . \quad (25.2)$$

(25.2) is a partial differential equation similar to (23.2) except that the partial differential with respect to x is the fourth instead of the second. The solution y can therefore be written

$$y = X \,.\, \sin \omega t \quad . \quad . \quad . \quad (25.3)$$

where X is a function of x only, i.e. X is independent of t. The phase-angle required for the most general solution is omitted as it is of no importance here. Substituting in (25.2),

$$EI\frac{\partial^4 X}{\partial x^4} = m\omega^2 X$$

or

$$\left.\begin{aligned} \frac{\partial^4 X}{\partial x^4} &= k^4 X \\ \text{where} \quad k^4 &= \frac{m\omega^2}{EI} \end{aligned}\right\} \quad . \quad . \quad . \quad (25.4)$$

Writing equation 25.4 in operator form,

$$D^4X = k^4X, \text{ where } D \equiv \frac{\partial}{\partial x}.$$

Hence $\qquad D = \pm k$ or $\pm ik \quad (i = \sqrt{-1})$.

The general solution in x must have four arbitrary constants, as (25.4) must be integrated four times to eliminate differential coefficients. By extension of the method of Section 2, Chapter I, the general solution is found to be of the form:

$$X = C_1e^{kx}+C_2e^{-kx}+C_3e^{ikx}+C_4e^{-ikx} \quad . \quad (25.5)$$

It is shown in Appendix I, sections 33 and 34, that

$$\begin{aligned} e^{\theta} &= \cosh\theta+\sinh\theta \\ e^{-\theta} &= \cosh\theta - \sinh\theta \\ e^{i\theta} &= \cos\theta+i.\sin\theta \\ e^{-i\theta} &= \cos\theta - i.\sin\theta \end{aligned}$$

Putting kx for θ, (25.5) can therefore be written in the form

$$X = A_1 \sin kx+A_2 \cos kx+A_3 \sinh kx+A_4 \cosh kx$$

and hence

$$y = (A_1 \sin kx+A_2 \cos kx+A_3 \sinh kx+A_4 \cosh kx) \sin \omega t \quad (25.6a)$$

The bending moment M is given by (25.1) and (25.6a) as:

$$M = EIk^2(-A_1 \sin kx - A_2 \cos kx+A_3 \sinh kx +A_4 \cosh kx) \sin \omega t \quad (25.6b)$$

and the shear force $\qquad F = \dfrac{\partial M}{\partial x}$, i.e.

$$F = EIk^3(-A_1 \cos kx+A_2 \sin kx+A_3 \cosh kx + A_4 \sinh kx) \sin \omega t \quad (25.6c)$$

From equations (25.6) the natural frequencies can be determined according to the end-conditions.

26. Natural frequencies of beam.

(i) Cantilever.

If the beam is built-in at one end, as in Fig. 37*a*, the end-conditions are that at $x = 0$, the displacement and slope are zero,

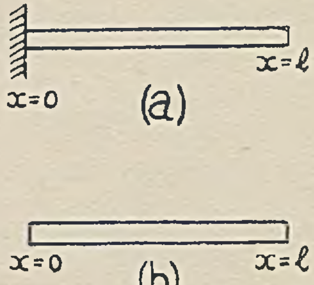


FIG. 37.—(*a*) Cantilever; (*b*) free-free beam.

while at $x = l$ the bending moment and shear force are zero, as there is no loading beyond the end of the beam.

I.e., at $$x = 0,\ y = 0 \text{ and } \frac{\partial y}{\partial x} = 0$$

at $$x = l,\ \frac{\partial^2 y}{\partial x^2} = 0 \text{ and } \frac{\partial^3 y}{\partial x_3} = 0.$$

Substituting these values in (25.6),

$$A_2 + A_4 = 0 \quad . \quad . \quad . \quad (26.1a)$$

$$A_1 + A_3 = 0 \quad . \quad . \quad . \quad (26.1b)$$

$$-A_1 \sin kl - A_2 \cos kl + A_3 \sinh kl + A_4 \cosh kl = 0 \quad (26.1c)$$

and

$$-A_1 \cos kl + A_2 \sin kl + A_3 \cosh kl + A_4 \sinh kl = 0 \quad (26.1d)$$

Elimination of the constants A_1, etc., from these four equations leads to the result:

$$\cos kl . \cosh kl + 1 = 0 \quad . \quad . \quad . \quad (26.2)$$

The first four roots of this equation are

$$kl = 1{\cdot}875,\ 4{\cdot}694,\ 7{\cdot}855,\ 10{\cdot}996$$

and the first four natural frequencies are therefore:

$$\omega/2\pi,$$

where $$\omega = \frac{\lambda}{l^2}\sqrt{\frac{EI}{m}} \text{ and } \lambda = 3{\cdot}54,\ 22{\cdot}4,\ 61{\cdot}7,\ 121 \quad . \quad (26.3)$$

(ii) Free-free beam.

If the beam is free at both ends, as in Fig. 37*b*, the end-conditions are :

$$M = F = 0 \text{ at both ends,}$$

i.e. $$\frac{\partial^2 y}{\partial x^2} = \frac{\partial^3 y}{\partial x^3} = 0 \text{ at both ends } (x = 0 \text{ and } x = l)$$

Hence
$$-A_2 + A_4 = 0 \quad . \quad . \quad . \quad (26.4a)$$
$$-A_1 + A_3 = 0 \quad . \quad . \quad . \quad (26.4b)$$
$$-A_1 \sin kl - A_2 \cos kl + A_3 \sinh kl + A_4 \cosh kl = 0 \quad (26.4c)$$
and
$$-A_1 \cos kl + A_2 \sin kl + A_3 \cosh kl + A_4 \sinh kl = 0 \quad (26.4d)$$

Elimination of the constants A_1, etc., from these equations leads to the result :

$$\cos kl . \cosh kl - 1 = 0 \quad . \quad . \quad . \quad (26.5)$$

The first four roots of this equation are

$$kl = 4{\cdot}730,\ 7{\cdot}853,\ 10{\cdot}996,\ 14{\cdot}137$$

and the first four natural frequencies are therefore

$$\omega/2\pi,$$

where $$\omega = \frac{\lambda}{l^2}\sqrt{\frac{EI}{m}} \text{ and } \lambda = 22{\cdot}4,\ 61{\cdot}7,\ 121,\ 200 \quad . \quad (26.6)$$

The natural frequencies of beams under other end-conditions can be found in a similar manner ; the subject is treated fully by Timoshenko (reference 7 in the Bibliography at the back of the book), who also discusses the validity of the various assumptions that have been made.

The deformation shapes of the cantilever, Fig. 37*a*, at the various natural frequencies are illustrated in Fig. 38. It will be seen that the continuous beam has the same property as multi-mass systems as regards the different modes of vibration, namely that each successive natural frequency involves an additional node.

The analytical solution of the problem of the uniform beam is relatively simple, owing to the fact that the quantities E and I in equations (25.1) and m in the equation $\frac{\partial F}{\partial x} = -m\ddot{y}$ are constant over the whole length of the beam. In a beam of variable cross-section or of non-uniform material the quantities

E, I and m are not all constant, and the differential equation corresponding to (25.2) is

$$\frac{\partial^2}{\partial x^2}\left[EI \cdot \frac{\partial^2 y}{\partial x^2}\right] = -m\ddot{y} \quad . \quad . \quad . \quad (26.7)$$

The difficulty of solution is not due to any abstruseness of the theory but to the complexity of the calculation. The equation

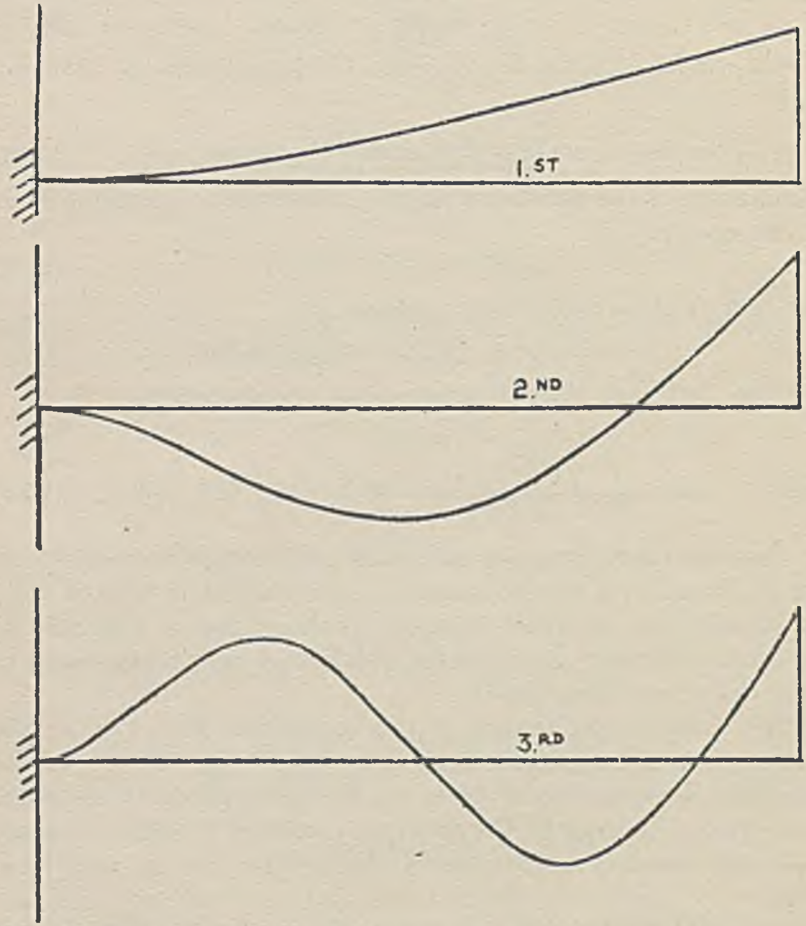


Fig. 38.—Deformation shapes of cantilever at first three natural frequencies.

(26.7) cannot be solved in terms of analytic functions if E, I and m are unspecified functions of x, and an approximate numerical method has to be developed to enable a solution to be obtained at all. Numerous authors have published methods of solution ;

an adequate treatment of the subject is given by Timoshenko (reference 8 in the Bibliography).

27. Rayleigh's method for heavy beams.

Rayleigh's approximate method for finding the lowest natural frequency of heavy uniform beams depends upon the energy properties of vibrating systems. When the system is at its maximum displacement the kinetic energy is instantaneously zero, as the system is instantaneously at rest, and the potential energy is at its maximum value ; for the potential energy is the work done against the restoring forces, and this is clearly a maximum at the maximum displacement. Similarly, when the system passes through its mean position (zero harmonic displacement) the kinetic energy is a maximum and the potential energy is zero.

Consider for example a single-mass system, the displacement of the mass m being $x = a.\sin \omega t$ against a spring force kx, as in Chapter I. The kinetic and potential energies are given by

$$\left.\begin{aligned} \text{K.E.} &= \tfrac{1}{2}m\dot{x}^2 = \tfrac{1}{2}ma^2\omega^2\cos^2\omega t & (a) \\ \text{P.E.} &= \int_0^x (kx)dx = \tfrac{1}{2}kx^2 = \tfrac{1}{2}ka^2\sin^2\omega t & (b) \end{aligned}\right\} \quad (27.1)$$

At the maximum displacement $\sin \omega t = \pm 1$, $\cos \omega t = 0$

Thus $(\text{P.E.})_{max.} = \tfrac{1}{2}ka^2$, $\text{K.E.} = 0$ . . (27.2a)

At the mean position $\sin \omega t = 0$, $\cos \omega t = \pm 1$,

and $(\text{K.E.})_{max.} = \tfrac{1}{2}ma^2\omega^2$, $\text{P.E.} = 0$. (27.2b)

The total energy in the system is the sum of the kinetic and potential energies, and this total remains constant throughout the motion in the absence of damping forces, as there is no dissipation. The maximum kinetic energy is thus equal to the maximum potential energy, and thus

$$\tfrac{1}{2}ka^2 = \tfrac{1}{2}ma^2\omega^2$$

whence

$$\omega^2 = \frac{k}{m} \quad . \quad . \quad . \quad . \quad (27.3)$$

and the frequency $\omega/2\pi$ is thus determined.

