ON CONNECTIONS BETWEEN INTEGRALS, INVERSE LAPLACE TRANSFORM AND MULTIPARAMETERS LIMITS OF SOME RATIONAL FUNCTIONS

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The scope of the paper is the presentation of two methods of generating certain multi-parametric limits. Both methods involve a parametric representation of rational functions f(x) performed in the procedure of calculating integral of f(x), and inverse Laplace transform of f(x).

Keywords: *rotational functions, inverse Laplace transformation*

1. INTRODUCTION

The paper is divided into four sections. In Section 2, which has an introductory character, we discuss integrals of the form $\int p(x)/(x^2+1)^n\,dx$. Basic reduction formulae of a recurrent nature are given, and, in specific cases, ready - made one's. In Section 3 the method of calculating these integrals is discussed, by applying a multi-parametric representation of the integrand [1-3]. By means of such technique it is easy to derive the values of the integral, yet in the form of a not that useful multi-parametric limit. In reality, it is possible, after standard calculations of the values of this integral and comparison of the two results, to obtain multi-parametric limits that are, by no means, trivial, which is demonstrated by a number of examples. A similar technical concept was used in the last Section for calculating Laplace inverse transform of the same rational functions, leading, consequently, to generating of a new family of multi-parametric limits.

2. ON CERTAIN INTEGRALS OF RATIONAL TYPE

Let us put:

$$I_n(p(x)) := \int \frac{p(x)}{(x^2+1)^n} dx \quad \text{and} \quad J_n(p(x)) := \int_0^x \frac{p(y)}{(y^2+1)^n} dy,$$

for every $n \in \mathbb{N}$ and $p \in C^m_{\mathbb{R}}(I)$, $I \subset \mathbb{R}$ be a nondegenerate interval, $m \in \mathbb{N}$.

Lemma 2.1. *The following formulae hold*

$$I_{n+1}(x^2 p(x)) = I_n(p(x)) - I_{n+1}(p(x)), \tag{1}$$

$$2n I_{n+1}(x^2 p(x)) = I_n(p(x) + x p'(x)) - \frac{x p(x)}{(x^2+1)^n}, \tag{2}$$

$$2n I_{n+1}(p(x)) = I_n((2n-1)p(x) - x p'(x)) + \frac{x p(x)}{(x^2+1)^n}, \tag{3}$$

$$I_n(p(x)) - I_{n-1}(p(x)) = \frac{x p(x)}{2(n-1)(x^2+1)^{n-1}} - \frac{1}{2(n-1)} I_{n-1}(p(x) + x p'(x)), \tag{4}$$

and

$$I_n(p(x)) - I_1(p(x)) = \sum_{k=1}^{n-1} \frac{x p(x)}{2k(x^2+1)^k} - \sum_{k=1}^{n-1} \frac{1}{2k} I_k(p(x) + x p'(x)). \tag{5}$$

Proof: By integrating by parts we get

$$I_n(p(x)) = \int (x)' p(x)(x^2+1)^{-n} dx = \frac{x p(x)}{(x^2+1)^n} - \int \left(\frac{x p'(x)}{(x^2+1)^n} - 2n \frac{x^2 p(x)}{(x^2+1)^{n+1}} \right) dx =$$

$$= \frac{x p(x)}{(x^2+1)^n} - I_n(x p'(x)) + 2n I_{n+1}(x^2 p(x)),$$

which implies (2) and by (1) the identity (3). The identity (4) from (3) follows, while (5) after adding (4) for indices $n, n-1, n-2, \ldots, 2$ follows.

Remark 2.2. *If* $p \in \mathbb{R}[x]$ *then the immediately application of the formula (3) is not an optimal procedure to finding the integral* $I_n(p(x))$.

First either it can be doing the decomposition

$$p(x) = xq_1(x^2) + q_2(x^2)$$

and then applied the formula (2), or it can be doing (many times) the decomposition

$$p(x) = xQ_1(x^2+1) + Q_2(x^2+1)$$

and reduced the problem to finding the integrals $I_m(x)$ *and* $I_m(1)$.