In the case of heavy beams, the expressions for the kinetic and potential energies are more complicated. Consider the elemental section of beam in Fig. 36. For sinusoidal motion $y = y_0 \sin \omega t$ the maximum kinetic energy $\delta(\text{K.E.})_{max.}$ of the section is $\delta(\text{K.E.})_{max.} = \frac{1}{2}m\omega^2 y_0^2 \delta x$, as the mass of the section is $m\delta x$. The maximum kinetic energy of the whole beam is therefore

$$(\text{K.E.})_{max.} = \tfrac{1}{2}m\omega^2\int_0^l y_0^2\,dx \quad . \quad . \quad (27.4a)$$

Referring to Fig. 39*a*, if the left-hand end of the section is fixed the bending moment M turns the right-hand end through the angle $\delta\theta$, and M is proportional to $\delta\theta$. The work done against M during this rotation is given by the shaded area in Fig. 39*b*,

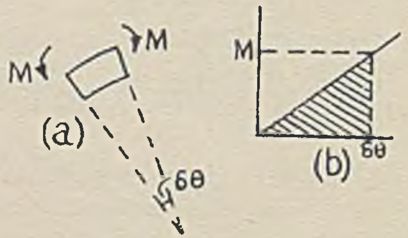


FIG. 39.—Derivation of potential energy formula for beam.

and is equal to $\frac{1}{2}M\delta\theta$. The potential energy of the section is thus

$$\delta(\text{P.E.}) = \tfrac{1}{2}M\delta\theta.$$

The slope of the displacement curve at the left-hand end of the section is $\frac{\partial y}{\partial x}$, and the slope at the right-hand end is

$$\frac{\partial y}{\partial x} + \left(\frac{\partial^2 y}{\partial x^2}\right).\delta x, \text{ hence } \delta\theta = \frac{\partial^2 y}{\partial x^2}\delta x$$

and
$$\delta(\text{P.E.}) = \tfrac{1}{2}M\frac{\partial^2 y}{\partial x^2}\delta x.$$

Substituting
$$M = EI\frac{\partial^2 y}{\partial x^2},$$

$$\delta(\text{P.E.}) = \tfrac{1}{2}EI\left(\frac{\partial^2 y}{\partial x^2}\right)^2\delta x.$$

The maximum potential energy of the whole beam is therefore

$$(\text{P.E.})_{max.} = \tfrac{1}{2}EI\int_0^l\left(\frac{\partial^2 y_0}{\partial x^2}\right)^2 dx \quad . \quad . \quad (27.4b)$$

The natural frequency $\omega/2\pi$ can be found if the deformation shape is known, by equating the two expressions (27.4). The procedure is to guess a deformation curve which satisfies the end-conditions and corresponds to the mode of vibration concerned. As an example, consider the system of Fig. 40, consisting of a heavy uniform beam simply supported at each end (so that the displacements at the ends are zero but the slopes are not

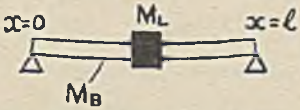


FIG. 40.—Simply-supported beam with central loading.

necessarily zero) and carrying a concentrated load at the centre of the span.

Let l = length of beam
m = mass per unit length of beam
M_B = mass of beam
M_L = mass of central load
y_C = maximum deflection at centre of span.

The end-conditions to be satisfied are that the displacements and bending moments at each end are zero, i.e.

$$y = 0 \text{ and } \frac{\partial^2 y}{\partial x^2} = 0 \text{ at } x = 0 \text{ and } x = l.$$

For the fundamental, or lowest, natural frequency the curve $y_0 = y_C \sin\frac{\pi x}{l}$ satisfies the end-conditions and the mode. From (27.4a) the maximum kinetic energy of the beam is

$$(\text{K.E.})_{max.} = \tfrac{1}{2}m\omega^2\int_0^l y_C^2 \sin^2\frac{\pi x}{l}dx = \tfrac{1}{4}m\omega^2 y_C^2 l$$
$$= \tfrac{1}{4}M_B\omega^2 y_C^2$$

and for the central load $(\text{K.E.})_{max.} = \tfrac{1}{2}M_L\omega^2 y_C^2$.
Hence for the whole system,

total $$(\text{K.E.})_{max.} = \tfrac{1}{2}\omega^2 y_C^2(\tfrac{1}{2}M_B + M_L) \quad . \quad (27.5a)$$

From (27.4b),

$$(\text{P.E.})_{max.} = \tfrac{1}{2}EI\int_0^l\left(\frac{\partial^2 y}{\partial x^2}\right)^2 dx = \frac{EIy_C^2\pi^4}{4l^4}\int_0^l 2.\sin^2\frac{\pi x}{l}dx.$$

The value of the definite integral is l, and hence

$$(\text{P.E.})_{max.} = \frac{EIy_C^2\pi^4}{4l^3} \quad . \quad . \quad (27.5b)$$

Equating the expressions in (27.5*a*, *b*),

$$\tfrac{1}{2}\omega^2 y_C{}^2(\tfrac{1}{2}M_B+M_L) = \frac{EIy_C{}^2\pi^4}{4l^3}$$

whence
$$\omega^2 = \frac{48{\cdot}7EI}{l^3(\frac{1}{2}M_B+M_L)} \quad . \quad . \quad . \quad (27.6)$$

From this formula the frequency $\omega/2\pi$ is obtained.

Two special cases deserve attention :

(i) If the central load is zero, the system reduces to a heavy beam simply supported at both ends, and (27.6) becomes

$$\omega^2 = \frac{97{\cdot}4EI}{M_B l^3}.$$

This result agrees with the exact expression derived by the method of Section 26.

(ii) If the beam is light compared with the central load, (27.6) becomes

$$\omega^2 = \frac{48{\cdot}7EI}{M_L l^3}.$$

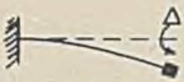
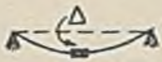
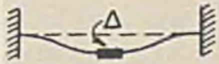
Comparing this formula with $\omega^2 = k/m$ for the system of Fig. 1, the effective spring-constant of the light beam is here given as 48·7 EI/l^3. If this spring-constant is calculated according to normal methods (see Appendix II, section 38) the numerical factor is obtained as 48 ; thus Rayleigh's method gives a value for ω^2 which is 1·4% in excess of the true value ; the frequency error is thus only 0·7%.

Rayleigh's method is very useful when it is desired to know the effect of neglecting the mass of a beam carrying concentrated loads ; in the case of a simply-supported beam carrying a central load, (27.6) shows that half the mass of the beam should be added to the central load if it is required to find the lowest natural frequency assuming the system to be simply a one-mass system similar to Fig. 1, k being the effective stiffness of the beam at the centre of the span.

Table VIII gives various formulae and results for three different types of loaded heavy beams :

(i) cantilever with end-loading,
(ii) simply-supported beam with central loading,
(iii) clamped-clamped beam with central loading. The table gives :

TABLE VIII

LENGTH OF BEAM ℓ, DEFLECTION y, MASS OF BEAM M_B, MASS OF LOAD M_L, MAX: DEFLECTION Δ			
SYSTEM:			
CONDITIONS $x = 0$, $x = \ell$	$y = 0 = \frac{\partial y}{\partial x}$; $\frac{\partial^2 y}{\partial x^2}$	$y = 0 = \frac{\partial^2 y}{\partial x^2}$	$y = 0 = \frac{\partial y}{\partial x}$
ASSUMED FUNCTION	$y = \Delta\left(1 - \cos\frac{\pi x}{2\ell}\right)$	$y = \Delta \sin\frac{\pi x}{\ell}$	$y = \frac{\Delta}{2}\left(1 - \cos\frac{2\pi x}{\ell}\right)$
$(K.E)_{max.}$	$\omega^2\Delta^2(\frac{1}{2}M_L + \cdot 113 M_B)$	$\omega^2\Delta^2(\frac{1}{2}M_L + \frac{1}{4}M_B)$	$\omega^2\Delta^2(\frac{1}{2}M_L + \frac{3}{16}M_B)$
$(P.E)_{max}$	$\frac{\pi^4 EI}{64\,\ell^3}\Delta^2$	$\frac{\pi^4 EI}{4\,\ell^3}\Delta^2$	$\frac{\pi^4 EI}{\ell^3}\Delta^2$
ω^2 $(F = \frac{\omega}{2\pi}$ CPS$)$	$\frac{3{\cdot}04 EI}{\ell^3(M_L + \cdot 226 M_B)}$	$\frac{48{\cdot}7 EI}{\ell^3(M_L + \frac{1}{2}M_B)}$	$\frac{195 EI}{\ell^3(M_L + \frac{3}{8}M_B)}$
FREQUENCY FOR LIGHT BEAM	$F = \frac{1}{2\pi}\sqrt{\frac{3EI}{M_L\ell^3}}$	$F = \frac{1}{2\pi}\sqrt{\frac{48EI}{M_L\ell^3}}$	$F = \frac{1}{2\pi}\sqrt{\frac{192EI}{M_L\ell^3}}$
CORRECTION FOR MASS OF BEAM	$0{\cdot}226\ M_B$	$0{\cdot}5\ M_B$	$0{\cdot}375\ M_B$
FREQUENCY ERROR	0·6 %	0·7 %	0·75 %

(*a*) end-conditions,
(*b*) Rayleigh function,
(*c*) maximum kinetic and potential energies,
(*d*) value of ω^2,
(*e*) frequency for light beam ($M_B = 0$),
(*f*) correction for mass of beam,
and (*g*) approximate frequency error.

EXERCISES V

1. Check by Rayleigh's method the expression given for the lowest natural frequency of a free-free beam in (26.6), and so obtain data for another column in Table VIII.

(The appropriate function is $y_0 = a - Y \sin \frac{\pi x}{l}$, in which a has to be found by determining the condition for zero total inertia force ($m\ddot{y}$) for the whole beam.)

2. Check the data given in columns (i) and (iii) of Table VIII.

3. Draw the swinging forms of a uniform heavy circular shaft at its first four natural frequencies, the shaft being free at both ends.

(Use the formula 23.6c in conjunction with the special end-conditions.)

4. Determine the natural frequencies of the heavy-shaft systems shown in Fig. 35*e*, *f*, *g*.

(Use the formula 24.3, putting in the particular end-conditions required

(*e*) $Z_A = 0$, $Z_B = J\omega^2$ giving $y = \frac{J_s}{J} \tan y$,

where $y = \omega l \sqrt{\frac{\rho}{G}}$ and J_s is the polar inertia of the shaft $= \rho l I$.

(*f*) $Z_A = \infty$, $Z_B = J\omega^2$ giving $y = -\frac{J_s}{J} \cot y$.

(*g*) $Z_A = -J_1\omega^2$, $Z_B = J_2\omega^2$, giving

$$K^2 J_1 J_2 \omega^4 + K(J_1 + J_2)\omega^2 \cot y - 1 = 0$$

where $K = l/GIy$.)

CHAPTER VI

COMPLEX VIBRATIONS

(Fourier Analysis)

28. Fourier series.

If a function $f(t)$ of any variable t is periodic with a t-period τ, so that if the variable t is increased by any multiple of τ the value of the function is unaltered (i.e. $f(t) = f(t+k\tau)$, where k is any integer), then the function can be expressed in the form:

$$f(t) = a_0+a_1 \cos \frac{2\pi t}{\tau}+a_2 \cos 2\frac{2\pi t}{\tau}+a_3 \cos 3\frac{2\pi t}{\tau}, \text{ etc.}$$

$$+b_1 \sin \frac{2\pi t}{\tau}+b_2 \sin 2\frac{2\pi t}{\tau}+b_3 \sin 3\frac{2\pi t}{\tau}, \text{ etc.}$$

The series can be expressed concisely as:

$$f(t) = a_0+\sum_{n=1}^{\infty} a_n \cos n\frac{2\pi t}{\tau}+\sum_{n=1}^{\infty} b_n \sin n\frac{2\pi t}{\tau}$$

where n is integral . . . (28.1)

The result is known as *Fourier's Theorem*, and the series is commonly termed a Fourier series.

If the variable t represents time, so that the function has a time-period τ, it is convenient to write ω for $2\pi/\tau$, and then

$$f(t) = a_0+\sum_{n=1}^{\infty} a_n \cos n\omega t+\sum_{n=1}^{\infty} b_n \sin n\omega t \quad . \quad (28.2)$$

The terms $a_n \cos n\omega t+b_n \sin n\omega t$ can be written as $r_n \sin n(\omega t+\phi_n)$, where $r_n{}^2 = a_n{}^2+b_n{}^2$ and $\tan n\phi_n = a_n/b_n$ so that the series can be put in the alternative form:

$$f(t) = a_0+\sum_{n=1}^{\infty} r_n \sin n(\omega t+\phi_n) \quad . \quad . \quad (28.3)$$

The quantity $\omega/2\pi$, being the reciprocal of the time-period τ, is termed the *fundamental frequency* of the function; the term in the series for which $n = 1$ is called the *fundamental component*, and the other terms ($n = 2$, 3, etc.) are known as the *harmonics*.

As an example of the synthesis of a periodic function by the addition of sine-waves, consider the function

$$f(\theta) = \sin\theta + \tfrac{1}{3}\sin 3\theta + \tfrac{1}{5}\sin 5\theta + \ldots$$
$$\ldots \frac{1}{2n-1}\sin(2n-1)\theta + \ldots \quad (28.4)$$

In Table IX are listed values of the various terms in (28.4) as far as the term involving 7θ, i.e. the 7th harmonic.

TABLE IX

θ/π	$\sin\theta$	$\frac{1}{3}\sin 3\theta$	$\frac{1}{5}\sin 5\theta$	$\frac{1}{7}\sin 7\theta$	sum
0	0·0000	0·0000	0·0000	0·0000	0·0000
0·1	0·3090	0·2697	0·2000	0·1156	0·8943
0·2	0·5878	0·3170	0·0000	− 0·1359	0·7689
0·3	0·8090	0·1030	− 0·2000	0·0441	0·7561
0·4	0·9511	− 0·1966	0·0000	0·0840	0·8385
0·5	1·0000	− 0·3333	0·2000	− 0·1429	0·7238
0·6	0·9511	− 0·1966	0·0000	0·0840	0·8385
0·7	0·8090	0·1030	− 0·2000	0·0441	0·7561
0·8	0·5878	0·3170	0·0000	− 0·1359	0·7689
0·9	0·3090	0·2697	0·2000	0·1156	0·8943
1·0	0·0000	0·0000	0·0000	0·0000	0·0000
1·1	− 0·3090	− 0·2697	− 0·2000	− 0·1156	− 0·8943
1·2	− 0·5878	− 0·3170	0·0000	0·1359	− 0·7689
1·3	− 0·8090	− 0·1030	0·2000	− 0·0441	− 0·7561
1·4	− 0·9511	0·1966	0·0000	− 0·0840	− 0·8385
1·5	− 1·0000	0·3333	− 0·2000	0·1429	− 0·7238
1·6	− 0·9511	0·1966	0·0000	− 0·0840	− 0·8385
1·7	− 0·8090	− 0·1030	0·2000	− 0·0441	− 0·7561
1·8	− 0·5878	− 0·3170	0·0000	0·1359	− 0·7689
1·9	− 0·3090	− 0·2697	− 0·2000	− 0·1156	− 0·8943
2·0	0·0000	0·0000	0·0000	0·0000	0·0000
2·1	0·3090	0·2697	0·2000	0·1156	0·8943
		etc.	etc.		

The values in the last column of Table IX are plotted in Fig. 41*a*. If more terms of the series are taken, the curve approximates more nearly to the shape of Fig. 41*b*,* i.e.

$$\left.\begin{aligned} f(\theta) &= \pi/4 \text{ for } 0 < \theta < 2\pi \\ f(\theta) &= -\pi/4 \text{ for } \pi < \theta < 2\pi \end{aligned}\right\} \quad . \quad . \quad (28.5)$$

It will be seen that the function (28.5) is such that

$$f(\theta) = -f(-\theta) = -f(2\pi - \theta) \quad . \quad . \quad (28.6)$$

* See note on Gibbs' phenomenon, p. 92.

so that the curve, Fig. 41*b*, is skew-symmetric about the values $\theta = 0$ and $\theta = \pi$. The result is due to the fact that all the terms in the series (28.4) are sine-terms and obey the law (28.6).

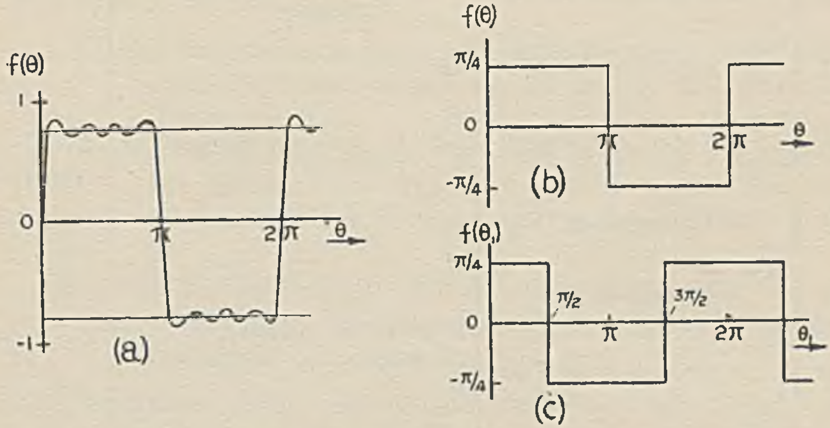


FIG. 41.—(*a*) Graph of the function (28·4), first four components : (*b*) sine-term function ; (*c*) cosine-term function.

Let the variable θ be changed to θ_1, where $\theta = \theta_1 + \pi/2$; the resulting graph is that of Fig. 41*c*, which is seen to be symmetrical about the lines $\theta_1 = 0$ and $\theta_1 = \pi$, so that

$$f(\theta_1) = f(-\theta_1) = f(2\pi - \theta_1) \quad . \quad . \quad (28.7)$$

Making the substitution in the series (28.4),

$$f(\theta_1) = \cos\theta_1 - \tfrac{1}{3}\cos 3\theta_1 + \tfrac{1}{5}\cos 5\theta, \text{ etc.} \quad . \quad (28.8)$$

as

$$\sin(\theta_1 + \pi/2) = \cos\theta$$
$$\sin 3(\theta_1 + \pi/2) = -\cos 3\theta_1$$
$$\sin 5(\theta_1 + \pi/2) = \cos 5\theta_1,$$
etc. etc.

The forms of (28.4) and (28.8) point to a general conclusion that if the function $f(t)$ in (28.2) is symmetrical about the lines $t = C$, $C + \pi/\omega$, etc., the series can be put into a form containing cosine terms only by making the substitution $t_1 = t - C$; and similarly if the function is skew-symmetric about these lines the series can be put into a form containing sine terms only.

29. Fourier analysis.

The process of determining the coefficients a and b in the series (28.2) for any particular function $f(t)$ is known as Fourier Analysis. The procedure depends upon the results :

$$
\left.\begin{array}{lll}
(a) & \displaystyle\int_0^{2\pi/\omega} \sin m\omega t . \sin n\omega t \, dt = 0 & \\
(b) & \displaystyle\int_0^{2\pi/\omega} \sin m\omega t . \cos n\omega t \, dt = 0 & \text{if } m \text{ and } n \text{ are unequal integers} \\
(c) & \displaystyle\int_0^{2\pi/\omega} \cos m\omega t . \cos n\omega t \, dt = 0 & \\
(d) & \displaystyle\int_0^{2\pi/\omega} \sin m\omega t . \cos m\omega t \, dt = 0 & \text{if } m \text{ is an integer} \\
(e) & \displaystyle\int_0^{2\pi/\omega} \sin \ m\omega t \, dt = 0 & \\
(f) & \displaystyle\int_0^{2\pi/\omega} \cos \ m\omega t \, dt = 0 & \text{if } m \text{ is an integer.} \\
(g) & \displaystyle\int_0^{2\pi/\omega} \sin^2 m\omega t \, dt = \pi/\omega & \\
(h) & \displaystyle\int_0^{2\pi/\omega} \cos^2 m\omega t \, dt = \pi/\omega &
\end{array}\right\} \quad (29.1)
$$

These results are proved in Appendix I, section 36.

Integrate (28.2) with respect to t over a cycle of the function, i.e. between the limits 0 and $2\pi/\omega$ for t.

$$\int_0^{2\pi/\omega} f(t)\, dt = \int_0^{2\pi/\omega} (a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega t + \sum_{n=1}^{\infty} b_n \sin n\omega t)\, dt.$$

The equations (29.1) show that every term on the right-hand side will be zero except that due to the integration of a_0. Hence

$$\int_0^{2\pi/\omega} f(t)\, dt = [a_0 t]_0^{2\pi/\omega} = a_0 \frac{2\pi}{\omega}$$

and

$$a_0 = \frac{\omega}{2\pi}\int_0^{2\pi/\omega} f(t)\, dt \quad . \quad . \quad (29.2a)$$

Now multiply (28.2) by $\cos m\omega t$ and integrate over a cycle ; thus

$$\int_0^{2\pi/\omega} f(t) \cos m\omega t\, dt = \int_0^{2\pi/\omega} a_m \cos^2 m\omega t\, dt = \frac{\pi}{\omega} a_m$$

as all the other terms integrate to zero by reason of the equations (29.1).

Hence

$$a_m = \frac{\omega}{\pi}\int_0^{2\pi/\omega} f(t) \cos m\omega t\, dt \quad . \quad . \quad (29.2b)$$

Similarly, by multiplying (28.2) by $\sin m\omega t$ and integrating over a cycle, it is found that

$$b_m = \frac{\omega}{\pi}\int_0^{2\pi/\omega} f(t) \sin m\omega t \, dt \qquad . \qquad . \qquad (29.2c)$$

The procedure for determining the Fourier coefficients a and b is thus as follows:

(i) The constant a_0 is found by evaluating the average value of the function over a cycle, for that is what the value (29.2a) is.

(ii) The cosine coefficients "a" are found by multiplying the function by the appropriate cosine function, integrating over a cycle, and multiplying the result by ω/π.

(iii) The sine coefficients "b" are found in a similar manner, except that the appropriate sine function is used instead of the cosine function.

The equations (29.2) are collected together below for easy reference.

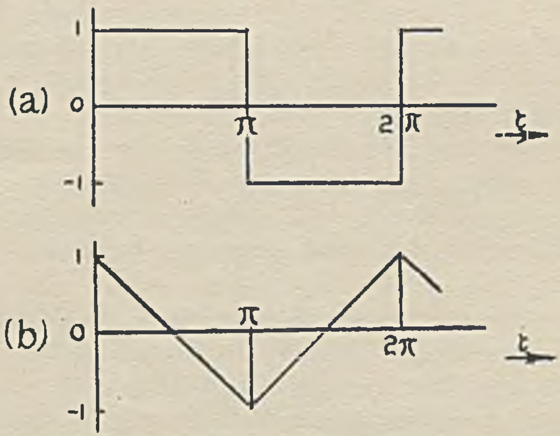


FIG. 42.—Periodic functions analysed in section 29.

$$\left.\begin{aligned} a_0 &= \frac{\omega}{2\pi}\int_0^{2\pi/\omega} f(t) \, dt && (a) \\ a_m &= \frac{\omega}{\pi}\int_0^{2\pi/\omega} f(t) \cos m\omega t \, dt && (b) \\ b_m &= \frac{\omega}{\pi}\int_0^{2\pi/\omega} f(t) \sin m\omega t \, dt && (c) \end{aligned}\right\} \qquad . \qquad (29.2)$$

As an example, the function graphed in Fig. 42a will be analysed in this manner.

$$\left.\begin{aligned} f(t) &= 1 \text{ for } 0 < t < \pi \\ f(t) &= -1 \text{ for } \pi < t < 2\pi \end{aligned}\right\} \quad . \quad . \quad (29.3)$$

Here $\omega = 1$. By (29.2*a*),

$$a_0 = \frac{1}{2\pi}\left[\int_0^\pi (1)\,dt + \int_\pi^{2\pi}(-1)\,dt\right] = 0.$$

By (29.2*b*),

$$a_m = \frac{1}{\pi}\int_0^\pi \cos mt\,dt - \frac{1}{\pi}\int_\pi^{2\pi}\cos mt\,dt = 0.$$

By (29.2*c*),

$$\begin{aligned} b_m &= \frac{1}{\pi}\int_0^\pi \sin mt\,dt - \frac{1}{\pi}\int_\pi^{2\pi}\sin mt\,dt \\ &= \frac{1}{\pi}\left[\frac{-\cos mt}{m}\right]_0^\pi - \frac{1}{\pi}\left[\frac{-\cos mt}{m}\right]_\pi^{2\pi} \\ &= \frac{2}{m\pi}(1 - \cos m\pi). \end{aligned}$$

If m is even, $\cos m\pi = 1$ and $b_m = 0$

If m is odd, $\cos m\pi = -1$ and $b_m = \frac{4}{m\pi}$.

The Fourier series representing the function of Fig. 42*a* is therefore :

$$f(t) = \frac{4}{\pi}(\sin t + \frac{1}{3}\sin 3t + \frac{1}{5}\sin 5t \;.\;.\;.) \quad . \quad (29.4)$$

This result agrees with the previous synthesis (Section 28, Fig. 41).

The Fourier analysis of a discontinuous function gives results which are accurate except at the discontinuities, where the Fourier function " jumps " too far ; this is known as Gibbs' phenomenon, an account of which is given by Karman and Biot (reference 9 in the Bibliography).

A slightly more complicated function is shown in Fig. 42*b*. By a suitable choice of the point $t = 0$ the function can be made either symmetrical or skew-symmetrical about $t = 0$, so that the series can be made to consist of cosine terms only or of sine terms only. Let the time-origin be as shown, so that the function is :

$$\left.\begin{aligned} f(t) &= 1 - \frac{2t}{\pi} && \text{for } 0 < t < \pi \\ f(t) &= -1 + \frac{2}{\pi}(t - \pi) = -3 + \frac{2t}{\pi} && \pi < t < 2\pi \end{aligned}\right\} \quad (29.5)$$

Dividing the integration into two stages, as before, and using the results :

$$\int t.\cos mt\,dt = t\frac{\sin mt}{m} - \int\frac{\sin mt}{m}\,dt$$
$$= t\frac{\sin mt}{m} + \frac{\cos mt}{m^2}$$

and

$$\int t.\sin mt\,dt = -t\frac{\cos mt}{m} - \int\frac{-\cos mt}{m}\,dt$$
$$= -t\frac{\cos mt}{m} + \frac{\sin mt}{m^2}$$

the Fourier series is obtained as :

$$f(t) = \frac{8}{\pi^2}(\cos t + \frac{1}{3^2}\cos 3t + \frac{1}{5^2}\cos 5t\ \ldots) \quad . \quad (29.6)$$

When it is required to carry out a Fourier analysis of a function which cannot be readily expressed in an analytic form such as (29.3) or (29.5), the integration is performed numerically as a summation. The ordinates, or values of the function, at a number of time-values equally spaced over the cycle are multiplied by the corresponding values of the appropriate cosine or sine function, and the products summed over the cycle. The results of this process are sufficiently accurate so long as the number of ordinates is greater than twice the index-number of the highest harmonic present. Thus a 48-ordinate analysis will give true results so long as there are no harmonics higher than the 23rd present. The numerical work is easy but tedious ; in order to reduce it to a minimum, various schemes of computation have been evolved, the most convenient one being that due to Runge, a full description of which is given by Den Hartog (reference 10 in the Bibliography at the back of the book).