Example 2.3. Let us set $J_n := J_n(1)$, $n \in \mathrm{N}$. Then, by (3) and by an easy induction argument we obtain the formula:

$$J_{n+1} = \frac{2n-1}{2n} J_n + \frac{x}{2n(x^2+1)^n} = 4^{-n}\binom{2n}{n}\left(\arctan x + x\sum_{k=1}^{n} \frac{4^{k-1}}{2k-1}\binom{2k-2}{k-1}^{-1}(x^2+1)^{-k}\right) =$$

$$= 4^{-n}\binom{2n}{n}\left(\arctan x + x\sum_{k=1}^{n} 4^{k-1}\left(\binom{2k}{2}c_{k-1}\right)^{-1}(x^2+1)^{-k}\right), \tag{6}$$

where c_n denotes the Catalan numbers, i.e, $c_n := \frac{1}{n+1}\binom{2n}{n}$, $n = 0,1,2,\dots$. We note that in [10] the following interesting trigonometric identities are generated from (6):

$$\binom{2n}{n}\sum_{k=1}^{n} 4^{k-1}\left(\binom{2k}{2}c_{k-1}\right)^{-1}\cos^{2k-1}(\varphi) = \sum_{k=1}^{n}\binom{2n}{n+k}\frac{\sin(2k\varphi)}{k\sin(\varphi)} \tag{7}$$

and

$$\binom{2n}{n}\sum_{k=0}^{n-1}\binom{2k}{k}^{-1}(2\cos(\varphi))^{2k} = \sum_{k=1}^{n}\binom{2n}{n+k}\frac{\cos(\varphi)\sin(2k\varphi) - 2k\cos(2k\varphi)\sin(\varphi)}{k\sin^3(\varphi)}. \tag{8}$$

Example 2.4. Now, let us set:

$$A_{m,n} := I_n(\arctan^m(x))$$

$m = 0,1,2,\dots$, $n = 1,2,3,\dots$. Then, we have:

$$A_{m,n} = A_{m,n-1} + I_n\left(x^2\arctan^m(x)\right) \tag{9}$$

and, by (2), we get

$$I_n\left(x^2\arctan^m(x)\right) = \frac{1}{2(n-1)}\left(I_{n-1}\left(\arctan^m(x) + \frac{2mx^2}{1+x^2}\arctan^{m-1}(x)\right) - \frac{x\arctan^m(x)}{(x^2+1)^n}\right) =$$

$$= \frac{1}{2(n-1)}A_{m,n-1} + \frac{m}{n-1}I_n\left(x^2\arctan^{m-1}(x)\right) - \frac{x\arctan^m(x)}{2(n-1)(x^2+1)^n},$$

which implies the following formula

$$A_{m,n} = \frac{2n-3}{2(n-1)} A_{m,n-1} - \frac{m}{n-1} I_n\left(x^2 \arctan^{m-1}(x)\right) + \frac{x \arctan^m(x)}{2(n-1)(x^2+1)^n}.$$

Using recursively of the equation (4) it may be convenient to apply the following auxiliary equations.

Lemma 2.5. *Let* $p \in C^n_{\mathrm{R}}(I)$. *Let us put:*

$$P_0(x) := p(x),$$

$$P_{k+1}(x) := P_k(x) + x\frac{d}{dx} P_k(x), \qquad k = 0,1,\ldots,n-1. \tag{10}$$

Then the following explicite formula hold (the proof by induction follows):

$$P_k(x) = \sum_{l=0}^{k} a_{l,k} x^l p^{(l)}(x), \tag{11}$$

where $a_{0,k} := 1$ *and*

$$a_{l,k} := \frac{1}{l!}\left((l+1)^k - \binom{l}{1} l^k + \binom{l}{2}(l-1)^k - \binom{l}{3}(l-2)^k + \ldots + (-1)^l \right)$$

for every $l = 1,\ldots,k$ *and* $k = 1,2,\ldots,n$. *For example:*

$$a_{1,k} = 2^k - 1, \quad a_{2,k} = \tfrac{1}{2}(3^k + 1) - 2^k, \quad a_{3,k} = \tfrac{1}{6}(4^k - 1) - \tfrac{1}{2}(3^k - 2^k),$$

for every $k \in \mathrm{N}$. *Moreover, we note that:*

$$a_{l,k+1} = (l+1)a_{l,k} + a_{l-1,k},$$

for every $l = 1,2,\ldots,k$.