A particular example of a complex wave with only two components is afforded by the phenomenon of *beating*. The effect occurs when two sine-waves of slightly different frequencies are added. Let the two functions be $a.\sin \omega t$ and $b.\sin(\omega+\Delta\omega)t$, where $\Delta\omega$ is small compared with ω. The sum of the two functions can be expressed in the form

$$(a+b.\cos\Delta\omega t)\sin\omega t + b.\sin\Delta\omega t.\cos\omega t,$$

or $r\sin(\omega t+\phi)$, where $r^2 = (a+b.\cos\Delta w t)^2 + (b.\sin\Delta\omega t)^2$

$$= a^2+b^2+2ab.\cos\Delta\omega t.$$

The amplitude r varies between the limits $(a+b)$ and $(a-b)$, the frequency of variation being $\Delta\omega/2\pi$, which is the difference between the frequencies of the two components. As an example of this phenomenon, the function $\sin 6t+2.\sin 5t$ is plotted in Fig. 43.

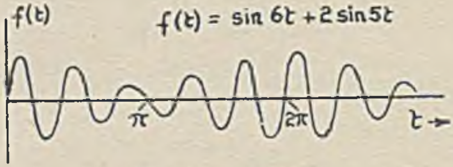


Fig. 43.—Beating.

30. Vibration under non-harmonic forces.

A sinusoidal force is termed a *harmonic* force. Periodic forces which are not sinusoidal, but which can however be expressed as a Fourier series of harmonic forces, are termed *non-harmonic*, and they are of very frequent occurrence in engineering. In the internal-combustion engine, for example, the gas-force acting on the piston in each cylinder is a periodic function with many harmonics present; the frequency of the fundamental component is the frequency of firing in each cylinder, i.e. half the number of revolutions per minute of the crankshaft in the case of four-stroke engines. These gas-forces are transmitted to the crank-shaft through connecting-rods of finite length, and the non-linear relation between the force on the piston and the resulting torque exerted at the crank-throw introduces further harmonics. The resultant torque in the crankshaft is therefore a Fourier series, the constant term (a_0) in which represents the steady torque available for the power-output of the engine, and the sine and cosine terms in which induce torsional vibration of the crankshaft and its associated inertias. As the frequency of the fundamental component in C.P.M. is $\frac{1}{2}\times$ R.P.M. of the crankshaft, there will in general be harmonic torque components at

$$(\tfrac{1}{2},\ 1,\ 1\tfrac{1}{2},\ 2,\ 2\tfrac{1}{2},\ 3,\ \ldots \text{ etc.}) \times \text{engine R.P.M.}$$

Many of these harmonics are practically balanced-out by suitable design of a multi-cylinder engine, but all the harmonics are present in a single-cylinder engine. Further information on this important subject may be found in textbooks on torsional vibration; the reader is referred to that by Dr. Ker Wilson (reference 11 in the Bibliography).

Suppose the mass in Fig. 9 to be acted upon by a complex force $P = P_1 \sin pt + P_2 \sin qt$, where p and q are not equal. Using the method of section 9,

$$[D^2+2\gamma D+\omega^2]x = \frac{P_1}{m}\sin pt+\frac{P_2}{m}\sin qt$$

and the displacement x is found to be of the form:

$$x = b_1 \sin (pt - \psi_1)+b_2 \sin (qt - \psi_2) \qquad (30.1)$$

where the quantities b and ψ are expressions similar to those in (9.3), subscript 1 indicating the frequency $p/2\pi$ and subscript 2 indicating the frequency $q/2\pi$. The motion of the mass is therefore the sum of two sine functions whose frequencies are the same as those of the two components of the exciting force.

In general, if the exciting force is a Fourier series in the form (28.3), i.e.

$$F = F_0+\sum_{n=1}^{\infty} r_n \sin n(pt+\phi_n) \quad . \quad . \quad (30.2)$$

then the displacement of the mass is given by

$$x = a_0+\sum_{n=1}^{\infty} a_n \sin n(pt+\psi_n) \quad . \quad . \quad (30.3a)$$

where the amplitude coefficients a_n are given in terms of the modulus of the dynamic stiffness $|z_n|$ at the frequency $np/2\pi$ as

$$a_n = \frac{r_n}{|z_n|} \quad . \quad . \quad . \quad (30.3b)$$

(see Chapter IV, end of Section 22).

The phase-angles ψ_n can easily be calculated, but are generally of far less importance than the amplitudes.

An important consideration is the work done by the force (30.2) on the displacement (30.3). Consider any particular harmonic of the displacement-function (30.3a), say the mth harmonic $x_m = a_m \sin . m(pt+\psi_m)$. The work done is $\int F . dx_m$; but $dx_m = a_m p . \cos m(pt+\psi_m)dt$, and the work done during a cycle of the fundamental component is

$$W_m = \int_0^{2\pi/p} a_m\left[F_0+\sum_{n=1}^{\infty} r_n \sin n(pt+\phi_n)\right]p . \cos m(pt+\psi_m)dt \quad (30.4)$$

and equations (29.1) show that all the terms on the right-hand side of (30.4) integrate to zero except that due to the mth harmonic in the force input. For any particular harmonic in the displacement function, the input energy that is required to make good the loss due to dissipation is therefore derived exclusively from the corresponding harmonic in the force. This result is of great practical importance, for it means that if one harmonic component of a periodic force input is varied, only the corresponding harmonic in the displacement-function will be altered.

EXERCISES VI

1. Determine the Fourier series for a function $f(t)$ which is such that

$$f(t) = a \sin kt \quad \text{for} \quad 0 < t < \frac{\pi}{k}$$

$$f(t) = 0 \quad \text{for} \quad \frac{\pi}{k} < t < 2\pi$$

k being integral.

2. Determine the Fourier series for a function $f(t)$ which is such that

$$f(t) = 1 \quad \text{for} \quad -t_1 < t < t_1$$
$$f(t) = 0 \quad \text{for} \quad t_1 < t < 2\pi - t_1$$

t_1 being less than π.

3. Develop a method of Fourier analysis by numerical integration with six ordinates, to evaluate the constants r and ψ in the function:

$$f(t) = a_0 + r_1 \sin(\omega t + \psi_1) + r_2 \sin 2(\omega t + \psi_2).$$

APPENDIX I

SOME PURE MATHEMATICS

31. Exponential function.

THE number e is defined as the limit of the expression $\left(1+\frac{1}{n}\right)^n$ as n tends to an infinite value. Thus

$$e = \underset{n \to \infty}{Lt} . \left(1+\frac{1}{n}\right)^n .$$

Expanding by the Binomial Theorem :

$$e = \underset{n \to \infty}{Lt} . \left\{1+\frac{n}{n}+\frac{n(n-1)}{n^2 \, |\underline{2}}+\frac{n(n-1)(n-2)}{n^3 \, |\underline{3}}+ \text{etc.}\right\}$$

i.e. $$e = 1+1+\frac{1}{|\underline{2}}+\frac{1}{|\underline{3}}+\frac{1}{|\underline{4}}+ \text{etc.} \quad . \quad . \quad (31.1)$$

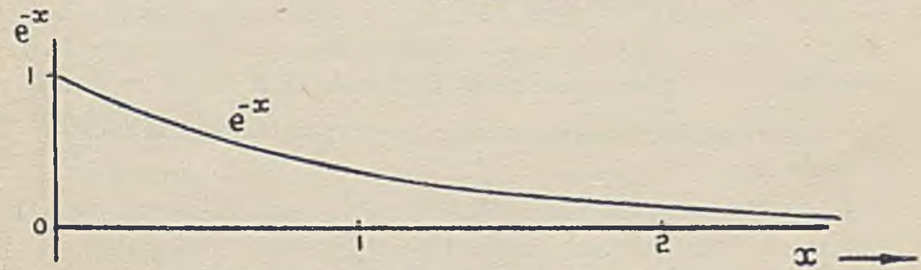


FIG. 44.—Decreasing exponential function e^{-x}.

The function e^x is similarly defined :

$$e^x = \left[\underset{n \to \infty}{Lt.} \left(1+\frac{1}{n}\right)^n\right]^x = \underset{n \to \infty}{Lt.} \left(1+\frac{1}{n}\right)^{nx}$$

and the result is obtained :

$$e^x = 1+x+\frac{x^2}{|\underline{2}}+\frac{x^3}{|\underline{3}}+\frac{x^4}{|\underline{4}}+ \text{etc.} \quad . \quad . \quad (31.2a)$$

Putting $-x$ for x,

$$e^{-x} = 1 - x+\frac{x^2}{|\underline{2}}-\frac{x^3}{|\underline{3}}+\frac{x^4}{|\underline{4}}- \text{etc.*} \quad . \quad (31.2b)$$

* A graph of e^{-x} is given in Fig. 44.

It is shown in textbooks on analysis that these series are convergent for all values of x. The function e^x is termed the *exponential* function and obeys the normal laws of indices.

Thus $$e^x . e^y = e^{x+y}, \qquad (e^x)^y = e^{xy}, \text{ etc.}$$

The exponential function is such that the series form can be differentiated term by term (see Whittaker and Watson, reference 12 in the Bibliography). Performing the differentiation, it is seen that the resulting function is identical with the original, i.e.

$$\frac{d}{dx}(e^x) = e^x \qquad . \quad . \quad . \quad (31.3)$$

32. Sine and cosine series.

Maclaurin's formula for the expansion of a function as a power series is:

$$f(x) = f(0) + xf'(0) + \frac{x^2}{\underline{|2}} f''(0) + \text{ etc.} . \qquad . \quad (32.1)$$

where dashes denote differentiation with respect to x and $f(0)$ is the value of $f(x)$ when $x = 0$, etc.

If $$f(x) = \sin x, \qquad f(0) = 0$$
$$f'(x) = \cos x, \qquad f'(0) = 1, \text{ etc.}$$

In this manner the function $\sin x$ can be expanded by Maclaurin's formula as:

$$\sin x = x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} - \frac{x^7}{\underline{|7}} + \text{ etc.} \quad . \quad . \quad (32.2a)$$

and similarly,

$$\cos x = 1 - \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}} - \frac{x^6}{\underline{|6}} + \text{ etc.} \quad . \quad (32.2b)$$

33. Exponential form of sine and cosine functions.

Let i stand for $\sqrt{-1}$; then

$$i^2 = -1, \qquad i^3 = -i, \qquad i^4 = 1, \qquad i^5 = i, \text{ etc.}$$

Thus $$e^{ix} = \left[1 - \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}} - \text{ etc.}\right] + i\left[x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} - \text{ etc.}\right]$$

i.e. $$e^{ix} = \cos x + i . \sin x \quad . \quad . \quad . \quad (33.1a)$$

Putting $-x$ for x,

$$e^{-ix} = \cos x - i . \sin x \quad . \quad . \quad (33.1b)$$

and hence

$$\left.\begin{aligned} \cos x &= \frac{e^{ix}+e^{-ix}}{2} \quad (a) \\ \sin x &= \frac{e^{ix}-e^{-ix}}{2i} \quad (b) \end{aligned}\right\} \quad . \quad . \quad (33.2)$$

From (33.2) it can be shown that $\cos^2 x + \sin^2 x = 1$. If the co-ordinates of a point in plane Cartesians are $x = r.\cos\theta$, $y = r.\sin\theta$, the point lies on the circle $x^2+y^2 = r^2$. The sine and cosine functions are therefore termed " circular functions."

34. Hyperbolic functions.

The functions $\cosh x$ and $\sinh x$ are defined :

$$\left.\begin{aligned} \cosh x &= \frac{e^{x}+e^{-x}}{2} \quad (a) \\ \sinh x &= \frac{e^{x}-e^{-x}}{2} \quad (b) \end{aligned}\right\} \quad . \quad . \quad (34.1)$$

and it can easily be shown that $\cosh^2 x - \sinh^2 x = 1$. If the co-ordinates of a point in plane Cartesians are $x = r\cosh\theta$, $y = r\sinh\theta$, the point lies on the hyperbola $x^2 - y^2 = r^2$. The functions cosh and sinh are therefore termed " hyperbolic functions."

35. Vectors.

A scalar quantity is one having magnitude but no direction ; examples are : numbers, temperatures, lengths, etc. A vector quantity is one having direction as well as magnitude ; displacements, velocities, accelerations and forces are all vectors. A scalar can be represented by the distance between two points ;

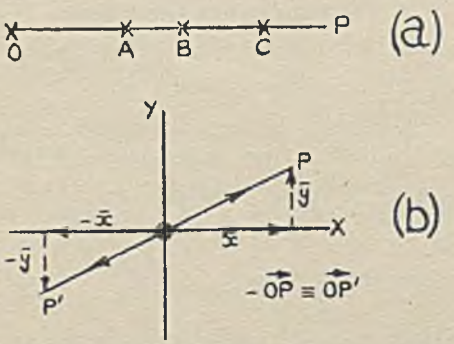


FIG. 45.—(a) Representation of scalar quantities on a straight line; (b) Representation of vector quantities on a plane.

if one point O is fixed, a set of scalars can be represented by the distances OA, OB, OC, etc., from O to points A, B, C, etc., on a straight line OP (Fig. 45*a*). A vector can be represented in magnitude and direction by the line joining two points. If one point O is fixed, a set of vectors in a plane XOY can be represented by the lines OP, OQ, etc., joining O to points P, Q, etc., in the plane XOY. Let the co-ordinates of the point P be (x, y) (Fig. 45*b*) and let N be the foot of the perpendicular from P to OX. A displacement from O to P is equivalent to a displacement from O to N together with a displacement from N to P; if $\bar{x}$ represents a displacement of magnitude x in the direction OX, and $\bar{y}$ represents a displacement of magnitude y in the direction OY, then

$$\overrightarrow{OP} \equiv \bar{x}+\bar{y} \quad . \quad . \quad . \quad (35.1)$$

As the point P is determined by the co-ordinates (x, y) the vector OP can be denoted by the *set* (x, y).

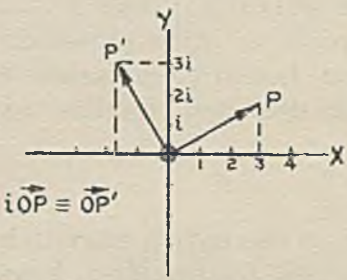


FIG. 46.—i as a versor-operator.

It can be seen from Fig. 45*b* that the vector $-(x, y)$ is in the same straight line POP' as the vector (x, y) but is in the opposite direction. Thus multiplication by -1 is equivalent to a rotation through two right-angles or π radians. By (33.1*a*),

$$e^{i\pi} = \cos \pi + i . \sin \pi = -1 \quad . \quad . \quad (35.2a)$$

Hence rotation of a vector through an angle π radians is effected by multiplying it by $e^{i\pi}$. By convention, rotation of a vector through one right-angle or $\pi/2$ radians is said to be effected by multiplying it by $e^{i\pi/2}$. Now, from (33.1*a*),

$$e^{i\pi/2} = \cos\frac{\pi}{2} + i . \sin\frac{\pi}{2} = i \quad . \quad . \quad (35.2b)$$

and hence rotation through one right-angle is by convention supposed to be effected by multiplication by i. That this convention is consistent can be easily shown. Real scalar quantities being measured along OX, Fig. 46, the quantities i, $2i$, $3i$, etc., are to be measured off along OY. Thus

$$\overrightarrow{OP} = x+iy.$$

Let $\quad x'+iy' = i(\overrightarrow{OP}) = i(x+iy) = (-y+ix) = \overrightarrow{OP'}.$

It is clear that OP' in Fig. 46 is at right-angles to OP.

In order to distinguish between the *number* i $(=\sqrt{-1})$ and the *operator* i (turning the associated vector through an angle $\pi/2$ in the positive sense), the symbol j is frequently used for the versor-operator. (A *versor*-operator is a turning-operator.)

If the polar co-ordinates of P are (r, θ), then

$$x+iy = r\cos\theta + i.r\sin\theta = re^{i\theta} \quad . \quad . \quad (35.3)$$

This exponential form is sometimes useful in calculations. The following alternative forms are of frequent occurrence:

$$\left.\begin{aligned} 1 &= \cos 0 + i\sin 0 = e^0 \\ -1 &= \cos\pi + i\sin\pi = e^{i\pi} \\ i &= \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = e^{i\pi/2} \\ -i &= \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2} = e^{i3\pi/2} \end{aligned}\right\} \quad . \quad . \quad (35.4)$$

Thus, for example, $i(x+iy) = ire^{i\theta} = re^{i(\theta+\pi/2)}$, a form of statement which shows clearly the turning effect of the operator i.

36. Integral formulae.

The equations (29.1) are easily derived.

$$\int_0^{2\pi/\omega} \sin m\omega t\,dt = -\frac{1}{m}\Big[\cos m\omega t\Big]_0^{2\pi/\omega} = 0 \quad . \quad (29.1e)$$

By ordinary trigonometry,

$$\sin m\omega t.\sin n\omega t = \tfrac{1}{2}\cos(m-n)\omega t - \tfrac{1}{2}\cos(m+n)\omega t.$$

If m and n are unequal,

$$\int_0^{2\pi/\omega} \sin m\omega t.\sin n\omega t\,dt = \tfrac{1}{2}\int_0^{2\pi/\omega}\cos(m-n)\omega t\,dt - \tfrac{1}{2}\int_0^{2\pi/\omega}\cos(m+n)\omega t\,dt$$

$$= \frac{1}{2(m-n)}\Big[\sin(m-n)\omega t\Big]_0^{2\pi/\omega} - \frac{1}{2(m+n)}\Big[\sin(m+n)\omega t\Big]_0^{2\pi/\omega}.$$

As both m and n are integers, both these terms are zero and the whole integral is zero. . . . . (29.1a)

If, however, $m = n$, then as

$$\sin^2 m\omega t = \tfrac{1}{2} - \tfrac{1}{2}\cos 2m\omega t,$$

$$\int_0^{2\pi/\omega} \sin^2 m\omega t \, dt = \tfrac{1}{2}\int_0^{2\pi/\omega} (1)\, dt - \tfrac{1}{2}\int_0^{2\pi/\omega} \cos 2m\omega t \, dt$$

$$= \frac{\pi}{\omega}. \qquad \qquad (29.1g)$$

The other equations (29.1) are obtained in a similar manner.

APPENDIX II

SOME APPLIED MATHEMATICS

37. Newton's second Law of Motion, units.

Newton's second Law of Motion states that the rate of change of momentum of a body is proportional to the impressed force. If the force in a certain direction is F, and the displacement in this direction is x, the resulting equation of motion is

$$F = \frac{d}{dt}(m\dot{x}) \quad . \quad . \quad . \quad (37.1a)$$

where m is the mass of the body. If the mass remains constant,

$$F = m\ddot{x} \quad . \quad . \quad . \quad . \quad (37.1b)$$

Use of the equation (37.1*b*) leads to two fundamentally different systems of units. In the scientific system the basic quantities are mass, length, and time (lbs., ft., secs.); (37.1*b*) then serves to *define* unit force as being that force which produces in unit mass a unit acceleration. This unit force is called the *poundal*, and since the weight of unit mass produces in the mass an acceleration g, 1 lb.wt. $= g$ poundals. In this equation g has the value 32·2 *ft.*/sec.^{2} for practical purposes. In the engineers' system, however, the basic quantities are force, length, and time (lbs. wt., ins., secs.) (37.1*b*) then serves to *define* unit mass as being that mass in which unit force produces unit acceleration. This unit mass is called the *slug*, and 1 slug $= g$ lbs. In this relation, g has the value 386 ins./sec.^{2}, as the unit of length is the inch.

> (In aerodynamics a " slug " equal to 32·2 lbs. is used, the basic units being lbs., ft., secs., as it is more convenient to make measurements in feet when such quantities as Young's modulus—usually given in lbs./in.^{2}—are not involved.)

Either system, the scientific or the engineers', may be used, but care must be taken to express all the quantities consistently in one system.

When a body rotates about an axis under the action of a torque T about that axis, the *angular* acceleration $\ddot{\theta}$ is proportional to the torque in the same way as a linear acceleration is proportional to the applied force. If the constant of proportionality is I, as yet undetermined, then

$$T = I\ddot{\theta} \qquad . \quad . \quad . \quad . \quad (37.2)$$

Consider a particle A of mass m at the end of a light arm OA of length r, rotating about O under the action of a force F at A normal to OA (Fig. 47). The velocity of A is $r\dot{\theta}$ normal to OA,

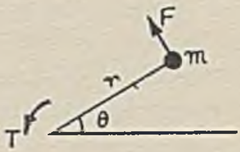


FIG. 47.—Derivation of $I = mr^2$ for a particle.

and the acceleration in the direction of the force F is therefore $r\ddot{\theta}$ as r is constant. By Newton's Second Law,

$$F = mr\ddot{\theta}.$$

The moment of the force F about O is $Fr = T$, and hence

$$T = mr^2\ddot{\theta},$$

and by comparison with (37.2) it is seen that

$$I = mr^2 \quad . \quad . \quad . \quad . \quad (37.3a)$$

This quantity I is termed the " moment of inertia " of the particle about O.

A continuous rigid body can be regarded as composed of elemental masses δm_i. The moment of inertia of the body about any axis is then defined as

$$I = \Sigma r_i^2 \delta m_i \qquad . \quad . \quad . \quad (37.3b)$$

where r_i is the distance from the mass δm_i to the axis of rotation, and the summation is extended over the whole body. The symbol J is frequently used for moments of inertia of solid bodies about an axis of rotation ; if the axis of rotation is also an axis of symmetry J is also termed the " polar inertia."

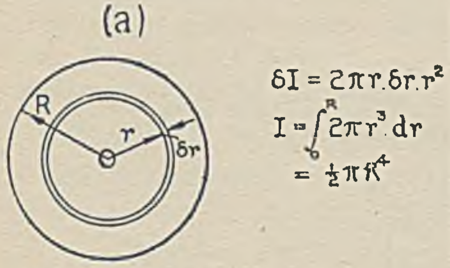
The dimensions of moments of inertia are those of mass × length^2 ; the scientific unit is therefore lbs.ft.^{2}, while the engineers' unit is slugs.ins.^{2} ; in terms of the basic units a slug is lbs.ins.$^{-1}$ secs.^{2}, being a force divided by an acceleration, and the engineers' unit of inertia is therefore lbs.ins.secs.^{2}

A somewhat analogous quantity is the " second moment of area " ; this is for a cross-sectional area what the moment of inertia is for a body; i.e. the second moment of area of any area about any axis is defined as

$$I = \Sigma r_i^2 \delta a_i \quad . \quad . \quad . \quad . \quad (37.4)$$

where δa_i is an elemental area at a distance r_i from the axis, and the summation is extended over the whole area. The units of I are clearly ins.^{4} or ft.^{4}

The polar second moment of area of a circular cross-section is found to be $\frac{1}{2}\pi R^4$, where R is the radius; this polar moment is with respect to an axis normal to the section and passing through the centre (Fig. 48a).



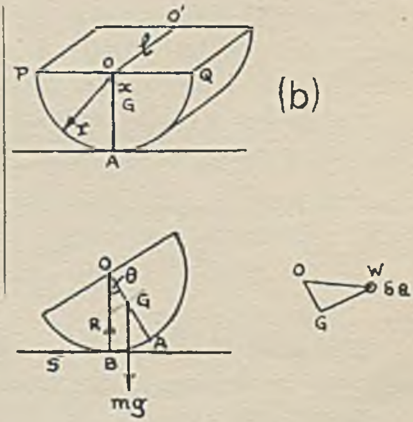


FIG. 48.—(a) Determination of polar second moment of area for circular cross-section; (b) Rolling oscillation of a solid half-cylinder.

Example.

To illustrate the method of application of Newton's Law in a complex problem, the natural frequency of small rolling oscillations of a solid half-cylinder (Fig. 48b) will be determined by ordinary dynamical methods, although the solution is most readily obtained by the energy method. Let the radius be r, the length l and the density ρ. The centre of area of the cross-section is on the line of symmetry OA, at a distance $OG = 4r/3\pi$ from the geometric centre O. (This result follows easily from one of the theorems of Pappus, which states that the volume of a solid of revolution is equal to the product of the generating area and the length of the path of the centre of this area; in this case the half-circle is rotated about the diameter PQ and generates a sphere. The generating area is $\frac{1}{2}\pi r^2$, and the length of the path of G is $2\pi.(OG)$; thus $\frac{1}{2}\pi r^2 . 2\pi.(OG) = \frac{4}{3}\pi r^3$, whence

$$OG = \frac{4r}{3\pi}.)$$

The moment of inertia J_0 of the body about the axis OO' is $\rho l I_0$, where I_0 is the second moment of area of the cross-section about O and is therefore $\frac{1}{4}\pi r^4$. Now $I_0 = \Sigma(OW)^2\delta a$, where W is the position of a typical elemental area δa. The second moment of area about G is easily shown to be $I_0 - (OG)^2.A$, A being the area of the cross-section $= \Sigma\delta a$. For

$$I_0 = \Sigma\{OG^2 + GW^2 - 2.OG.GW.\cos \underline{|OGW}\}\delta a$$

$$= OG^2.A + I_G - 2.OG\Sigma GW.\cos \underline{|OGW}.\delta a$$

where $I_G = \Sigma GW^2.\delta a$ = second moment of area about G. The third term vanishes as G is the centre of area of the cross-section, and thus $I_G = I_0 - (OG)^2.A$. (This is a perfectly general result, which holds true for any area.)