3. CONNECTIONS BETWEEN INTEGRALS AND MULTIPARAMETER LIMITS

The theoretical grounds for our deliberations are given by the following lemma:

Lemma 3.1. *Let* $f(x,\varepsilon) \in C([0,a])$, $\varepsilon \in \mathrm{R}$. *Suppose that* $f(x,\varepsilon) \to f(x,1)$ *when* $\varepsilon \to 1$ *in* $C([0,a])$. *Let us put*

$$F(x,\varepsilon) := \int_{\cdot}^{x} f(z,\varepsilon)dz, \qquad x \in [\cdot, a], \qquad \varepsilon \in \mathrm{R}.$$

Then $F(x,\varepsilon) \to F(x,1)$ *when* $\varepsilon \to 1$ *in* $C^1([0,a])$.

Remark 3.2. *It can be easily proved the many parameters version of this lemma, which will be applied permanently in the exercises below.*

Remark 3.3. *In this paper all kinds of the convergence of the respective functions will be mean either as uniformly convergence on some interval of* $\mathbb{R}$ *or as pointwise convergence.*

Corollary 3.4. *We have*

$$J_2(ax+b) = \lim_{\varepsilon,\delta\to 1}\int_0^x \frac{ay+b}{(y^2+\varepsilon^2)(y^2+\delta^2)}dy =$$

$$= \lim_{\substack{\varepsilon,\delta\to 1\\ \varepsilon\neq\delta}} \frac{1}{\delta^2-\varepsilon^2}\int_0^x \left(\frac{ay+b}{y^2+\varepsilon^2} - \frac{ay+b}{y^2+\delta^2}\right)dy =$$

$$= \lim_{\substack{\varepsilon,\delta\to 1\\ \varepsilon\neq\delta}} \frac{1}{\delta^2-\varepsilon^2}\left(\frac{a}{2}\ln\left(\frac{\delta^2(x^2+\varepsilon^2)}{\varepsilon^2(x^2+\delta^2)}\right) + \frac{b}{\varepsilon}\arctan\frac{x}{\varepsilon} - \frac{b}{\delta}\arctan\frac{x}{\delta}\right).$$

Hence, by (3), we get

$$\frac{bx-a}{2(x^2+1)} + \frac{b}{2}\arctan x =$$

$$= \lim_{\substack{\varepsilon,\delta\to 1\\ \varepsilon\neq\delta}} \frac{1}{\delta^2-\varepsilon^2}\left(\frac{a}{2}\ln\left(\frac{x^2+\varepsilon^2}{x^2+\delta^2}\right) + \frac{b}{\varepsilon}\arctan\frac{x}{\varepsilon} - \frac{b}{\delta}\arctan\frac{x}{\delta}\right) =$$

$$= \lim_{\varepsilon\to 1} \frac{1}{1-\varepsilon^2}\left(\frac{a}{2}\ln\left(\frac{x^2+\varepsilon^2}{x^2+1}\right) + \frac{b}{\varepsilon}\arctan\frac{x}{\varepsilon} - b\arctan x\right),$$

i.e.,

$$\frac{x}{x^2+1} + \arctan x = \lim_{\substack{\varepsilon,\delta\to 1\\ \varepsilon\neq\delta}} \frac{1}{\delta^2-\varepsilon^2}\left(\frac{1}{\varepsilon}\arctan\frac{x}{\varepsilon} - \frac{1}{\delta}\arctan\frac{x}{\delta}\right) =$$

$$= \lim_{\varepsilon\to 1} \frac{1}{1-\varepsilon^2}\left(\frac{1}{\varepsilon}\arctan\frac{x}{\varepsilon} - \arctan x\right),$$

since

$$\lim_{\substack{\varepsilon,\delta\to 1\\ \varepsilon\neq\delta}} \frac{1}{\delta^2-\varepsilon^2}\ln\left(\frac{x^2+\varepsilon^2}{x^2+\delta^2}\right) = \lim_{\substack{\varepsilon,\delta\to 1\\ \varepsilon\neq\delta}}\left(\frac{-1}{x^2+\delta^2}\frac{\ln\left(1+\frac{\varepsilon^2-\delta^2}{x^2+\delta^2}\right)}{\frac{\varepsilon^2-\delta^2}{x^2+\delta^2}}\right) = \frac{-1}{x^2+1}.$$