Let $OG = x$; the moment of inertia J_G of the body about an axis parallel to OO' and passing through G is given by:

$$J_G = I_G\rho l = (\tfrac{1}{4}\pi r^4 - x^2\tfrac{1}{2}\pi r^2)\rho l$$

$$= \pi r^2\rho l\left(\frac{r^2}{4} - \frac{8r^2}{9\pi^2}\right) = \pi r^4\rho l(\tfrac{1}{4} - 8/9\pi^2).$$

The solid is rolled to one side of the static position, without slipping on the supporting horizontal plane. Let B be the point of contact when the angular displacement is θ, and let the resolved reactions at B be R and S as shown. The vertical distance moves by G is $x(1 - \cos\theta) = \frac{1}{2}x\theta^2$, as θ is small; the horizontal displacement of G is $r.\tan\theta - x.\sin\theta = (r - x)\theta$. Applying Newton's Law for three displacements, i.e. for vertical and horizontal displacements of G and for the rotation about G,

$$\left.\begin{aligned}
S &= m\frac{d^2}{dt^2}[(r-x)\theta] = m(r-x)\ddot{\theta} && (a)\\
R - mg &= m\frac{d^2}{dt^2}[\tfrac{1}{2}x\theta^2] = m\frac{d}{dt}[x\theta\dot{\theta}] &&\\
&= mx\dot{\theta}^2 + mx\theta.\ddot{\theta} && (b)\\
\text{and}\quad Rx.\sin\theta + S(r - x.\cos\theta) &= -J_G\ddot{\theta} &&\\
\text{i.e.}\quad Rx\theta + S(r-x) &= -J_G\ddot{\theta} && (c)
\end{aligned}\right\} \quad (37.5)$$

as θ is small.

Elimination of R and S leads to the equation:

$$m[gx\theta + x^2\theta\dot{\theta}^2 + x^2\theta^2\ddot{\theta} + (r-x)^2\ddot{\theta}] = -J_G\ddot{\theta} \quad . \quad (37.6)$$

Putting $\theta = a \sin \omega t$, and considering the instant when $\cos \omega t = 0$, $\sin \omega t = 1$, the equation is obtained :

$$m(gxa - x^2a^3\omega^2) - m(r - x)^2a\omega^2 = +J_Ga\omega^2,$$

and omitting terms containing powers of a higher than the first,

$$mgx - m(r - x)^2\omega^2 = J_G\omega^2$$

whence
$$\omega^2 = \frac{8g/r}{9\pi - 16} \quad . \quad . \quad . \quad (37.7)$$

by substituting for m and J_G; the natural frequency $\omega/2\pi$ is thus obtained.

38. Bending of Beams.

Fig. 49 represents a short section of a beam which is deformed by bending ; circular deformation is assumed so that all plane cross-sections remain plane and undistorted and all pass through a line normal to the plane of the longitudinal section Fig. 49a,

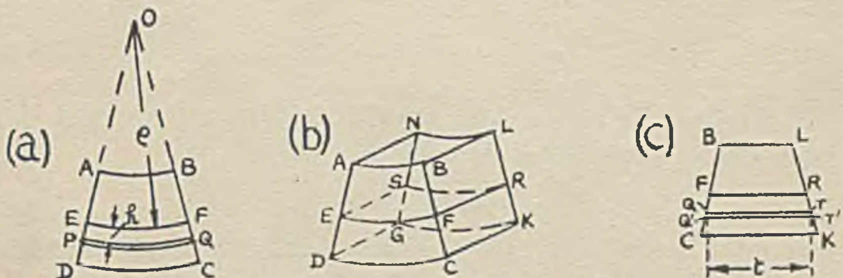


Fig. 49.—Bending of beams : (a) longitudinal section ; (b) perspective view (c) transverse section.

the intersection of this line with the plane of Fig. 49a being O. Considering the beam to be composed of fibres originally parallel to AB and DC in the undeformed condition, these fibres are deformed into circular arcs, and those lying in the plane of the diagram are concentric at O. The fibres nearer O are compressed and those further from O are extended. There is a plane $EFRS$ such that all the fibres in it are merely bent, without extension or compression. This plane is termed the " neutral plane," and the line of intersection of the neutral plane with a cross-section is termed the " neutral axis."

Let E = Young's modulus for the material
ρ = radius of curvature of deformed neutral plane = OE
h = distance of slice PQ from neutral plane
δh = thickness of slice
e = extension per unit length of fibre PQ
l = undeformed length of $PQ = EF$
M = bending moment at $BCKL$
s = stress on area $QQ'T'T$
t = width of area $QQ'T'T = QT$

By the properties of similar figures,

$$\frac{\text{arc } PQ}{\text{arc } EF} = \frac{OP}{OE} = \frac{\rho+h}{\rho} = 1+\frac{h}{\rho} \quad . \quad . \quad (38.1)$$

but

$$\frac{\text{arc } PQ}{\text{arc } EF} = \frac{l(1+e)}{l} = 1+e \quad . \quad . \quad . \quad (38.2)$$

Hence

$$e = \frac{h}{\rho}.$$

Now

$$e = \text{tensile strain} = \frac{\text{tensile stress}}{E} = \frac{s}{E}$$

hence

$$s = \frac{Eh}{\rho} \quad . \quad . \quad . \quad . \quad (38.3)$$

The tensile force on the area $QQ'T'T$ is s(area) $= st\delta h$ and this force has a moment δM about FR, where

$$\delta M = sth\delta h.$$

The total moment for the whole area $BCKL$ is the bending moment M; thus

$$M = \Sigma sth\delta h$$

or, by (38.3),

$$M = \frac{E}{\rho}\Sigma th^2\delta h \quad . \quad . \quad . \quad . \quad (38.4)$$

or

$$M = \frac{EI}{\rho} \text{ where } I = \Sigma th^2\delta h \quad . \quad . \quad (38.5)$$

(see Section 37)

The resultant tensile force on the area $BCKL$ is zero, hence

$$\Sigma st\delta h = 0$$

i.e.

$$\frac{E}{\rho}\Sigma ht\delta h = 0.$$

This last equation expresses the fact that the neutral axis FR passes through the centre of area of the cross-section $BCKL$.

If the distances along the undeformed beam are denoted by x, and y is the deflection of the beam from its undeformed position, the radius of curvature is given by

$$\frac{1}{\rho} = \frac{\dfrac{d^2y}{dx^2}}{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}.$$

For small displacements for which $\dfrac{dy}{dx}$ is small, this formula reduces to $\dfrac{1}{\rho} = \dfrac{d^2y}{dx^2}$, and (38.5) becomes

$$M = EI\frac{d^2y}{dx^2} \qquad (38.6)$$

The deflection curve of a loaded beam is easily obtained. Consider, for example, a light loaded cantilever, the loading being concentrated at the free end (Fig. 50). At a distance x

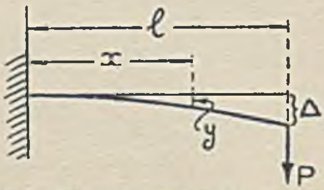


Fig. 50.—Loaded light cantilever.

from the built-in end the bending moment is given by

$$M = P(l-x)$$

i.e.

$$EI\frac{d^2y}{dx^2} = P(l-x)$$

hence

$$EI\frac{dy}{dx} = Plx - \frac{Px^2}{2} + A$$

$$EIy = \frac{Plx^2}{2} - \frac{Px^3}{6} + Ax + B.$$

At the built-in end ($x = 0$) the displacement and slope are zero; hence $A = B = 0$,

$$EIy = \frac{Plx^2}{2} - \frac{Px^3}{6}.$$

If Δ is the end-deflection at $x = l$, it is found that

$$\Delta = \frac{Pl^3}{3EI}.$$

The linear stiffness of the beam to an end-load P is thus

$$\frac{P}{\Delta} = \frac{l}{3EI} \quad . \quad . \quad . \quad . \quad (38.7)$$

39. Torsion of a circular shaft.

The shear modulus G of a material is defined thus: let $ABCD$ be a small cube of the material (Fig. 51*a*), and while the face

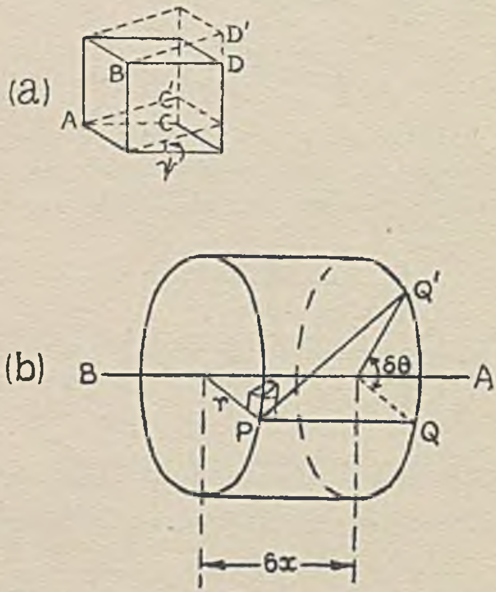


Fig. 51.—(*a*) Shearing deformation of small cube. (*b*) Torsion of circular shaft.

AB is fixed, let the face CD be subjected to a shearing stress q, so that the cube deforms into a rhomboid prism $ABC'D'$. Let the angle CAC' be ψ, then

$$G = \frac{q}{\psi} \quad . \quad . \quad . \quad . \quad (39.1)$$

Now consider a short section δx of a uniform circular shaft (Fig. 51*b*) of radius r. Under torsion the generator PQ becomes PQ'; let the angle QOQ' be $\delta\theta$, O being the centre of the cross-section through Q and Q'. Imagining the small cube of Fig. 51*a* to be at P, the shear strain ψ is given by

$$\tan \psi = r\frac{\delta\theta}{\delta x}$$

i.e. $$\psi = r\frac{\delta\theta}{\delta x}, \text{ as } \psi \text{ is small.}$$

Let the area of that side of the cube which is included in the cross-section of the shaft at P be δa, on which the shear stress is $q = G\psi$ (by 39.1). The shear force on δa is then $q\delta a$, and by the above formula this is equal to

$$Gr\frac{\delta\theta}{\delta x}\delta a\,;$$

the moment of this force about the axis AB is

$$\delta T = Gr^2\frac{\delta\theta}{\delta x}\delta a,$$

and the total torque transmitted by the section is

$$T = \Sigma Gr^2\delta a\frac{\delta\theta}{\delta x} = GI\frac{\delta\theta}{\delta x} \quad . \quad . \quad . \quad (39.2)$$

where I is the second moment of area of cross-section (see Section 37), as the summation is extended over the whole cross-section.

APPENDIX III

FREQUENCY EQUATIONS AND NUMERICAL SOLUTIONS

40. Determinants.

The equations of motion for a complicated vibrating system contain a number of variables representing linear and angular displacements; elimination of these variables in order to obtain a frequency equation similar to (16.5) is often best performed by means of a determinant. The mathematical properties of determinants are fully treated in standard intermediate text-books on Algebra; sufficient will be restated here to enable the method to be explained.

The set of simultaneous linear equations in three variables

$$\left.\begin{aligned} a_1x+b_1y+c_1z &= 0 \\ a_2x+b_2y+c_2z &= 0 \\ a_3x+b_3y+c_3z &= 0 \end{aligned}\right\} \qquad (40.1a)$$

has a solution $x = y = z = 0$, and may have another solution in which at least one of the variables x, y, z is not zero. Suppose z is not zero; the set can then be written:

$$\left.\begin{aligned} a_1X+b_1Y+c_1 &= 0 \\ a_2X+b_2Y+c_2 &= 0 \\ a_3X+b_3Y+c_3 &= 0 \\ \text{where} \quad X = x/z, \quad Y &= y/z \end{aligned}\right\} \qquad (40.1b)$$

Solving the second and third equations of this transformed set for X, Y,

$$\frac{X}{b_2c_3 - c_2b_3} = \frac{-Y}{a_2c_3 - c_2a_3} = \frac{1}{a_2b_3 - b_2a_3}$$

i.e.
$$\frac{X}{\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}} = \frac{-Y}{\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}} = \frac{1}{\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}}$$

where
$$\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} \equiv b_2c_3 - c_2b_3, \text{ etc.} \qquad (40.2)$$

(40.2) *defines* a second-order *determinant*, represented symbolically by the left-hand side of the equation. Substitution of the values of X and Y in the first equation of (40.1b) leads to

$$a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} = 0 \qquad (40.3a)$$

and this last equation is expressed in the form :

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0 \qquad (40.3b)$$

This arrangement of letters in three rows and three columns is called a third-order determinant, and is defined to be identical with the left-hand side of (40.3a). (40.3b) expresses the condition that the set of equations (40.1a) should have a solution in x, y, z other than $x = y = z = 0$.