Corollary 3.5. *Let* $p\in\mathbb{R}[x]$, $\deg p \le 5$ *and let*

$$p(x) = xq_1(x^2) + q_2(x^2).$$

Then the following decomposition hold:

$$\frac{p(x)}{(x^2+\varepsilon^2)(x^2+\delta^2)(x^2+\gamma^2)} = \frac{xA_1 + A_1^*}{x^2+\varepsilon^2} + \frac{xA_2 + A_2^*}{x^2+\delta^2} + \frac{xA_3 + A_3^*}{x^2+\gamma^2}, \tag{12}$$

where ε, δ, γ *are distinct elements of* $\mathbb{R}$ *and:*

$$A_1 = \frac{q_1(-\varepsilon^2)}{(\delta^2-\varepsilon^2)(\gamma^2-\varepsilon^2)}, \quad A_2 = \frac{q_1(-\delta^2)}{(\varepsilon^2-\delta^2)(\gamma^2-\delta^2)},$$

$$A_3 = \frac{q_1(-\gamma^2)}{(\varepsilon^2-\gamma^2)(\delta^2-\gamma^2)},$$

$$A_1^* = \frac{q_2(-\varepsilon^2)}{(\delta^2-\varepsilon^2)(\gamma^2-\varepsilon^2)}, \quad A_2^* = \frac{q_2(-\delta^2)}{(\varepsilon^2-\delta^2)(\gamma^2-\delta^2)},$$

$$A_3^* = \frac{q_2(-\gamma^2)}{(\varepsilon^2-\gamma^2)(\delta^2-\gamma^2)}.$$

Hence, we get:

$$J_3(p(x)) = \lim_{\varepsilon,\delta,\gamma\to 1}\Big(\frac{1}{2}\big(A_1\ln(x^2+\varepsilon^2) + A_2\ln(x^2+\delta^2) + A_3\ln(x^2+\gamma^2)\big) +$$

$$+ A_1^*\frac{1}{\varepsilon}\arctan\frac{x}{\varepsilon} + A_2^*\frac{1}{\delta}\arctan\frac{x}{\delta} + A_3^*\frac{1}{\gamma}\arctan\frac{x}{\gamma}\Big). \tag{13}$$

But from (3) it can be generated the formula

$$J_3(ax+b) = \frac{3}{8}b\arctan(x) + \frac{a}{4} + \frac{3bx}{8(x^2+1)} + \frac{bx-a}{4(x^2+1)^2},$$

which by (13) implies

$$\frac{-1}{2(x^2+1)^2} = \lim_{\substack{\varepsilon,\delta,\gamma\to 1\\ \varepsilon\neq\delta\ \ \varepsilon\neq\gamma\ \ \delta\neq\gamma}}\left(\frac{\ln(x^2+\varepsilon^2)}{(\delta^2-\varepsilon^2)(\delta^2-\gamma^2)} + \frac{\ln(x^2+\delta^2)}{(\varepsilon^2-\delta^2)(\gamma^2-\delta^2)} + \frac{\ln(x^2+\gamma^2)}{(\varepsilon^2-\gamma^2)(\delta^2-\gamma^2)}\right), \tag{14}$$