The rule for expanding a three-row determinant, as indicated in (40.3a), can be stated as follows: "Associate with each element a_1, b_1, c_1 of the *top* row the signs $+$, $-$, $+$ alternately, always giving the left-hand element the positive sign; multiply each such element, with its associated sign, by the determinant formed by omitting the row and column containing the element, and sum the products so obtained." Thus the determinant formed by omitting the row and column containing b_1 is $\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$ and the sign to be associated with b_1 is negative; similarly for a_1 and c_1.

It is convenient to express a determinant concisely by writing down the elements of the diagonal which runs from the top left corner to the bottom right corner; thus the determinant in (40.3b) is represented by $|a_1 \; b_2 \; c_3|$ and the three-row determinant is defined:

$$\left.\begin{aligned} |a_1 \; b_2 \; c_3| &\equiv a_1|b_2 \; c_3| - b_1|a_2 \; c_3| + c_1|a_2 \; b_3| \\ \text{where} \quad |b_2 \; c_3| &\equiv b_2c_3 - c_2b_3, \text{ etc.} \end{aligned}\right\} \quad . \qquad (40.4)$$

The following rules of manipulation are proved in Algebra textbooks, and may easily be demonstrated in particular cases by performing them on numerical determinants:

(i) The value of a determinant is unaltered if columns and rows are interchanged. Thus

$$\begin{vmatrix} 7 & 5 & 3 \\ 0 & 1 & 0 \\ 5 & 8 & 2 \end{vmatrix} = 7(2 - 0) - 5(0) + 3(0 - 5) = -1$$

and
$$\begin{vmatrix} 7 & 0 & 5 \\ 5 & 1 & 8 \\ 3 & 0 & 2 \end{vmatrix} = 7(2-0) - 0 + 5(0-3) = -1.$$

(ii) If two rows or columns are interchanged the value of the determinant is multiplied by -1. Thus

$$\begin{vmatrix} 0 & 1 & 0 \\ 7 & 5 & 3 \\ 5 & 8 & 2 \end{vmatrix} = -1(14-15) = 1.$$

(iii) If two rows or two columns are identical, the determinant has zero value; this follows from (ii). Thus

$$\begin{vmatrix} 0 & 1 & 0 \\ 7 & 5 & 3 \\ 7 & 5 & 3 \end{vmatrix} = 1(21-21) = 0.$$

(iv) If all the elements of a row or column are multiplied by the same constant, the value of the determinant is multiplied by that constant; thus

$$\begin{vmatrix} 14 & 10 & 6 \\ 0 & 1 & 0 \\ 15 & 24 & 6 \end{vmatrix} = 2 \times \begin{vmatrix} 7 & 5 & 3 \\ 0 & 1 & 0 \\ 15 & 24 & 6 \end{vmatrix} = 2 \times 3 \times \begin{vmatrix} 7 & 5 & 3 \\ 0 & 1 & 0 \\ 5 & 8 & 2 \end{vmatrix} = -6$$

(v) If the elements of a row or column are each expressed as the sum of two numbers, the determinant can be expressed as the sum of two determinants, the other rows or columns remaining unaltered. Thus

$$\begin{vmatrix} 7 & 5 & 3 \\ 0 & 1 & 0 \\ 5 & 8 & 2 \end{vmatrix} = \begin{vmatrix} 5+2 & 8-3 & 2+1 \\ 0 & 1 & 0 \\ 5 & 8 & 2 \end{vmatrix}$$

$$= \begin{vmatrix} 5 & 8 & 2 \\ 0 & 1 & 0 \\ 5 & 8 & 2 \end{vmatrix} + \begin{vmatrix} 2 & -3 & 1 \\ 0 & 1 & 0 \\ 5 & 8 & 2 \end{vmatrix} = - \begin{vmatrix} 0 & 1 & 0 \\ 2 & -3 & 1 \\ 5 & 8 & 2 \end{vmatrix} = -1$$

(vi) If to the elements of any row, or column, are added the same multiples of the corresponding elements in any other row, or column, the value of the determinant is unaltered. Thus

$$\begin{vmatrix} 17 & 21 & 7 \\ 0 & 1 & 0 \\ 5 & 8 & 2 \end{vmatrix} = \begin{vmatrix} 7+10 & 5+16 & 3+4 \\ 0 & 1 & 0 \\ 5 & 8 & 2 \end{vmatrix}$$

$$= \begin{vmatrix} 7 & 5 & 3 \\ 0 & 1 & 0 \\ 5 & 8 & 2 \end{vmatrix} + 2 \begin{vmatrix} 5 & 8 & 2 \\ 0 & 1 & 0 \\ 5 & 8 & 2 \end{vmatrix} = \begin{vmatrix} 7 & 5 & 3 \\ 0 & 1 & 0 \\ 5 & 8 & 2 \end{vmatrix} = -1$$

These rules are very useful for determining the value of a numerical determinant. For example, let

$$\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 6 & 5 \\ 7 & 8 & 9 \end{vmatrix}$$

Subtracting the top row from the bottom row (rule vi)

$$\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 6 & 5 \\ 6 & 6 & 6 \end{vmatrix} = 6 \begin{vmatrix} 1 & 2 & 3 \\ 4 & 6 & 5 \\ 1 & 1 & 1 \end{vmatrix} = -6 \begin{vmatrix} 1 & 1 & 1 \\ 4 & 6 & 5 \\ 1 & 2 & 3 \end{vmatrix}$$

(rule iv) (rule ii)

Subtracting the left-hand column from the other two, (rule vi),

$$\Delta = -6 \begin{vmatrix} 1 & 0 & 0 \\ 4 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix} = -6(4-1) = -18.$$

All the results stated in this section are capable of extension to determinants of higher order. Thus, for example, the definition of a fourth-order determinant is :

$$|a_1b_2c_3d_4| = a_1|b_2c_3d_4| - b_1|a_2c_3d_4| + c_1|a_2b_3d_4| - d_1|a_2b_3c_4| \quad . \quad (40.5)$$

and the equation $|a_1b_2c_3d_4| = 0$ is the condition that the set of four simultaneous linear equations

$$\begin{cases} a_1x + b_1y + c_1z + d_1w = 0 \\ a_2x + b_2y + c_2z + d_2w = 0 \\ a_3x + b_3y + c_3z + d_3w = 0 \\ a_4x + b_4y + c_4z + d_4w = 0 \end{cases}$$

has a solution in x, y, z, w other than $x = y = z = w = 0$.

41. Frequency equations.

Suppose that the equations of free undamped motion of a vibrating body having three degrees of freedom are :

$$\left.\begin{aligned} a_1X + b_1Y + c_1z + m\ddot{X} &= 0 \\ a_2X + b_2Y + c_2z + m\ddot{Y} &= 0 \\ a_3X + b_3Y + c_3z + J\ddot{z} &= 0 \end{aligned}\right\} \quad . \quad . \quad (41.1a)$$

where m is the mass of the body, J its moment of inertia with respect to the rotation z, and X and Y represent linear displacements. For sinusoidal motion $X = A \sin \omega t$, etc., $\ddot{X} = -\omega^2 X$, etc., and the equations of motion can be written as

$$\left.\begin{aligned} (a_1 - m\omega^2)X + b_1Y + c_1z &= 0 \\ a_2X + (b_2 - m\omega^2)Y + c_2z &= 0 \\ a_3X + b_3Y + (c_3 - J\omega^2)z &= 0 \end{aligned}\right\} \quad . \quad . \quad (41.1b)$$

and there is a solution other than $X = Y = z = 0$ if

$$\begin{vmatrix} (a_1 - m\omega^2) & b_1 & c_1 \\ a_2 & (b_2 - m\omega^2) & c_2 \\ a_3 & b_3 & (c_3 - J\omega^2) \end{vmatrix} = 0$$

Expansion of this determinant by (40.4) gives an equation of the form

$$P\omega^6 + Q\omega^4 + R\omega^2 + S = 0 \quad . \quad . \quad (41.2)$$

where P, Q, R, S are expressions involving the constants m, J, a_1, etc.

Similarly, the equations of motion (free and undamped) of a body having six degrees of freedom (Fig. 23, Chapter III) are of the form

$$a_1X + b_1Y + c_1Z + d_1x + e_1y + f_1z + m\ddot{X} = 0,$$
$$\text{etc. etc.}$$

The resulting frequency equation analogous to (41.2) is of degree 6 in ω^2. As noted in Chapter III, section 16, however, symmetry in the spring-mounting scheme may result in certain modes of vibration being uncoupled. In a standard type of in-line aero-engine mounting, for example, if X, Y, Z represent displacements vertically, transversely and fore-and-aft respectively, and x, y, z are rotations about corresponding axes (see Fig. 23), the following pairs of displacements are uncoupled:

X and x	X and Y
Y ,, y	X ,, Z
Z ,, z	Y ,, Z
X ,, z	x ,, y
Z ,, x	y ,, z

The equations of motion then reduce to the set:

$$\left.\begin{array}{llllll} a_1X & & & +e_1y & & +m\ddot{X} = 0 \\ & b_2Y & +d_2x & & +f_2z & +m\ddot{Y} = 0 \\ & c_3Z & & +e_3y & & +m\ddot{Z} = 0 \\ & b_4Y & +d_4x & & +f_4z & +J_x\ddot{x} = 0 \\ a_5X & +c_5Z & & +e_5y & & +J_y\ddot{y} = 0 \\ & b_6Y & +d_6x & & +f_6z & +J_z\ddot{z} = 0 \end{array}\right\} \quad (41.3)$$

where J_x, J_y, J_z are the moments of inertia of the body with respect to rotations x, y, z respectively. The first, third and fifth equations contain only X, Z and y, and the second, fourth and sixth equations contain only Y, x and z; the two resulting determinantal equations are therefore:

$$\left.\begin{array}{l}\begin{vmatrix}(a_1 - m\omega^2) & 0 & e_1 \\ 0 & (c_3 - m\omega^2) & e_3 \\ a_5 & c_5 & (e_5 - J_y\omega^2)\end{vmatrix} = 0 \\ \\ \begin{vmatrix}(b_2 - m\omega^2) & d_2 & f_2 \\ b_4 & (d_4 - J_x\omega^2) & f_4 \\ b_6 & d_6 & (f_6 - J_z\omega^2)\end{vmatrix} = 0\end{array}\right\} \quad (41.4)$$

and these lead to two cubic equations in ω^2; the six natural frequencies are thus determined.

42. Numerical solutions.

Suppose that a cubic equation in ω^2 is obtained by the method of Section 41. It is required to find the three natural frequencies. Let the equation be, for example,

$$\omega^6 - 6{\cdot}51 \times 10^6\omega^4 + 13{\cdot}19 \times 10^{12}\omega^2 - 7{\cdot}99 \times 10^{18} = 0 \quad (42.1)$$

Put $$x = \omega^2 \times 10^{-6}$$

then $$x^3 - 6{\cdot}51x^2 + 13{\cdot}19x - 7{\cdot}99 = 0 \quad . \quad . \quad (42.2)$$

Let the left-hand side of (42.2) be represented by y.

$$\left.\begin{array}{lll}\text{When } x = 0, & y = & -\ 7{\cdot}99 \\ \quad\quad\ \ 1 & & -\ 0{\cdot}31 \\ \quad\quad\ \ 2 & & \ \ \ 0{\cdot}35\end{array}\right\};$$ there is therefore a root between 1 and 2. Put $x = 1 + h$, and substitute in (42.2), neglecting powers of h higher than the first. Thus:

$$1 + 3h - 6{\cdot}51(1 + 2h) + 13{\cdot}19(1 + h) - 7{\cdot}99 = 0$$

i.e. $$-\ 0{\cdot}31 + 3{\cdot}17h = 0, \quad h = 0{\cdot}1 \text{ approximately,}$$

and $x = 1 + h = 1{\cdot}1$ approximately. Now put $x = 1{\cdot}1 + k$ and solve for k in a similar manner; it is found that k is $0{\cdot}01$ approximately, so that a nearer approximation to x is $1{\cdot}11$. The process is one of applying a series of decreasing corrections to an approximate value of the root found by inspection or graphical methods; the next correction in the series $0{\cdot}1$, $0{\cdot}01$, . . . is found to be $-\ 0{\cdot}002$, so that the value $1{\cdot}11$ is correct to three significant figures. The equation is now divided through by $(x - 1{\cdot}11)$, i.e. the left-hand side is factorised with $(x - 1{\cdot}11)$ as one factor:

$$(x - 1{\cdot}11)(x^2 - 5{\cdot}40x + 7{\cdot}20) = 0,$$

and the other two roots are found by solving the quadratic equation

$$x^2 - 5{\cdot}40x + 7{\cdot}20 = (x - 3{\cdot}0)(x - 2{\cdot}4) = 0.$$

Hence the three roots in x are : 1·11, 2·4, 3·0, and the corresponding values of ω^2 are :

$$\omega^2 = 1{\cdot}11 \times 10^6,\ 2{\cdot}4 \times 10^6,\ 3{\cdot}0 \times 10^6$$

and $$F = \omega/2\pi = 167,\ 247,\ 275 \text{ c.p.s.} \qquad (42.3)$$

The method is often stated in another form. Let $x = a_1$ be an approximate solution to the equation $f(x) = 0$. Let the function be represented in the neighbourhood of $x = a_1$ by the curve of Fig. 52 ; $Q_1R = f(a_1)$ and if the approximate root a_1