and

$$\frac{3}{8}\arctan(x) + \frac{3x}{8(x^2+1)} + \frac{x}{4(x^2+1)^2} =$$

$$= \lim_{\substack{\varepsilon,\delta,\gamma\to 1\\ \varepsilon\neq\delta\ \ \varepsilon\neq\gamma\ \ \delta\neq\gamma}}\left(\frac{\arctan\frac{x}{\varepsilon}}{\varepsilon(\delta^2-\varepsilon^2)(\gamma^2-\varepsilon^2)} + \frac{\arctan\frac{x}{\delta}}{\delta(\varepsilon^2-\delta^2)(\gamma^2-\delta^2)} + \frac{\arctan\frac{x}{\gamma}}{\gamma(\varepsilon^2-\gamma^2)(\delta^2-\gamma^2)}\right), \tag{15}$$

since ($n \in \mathbb{Z}$):

$$\lim_{\substack{\varepsilon,\delta,\gamma\to 1\\ \varepsilon\neq\delta\ \ \varepsilon\neq\gamma\ \ \delta\neq\gamma}}\left(\frac{\varepsilon^n\ln\varepsilon}{(\delta^2-\varepsilon^2)(\gamma^2-\varepsilon^2)}+\frac{\delta^n\ln\delta}{(\varepsilon^2-\delta^2)(\gamma^2-\delta^2)}+\frac{\gamma^n\ln\gamma}{(\varepsilon^2-\gamma^2)(\delta^2-\gamma^2)}\right)=$$

(for $\gamma=\varepsilon+x$, $\delta=\varepsilon-x$, $x\to 0$, $\varepsilon\to 1$, by the fact that these limits exists)

$$=\lim_{\substack{x\to 0\\ \varepsilon\to 1}}\frac{(2n(n-2)\ln\varepsilon+4(n-1))\varepsilon^{n-1}x^2+o(x^2)}{4\varepsilon x^2(4\varepsilon^2-x^2)}=\frac{1}{4}(n-1).$$

Remark 3.6. *We note that from (15) when* $x\to\infty$ *we obtain*

$$\lim_{\substack{\varepsilon,\delta,\gamma\to 1\\ \varepsilon\neq\delta\ \ \varepsilon\neq\gamma\ \ \delta\neq\gamma}}\left(\frac{1}{\varepsilon(\delta^2-\varepsilon^2)(\gamma^2-\varepsilon^2)}+\frac{1}{\delta(\varepsilon^2-\delta^2)(\gamma^2-\delta^2)}+\frac{1}{\gamma(\varepsilon^2-\gamma^2)(\delta^2-\gamma^2)}\right)=\frac{3}{8}.$$

Remark 3.7. *The above procedure may be used in any other case. For example, in view of function* $f(x)=(x^4+1)^{-2}$*, it provides, successively:*

$$\int\frac{dx}{(x^4+1)^2}=\frac{x}{4(x^4+1)}+\frac{3}{4}\int\frac{dx}{x^4+1},\tag{16}$$

$$\int\frac{dx}{(x^4+\varepsilon^4)(x^4+\delta^4)}=\frac{1}{\delta^4-\varepsilon^4}\int\left(\frac{1}{x^4+\varepsilon^4}-\frac{1}{x^4+\delta^4}\right)dx,\tag{17}$$

and

$$\int\frac{dx}{x^4+\varepsilon^4}\overset{[4,8,11]}{=}\int\begin{vmatrix}-\sqrt{2}\varepsilon & \sqrt{2}\varepsilon\\ \varepsilon^2 & \varepsilon^2\end{vmatrix}^{-1}\left(\frac{x-\sqrt{2}\varepsilon}{x^2-\sqrt{2}\varepsilon x+\varepsilon^2}-\frac{x+\sqrt{2}\varepsilon}{x^2+\sqrt{2}\varepsilon x+\varepsilon^2}\right)dx=$$

$$=\frac{\sqrt{2}}{4\varepsilon^3}\left(\arctan\left(\frac{\varepsilon x\sqrt{2}}{\varepsilon^2-x^2}\right)+\frac{1}{2}\ln\left|\frac{x^2+\sqrt{2}\varepsilon x+\varepsilon^2}{x^2-\sqrt{2}\varepsilon x+\varepsilon^2}\right|\right),\tag{18}$$

where we use the identity $\arctan x+\arctan y=\arctan\left(\frac{x+y}{1-xy}\right)$ *(in the case* $xy<1$*). Hence, like in the Corollary 3.5, the following limits can be generated:*

$$\lim_{\substack{\varepsilon,\delta\to 1\\ \varepsilon\neq\delta}}\frac{1}{\delta-\varepsilon}\left(\varepsilon^{-3}\arctan\left(\frac{\varepsilon x\sqrt{2}}{\varepsilon^2-x^2}\right)-\delta^{-3}\arctan\left(\frac{\delta x\sqrt{2}}{\delta^2-x^2}\right)\right)=3\arctan\frac{\sqrt{2}x}{1-x^2},\tag{19}$$

and

$$\lim_{\substack{\varepsilon,\delta\to 1\\ \varepsilon\neq\delta}} \frac{1}{\delta-\varepsilon}\left(\delta^{-3}\ln\left|\frac{x^2-\sqrt{2}\delta x+\delta^2}{x^2+\sqrt{2}\delta x+\delta^2}\right|-\varepsilon^{-3}\ln\left|\frac{x^2-\sqrt{2}\varepsilon x+\varepsilon^2}{x^2+\sqrt{2}\varepsilon x+\varepsilon^2}\right|\right)=$$

$$=\frac{4\sqrt{2}x}{x^4+1}+3\ln\left|\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}\right|. \tag{20}$$

4. INVERSE LAPLACE TRANSFORM AND MULTIPARAMETERS LIMITS

4.1 Explicit equations

In the following Lemma the explicit analytic formulae for the polynomials $P_{\rho,n}(t)$ and $Q_{\rho,n}(t)$ are given. We note that any classical Laplace transform monograph does not contain this formulae, in many manuals and monographs [5-7] only the compact description (27) may be found - see Remark 4.2 below.

Lemma 4.1. *For every $n\in\mathbb{N}$ and $\rho=1,s$, there exists polynomials $P_{\rho,n}$, $Q_{\rho,n}\in\mathbb{Z}[t]$ such that*

$$(2(n-1))!!\,L^{-1}[\frac{\rho}{(s^2+1)^n}](t)=P_{\rho,n}(t)\sin(t)+Q_{\rho,n}(t)\cos(t).$$

The following recurrent relations hold:

$$P_{1,1}(t)\equiv 1, P_{s,1}(t)\equiv 0,$$

$$P_{s,n+1}(t)=tP_{1,n}(t)$$

and

$$P_{1,n+1}(t)=-tP_{s,n}(t)+(2n-1)P_{1,n}(t),\quad \text{for}\quad n=1,2\ldots$$

$$=-t^2P_{1,n-1}(t)+(2n-1)P_{1,n}(t),\quad \text{for}\quad n=2,3\ldots. \tag{21}$$

Hence $P_{1,n}(t)$, $n\in\mathbb{N}$ are even functions, however polynomials $P_{s,n}(t)$, $n=2,3,\ldots$ are odd functions.

More precisely, by (21), it can be deduced the following explicite form of the polynomials $P_{1,n}(t)$, $n\in\mathbb{N}$:

$$P_{1,2n}(t)=(-1)^{n-1}\binom{n}{1}\frac{(2n-1)!!}{(2n-3)!!}t^{2(n-1)}+$$

$$+(-1)^{n-2}\binom{n+1}{3}\frac{(2n+1)!!}{(2n-5)!!}t^{2(n-2)}+$$

$$+(-1)^{n-3}\binom{n+2}{5}\frac{(2n+3)!!}{(2n-7)!!}t^{2(n-3)}+\ldots+(4n-3)!!, \quad (22)$$

and

$$P_{1,2n+1}(t)=(-1)^n t^{2n}+(-1)^{n-1}\binom{n+1}{2}\frac{(2n+1)!!}{(2n-3)!!}t^{2(n-1)}+$$

$$+(-1)^{n-2}\binom{n+2}{4}\frac{(2n+3)!!}{(2n-5)!!}t^{2(n-2)}+$$

$$+(-1)^{n-3}\binom{n+3}{6}\frac{(2n+5)!!}{(2n-7)!!}t^{2(n-3)}+\ldots+(4n-1)!!. \quad (23)$$