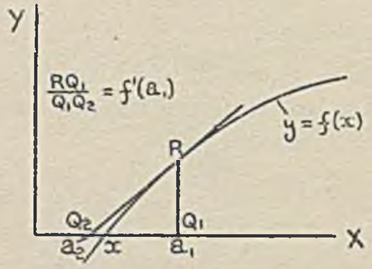


Fig. 52.—Approximation to solution of a numerical equation.

is sufficiently close to the true value x, a closer approximation is found by drawing the tangent RQ_2 at R and finding the point Q_2 in which the tangent cuts the x-axis. Now $Q_1R = Q_1Q_2f'(a_1)$ where $f'(a_1)$ means the value of $\frac{d}{dx}f(x)$ when $x = a_1$. Thus

$$a_2 = a_1 - Q_1Q_2 = a_1 - \frac{f(a_1)}{f'(a_1)} \qquad (42.4)$$

Putting a_2 for a_1, a still closer approximation $x = a_3$ is found in a similar manner. As an illustration, consider this method applied to the equation (42.2).

$$f(x) = x^3 - 6{\cdot}51x^2 + 13{\cdot}19x - 7{\cdot}99$$
$$f'(x) = 3x^2 - 13{\cdot}02x + 13{\cdot}19$$
$$a_1 = 1,\ f(a_1) = -0{\cdot}31 \text{ and } f'(a_1) = 3{\cdot}17$$

Hence $$a_2 = 1 - \frac{-0{\cdot}31}{3{\cdot}17} = 1{\cdot}1 \text{ approx.; as before.}$$

It is evident that the two methods are equivalent ; the second form of the procedure is more easily applied to higher order equations. It is necessary in any case to obtain as a first approximation a value which is not too widely different from the true solution, particularly if the curvature of the graph $y = f(x)$ is large in the neighbourhood of the root.

43. Normalising of numerical equations.

It frequently happens in the analysis of an engineering vibration problem that a large number of fairly simple equations (quadratics and cubics) require to be solved. In such circumstances much time can be saved by *normalising* the equations and making use of special tables. The process, which was known to the Babylonians *circa* 2000 B.C. for certain types of cubic equations, will be illustrated here by applying it to the general quadratic equation,

$$ax^2+bx+c = 0 \quad . \quad . \quad . \quad (43.1)$$

Let $x = yd$, so that $ay^2d^2+byd+c = 0$,

i.e. $$y^2+\frac{b}{ad}y+\frac{c}{ad^2} = 0,$$

and choose d so that the coefficient of y is unity, i.e. $d = b/a$. Then

$$\left.\begin{aligned} y^2+y+R &= 0 \\ \text{where} \quad R &= ac/b^2 \end{aligned}\right\} \quad . \quad . \quad . \quad (43.2)$$

The roots in y are

$$\left.\begin{aligned} y &= -\tfrac{1}{2}(1 \pm \sqrt{1-4R}) \\ \text{and} \quad x &= \frac{b}{a}y \end{aligned}\right\} \quad . \quad . \quad (43.3)$$

Special tables can be constructed giving the two values of y corresponding to any value of R; for real roots it is only necessary to consider values of $R < 0{\cdot}25$, and the range of negative values included will depend to some extent on the nature of the problem which gives rise to the equation. If the roots in x are essentially positive, as will be the case if x is proportional to ω^2, $\omega/2\pi$ being a natural frequency, then R must be positive and is restricted to the range $0 \leqslant R \leqslant 0{\cdot}25$. Tables giving the values of y corresponding to this range of values of R are in course of preparation.

The method of normalising equations is of no great use unless a large number of equations of the same type have to be solved, but the time saved in such cases is considerable, apart from the fact that the results are easily tabulated together with the incidental working.

MISCELLANEOUS EXERCISES

(Numerical examples on the work of Chapters I–IV.)

All frequencies should be given in C.P.M.

CHAPTER I

Find the natural frequency of a system similar to Fig. 1, in which the constants are:

1. $k = 4{,}000$ lbs./in., $m = 2$ slugs.

2. $k = 3{,}000{,}000$ poundals/ft., $m = 150$ lbs.

3. $k = 7{,}500$ lbs./in., $m = 200$ lbs.

4. $k = 3{,}750$ lbs./in., $m = 100$ lbs.

Find the natural frequency of a system similar to Fig. 4, in which the constants are:

5. $J_1 = J_2 = 2$, $C = 3{,}500{,}000$.

6. $J_1 = 3$, $J_2 = 4$, $C = 1{,}700{,}000$.

7. $J_1 = 8$, $J_2 = 6$, $C = 3{,}400{,}000$.

(inertias in lbs.ins.secs.^{2}, stiffnesses in lbs.ins./radian.)

8. Determine the period of oscillation of a compound pendulum similar to Fig. 3c, in which the constants are:

$$m = 10 \text{ lbs.}, \quad I_A = 3{\cdot}73 \text{ lbs.ins.secs.}^2, \quad h = 8 \text{ inches.}$$

CHAPTER II

Find the resonant frequency, and the natural frequency if the damping is neglected, of a system similar to Fig. 9, in which the constants are:

9. $k = 200$ lbs./in., $m = 2$ slugs, $c = 40\%$ critical.

10. $k = 3{,}500$ lbs./in., $m = 1{\cdot}2$ slugs, $c = 12$ lbs.ins.$^{-1}$-secs.

11. $k = 40{,}000$ lbs./in., $m = 3$ slugs, $c = 20\%$ critical.

12. Find the resonant frequency of a torsional system equivalent to Fig. 9, in which the inertia is 10 lbs.ins.secs.^{2}, the shaft stiffness is 4,500,000 lbs.ins./radian, and the damping force is 45% of the critical value.

13. Find the force transmitted to the base in a system similar to that of Section 12, Example I, in which the mass is 5,000 lbs., the springs have a total stiffness of 12,500 lbs./in., the damping force is 25% of the critical value, and the applied force has a maximum value of 750 lbs. and a frequency of 1,500 C.P.M.

CHAPTER III

Find the natural frequencies of the system of Fig. 16, in which the constants are:

14. $k_1 = k_2 = 1{,}000$ lbs./in., $m_1 = m_2 = 200$ lbs.

15. $k_1 = 300$, $k_2 = 450$ lbs./in., $m_1 = 20$, $m_2 = 30$ lbs.

16. $k_1 = 600$, $k_2 = 900$ lbs./in., $m_1 = 40$, $m_2 = 60$ lbs.

Determine the ratio a_2/a_1 at the natural frequencies in nos. **14–16.**

Find the natural coupled and uncoupled frequencies of the system of Fig. 21, in which the constants are:

17. $m = 1{\cdot}2$ slugs, $J = 40$ lbs.ins.secs.^{2}, $k_1 = 2{,}000$, $k_2 = 3{,}000$ lbs./in., $d_1 = d_2 = 5$ ins.

18. $m = 1{\cdot}5$ slugs, $J = 200$ lbs.ins.secs.^{2}, $k_1 = 3{,}000$, $k_2 = 4{,}500$ lbs./in., $d_1 = 6$, $d_2 = 4$ ins.

19. $m = 2{\cdot}25$ slugs, $J = 300$ lbs.ins.secs.^{2}, $k_1 = k_2 = 4{,}000$ lbs./in., $d_1 = 6$, $d_2 = 4$ ins.

20. $m = 2{\cdot}475$ slugs, $J = 330$ lbs.ins.secs.^{2}, $k_1 = k_2 = 8{,}800$ lbs./in., $d_1 = 6$, $d_2 = 4$ ins.

Chapter IV

21. Find the lowest natural frequency of a seven-mass torsional system in which:

$$J_1 = J_2 = J_3 = J_4 = J_5 = J_6 = 0{\cdot}39,\ J_7 = 11{\cdot}1 \text{ lbs.ins.secs.}^2, \text{ and}$$
$$C_1 = C_2 = C_3 = C_4 = C_5 = 7{\cdot}25,\ C_6 = 4{\cdot}55 \text{ lbs.ins./micro-radian.}$$

Determine the swinging form at this frequency.

22. Find the lowest natural frequency of a geared torsional system which is as follows:

(inertias in lbs.ins.secs.^{2}, stiffnesses in lbs.ins./micro-radian.)

$J_1 = 0{\cdot}505$	$C_1 = 8{\cdot}788$
$J_2 = 0{\cdot}521$	$C_2 = 8{\cdot}793$
$J_3 = 0{\cdot}501$	$C_3 = 8{\cdot}793$
$J_4 = 0{\cdot}501$	$C_4 = 8{\cdot}793$
$J_5 = 0{\cdot}521$	$C_5 = 8{\cdot}788$
$J_6 = 0{\cdot}556$	$C_6 = 2{\cdot}751$
$J_7 = 0{\cdot}121$	

(J_7 geared to J_8 with a $0{\cdot}477 : 1$ *reduction* gear)

$J_8 = 1{\cdot}027$	$C_7 = 11{\cdot}182$
$J_9 = 3{\cdot}060$	

Determine the swinging form at this frequency.

ANSWERS TO EXERCISES

Chapter I

1. $x = C_1 e^{\alpha t} + C_2 e^{\beta t}$, where $\binom{\alpha}{\beta} = \frac{1}{2}(-P \pm \sqrt{P^2 - 4Q})$.

2. $(1 + J_1/J_2)A_1 = \sqrt{R^2 + S^2/\omega^2}$, $\tan\phi = \omega R/S$, $A = \dot{\Theta} - S/(1 + J_1/J_2)$, $B = \Theta_1 - R/(1 + J_1/J_2)$, where $R = \Theta_1 - \Theta_2$, $S = \dot{\Theta}_1 - \dot{\Theta}_2$, and $\omega^2 = C(J_1 + J_2)/J_1 J_2$. For no vibration, $\omega^2 R^2 + S^2 = 0$; for no steady rotation, $J_1\Theta_1 + J_2\Theta_2 = 0 = J_1\dot{\Theta}_1 + J_2\dot{\Theta}_2$.

3. $y = C_1 e^{kx} + C_2 e^{-kx} + C_3 e^{ikx} + C_4 e^{-ikx}$.

Chapter II

1. For type (b), $\dot{x}_0/x_0 < -\gamma$, where x_0 and $\dot{x}_0$ are initial displacement and velocity.

Chapter III

2. $a_2 = \dfrac{k_2}{k_2 - m_2 p^2} a_1$, a_1 being given by (15.3).

3. $k_1 = 0$.

Chapter IV

1. First six natural frequencies are approximately 1,900, 4,100, 6,100, 8,620, 9,200, and 9,550 C.P.M.; seventh is outside range of Table III (Note: $Z_{1A} = Z_{2B} = 0$ is a solution of (22.6) giving three of the frequencies.)

Chapter V

1. (a) $y = a$, $\dfrac{\partial^2 y}{\partial x^2} = 0$ at each end; (b) $y_0 = a - Y \sin \dfrac{\pi x}{l}$; a is found to be $2Y/\pi$ by determining condition for zero total inertia force ($m\ddot{y}$) for the whole beam; (c) $(\text{K.E.})_{max.} = 0{\cdot}048\ \omega^2 Y^2 M_B$, $(\text{P.E.})_{max.} = EI\pi^4 Y^2/4l^3$; (d) $\omega^2 = 519\ EI/M_B l^3$.

Chapter VI

1. $a_0 = a/k\pi$, $a_k = 0$, $b_k = a/2k$; and for $m \neq k$,

$$a_m = \frac{-ak}{\pi(m^2 - k^2)}(1 + \cos m\pi/k);\quad b_m = \frac{-ak}{\pi(m^2 - k^2)} \sin m\pi/k$$

2. $a_0 = t_1/\pi$; $a_n = (2/n\pi)\sin nt_1$; $b_n = 0$.

Miscellaneous Set

(Frequencies in C.P.M.)

1. 427. 2. 1,350. 3. 1,150. 4. 1,150. 5. 17,900.
6. 9,500. 7. 9,500. 8. 1·36 secs.

9.	Resonant,	78·6	natural,	95·5
10.	,,	511	,,	516

11. Resonant, 1,070 natural, 1,110.

12. 4,940. **13.** 550 lbs.

14. 259, 678 ; ratios : 1·615, − 0·618.

15. 510, 1,260 ; „ 1·98, − 0·498.

16. As for **15.**

17. Coupled, 503, 642 ; uncoupled, 534, 615.

18. „ 286, 675 ; „ same as coupled.

19. „ 206, 585 ; „ 251, 570.

20. „ 292, 828 ; „ 355, 807.

21. 9,940 ;

A_1	A_2	A_3	A_4	A_5	A_6	A_7
1	0·942	0·829	0·667	0·467	0·239	− 0·145

22. 11,990 ;

A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9
1	0·909	0·734	0·492	0·207	− 0·099	− 1·042	− 0·497	− 0·875

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