For example, we have:

$$P_{1,2}(t)=1, \qquad P_{1,3}(t)=-t^2+3, \qquad P_{1,4}(t)=-6t^2+15,$$

$$P_{1,5}(t)=t^4-45t^2+105, \qquad P_{1,6}(t)=15t^4-420t^2+945,$$

$$P_{1,7}(t)=-t^6+210t^4-4725t^2+10395.$$

The following recurrent relations hold:

$$Q_{1,1}(t)\equiv 0, Q_{s,1}(t)\equiv 1,$$

$$Q_{s,n+1}(t)=tQ_{1,n}(t)$$

and

$$Q_{1,n+1}(t)=-tQ_{s,n}+(2n-1)Q_{1,n}(t), \qquad \text{for} \quad n=1,2\ldots. \quad (24)$$

Hence, it can be deduced the following explicite form of polynomials $Q_{1,n}(t)$*,* $n\in\mathbb{N}$*:*

$$Q_{1,2n}(t)=(-1)^n t^{2n-1}+(-1)^{n-1}\binom{n}{2}\frac{(2n+1)!!}{(2n-3)!!}t^{2n-3}+$$

$$+(-1)^{n-2}\binom{n+1}{4}\frac{(2n+3)!!}{(2n-5)!!}t^{2n-5}+$$

$$+(-1)^{n-3}\binom{n+2}{6}\frac{(2n+5)!!}{(2n-7)!!}t^{2n-7}+\ldots-(4n-3)!!t, \quad (25)$$

and

$$Q_{1,2n+1}(t)=(-1)^n\binom{n}{1}\frac{(2n+1)!!}{(2n-1)!!}t^{2n-1}+$$

$$+(-1)^{n-1}\binom{n+1}{3}\frac{(2n+3)!!}{(2n-3)!!}t^{2n-3}+$$

$$+(-1)^{n-2}\binom{n+2}{5}\frac{(2n+5)!!}{(2n-5)!!}t^{2n-5}+\ldots-(4n-1)!!t. \tag{26}$$

For example, we have:

$$Q_{1,2}(t)=-t,\qquad Q_{1,3}(t)=-3t,\qquad Q_{1,4}(t)=t^3-15t,$$

$$Q_{1,5}(t)=10t^3-105t,\qquad Q_{1,6}(t)=-t^5+105t^3-945t,$$

$$Q_{1,7}(t)=-21t^5-1260t^3-10395t.$$

Remark 4.2. *It can be generated the formulae ($\rho=1,s$; $n\in\mathrm{N}$) [1, 3, 5, 7]:*

$$L^{-1}\Big[\frac{\rho}{(s^2+a^2)^n}\Big]=\begin{cases}\dfrac{\sqrt{\pi}}{(n-1)!}\left(\dfrac{t}{2a}\right)^{n-\frac{1}{2}}J_{n-\frac{1}{2}}(at) & \text{for } \rho=1,\\[2ex] \dfrac{\sqrt{\pi}\,a}{(n-1)!}\left(\dfrac{t}{2a}\right)^{n-\frac{1}{2}}J_{n-\frac{3}{2}}(at) & \text{for } \rho=s,\end{cases} \tag{27}$$

where $J_{n-\frac{1}{2}}(z)$ is the respective Bessel function:

$$J_{n-\frac{1}{2}}(z):=\sum_{r=0}^{\infty}(-1)^r\frac{\left(\frac{z}{2}\right)^{2r+n-1/2}}{r!\,\Gamma\left(r+n+\frac{1}{2}\right)}.$$

4.2 The technique of a parametric representation of the Laplace transform of some rational functions

The following theorem holds for the inverse Laplace transform (for more details see [2, 6, 9]).

Theorem 4.3. *Suppose that a given complex function $F(s)$, for $\Re(s)>\sigma_0$, is defined and satisfies three following conditions* [9-11]*:*

1. $F(s)$ *is holomorphic in the halfplane* $\Re(s)>\sigma_0$*;*
2. $\forall\varepsilon>0\qquad\exists R=R(\varepsilon)>0\qquad\forall s\in\mathrm{C}$

   if $\Re(s)\ge\sigma_1>\sigma_0$ *and* $|s|>R\ \Rightarrow\ |F(s)|<\varepsilon$*;*
3. *If* $\sigma\ge\sigma_1>\sigma_0$ *then*

$$\int_{\sigma-i\infty}^{\sigma+i\infty} |F(s)||ds| = \int_{-\infty}^{\infty} |F(\sigma+i\xi)|d\xi < \infty;$$

Then for every $\sigma \geq \sigma_1 > \sigma_0$ *we have*

$$f(t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{st}F(s)ds.$$

Hence, it can be obtained the following result, which is fundamental for this section:

Theorem 4.4. *Let* $p \in \mathbb{C}[s]$ *and* $n \in \mathbb{N}$. *If* $2(n-1) \geq \deg(p)$ *then the following formula hold*

$$L^{-1}\left[\frac{p(s)}{(s^2+1)^n}\right] = L^{-1}\left[\lim_{\substack{\varepsilon_k \to 1 \\ k=1,2,\ldots,n}} \frac{p(s)}{\prod_{k=1}^{n}(s^2+\varepsilon_k^2)}\right] = \lim_{\substack{\varepsilon_k \to 1 \\ k=1,2,\ldots,n}} L^{-1}\left[\frac{p(s)}{\prod_{k=1}^{n}(s^2+\varepsilon_k^2)}\right].$$

Moreover, if $p(s) = sq_1(s^2) + q_2(s^2)$ *then*

$$L^{-1}\left[\frac{p(s)}{(s^2+1)^n}\right] = \lim_{\substack{\varepsilon_k \to 1 \\ k=1,2,\ldots,n}} L^{-1}\left[\sum_{k=1}^{n}\frac{A_k s + B_k}{s^2+\varepsilon_k^2}\right] =$$

$$= \lim_{\substack{\varepsilon_k \to 1 \\ k=1,2,\ldots,n}} \sum_{k=1}^{n}\left(A_k\cos(\varepsilon_k t) + \frac{B_k}{\varepsilon_k}\sin(\varepsilon_k t)\right), \tag{28}$$

where

$$A_k = \frac{q_1(-\varepsilon_k^2)}{\prod_{\substack{1\leq l\leq n \\ l\neq k}}(\varepsilon_l^2-\varepsilon_k^2)} \quad \text{and} \quad B_k = \frac{q_2(-\varepsilon_k^2)}{\prod_{\substack{1\leq l\leq n \\ l\neq k}}(\varepsilon_l^2-\varepsilon_k^2)}$$

for every $k = 1,2,\ldots,n$. *In all above formulae it is assumed that* ε_k, $k = 1,2,\ldots,n$ *are pairwise different.*

Now it will be presented the sequence of examples illustrating the action of the procedure of parametric representation described in Theorem 4.4 for calculating of the inverse Laplace transform [12].

Example 4.5. We have (for every $s \in \mathbb{C}$, $s \neq 0$):

$$\frac{1}{(s^2+1)^2} = \lim_{\varepsilon,\delta\to 1}\frac{1}{(s^2+\varepsilon^2)(s^2+\delta^2)} = \lim_{\substack{\varepsilon,\delta\to 1 \\ \varepsilon\neq\delta}}\frac{1}{\delta^2-\varepsilon^2}\left(\frac{1}{s^2+\varepsilon^2}-\frac{1}{s^2+\delta^2}\right)$$

Hence, by (28), we get:

$$L^{-1}\left[\frac{1}{(s^2+1)^2}\right]=\lim_{\substack{\varepsilon,\delta\to 1\\ \varepsilon\neq\delta}}\frac{1}{\delta^2-\varepsilon^2}L^{-1}\left[\frac{1}{s^2+\varepsilon^2}-\frac{1}{s^2+\delta^2}\right]=$$

$$=\lim_{\substack{\varepsilon,\delta\to 1\\ \varepsilon\neq\delta}}\frac{1}{\delta^2-\varepsilon^2}\left(\frac{\sin(\varepsilon t)}{\varepsilon}-\frac{\sin(\delta t)}{\delta}\right),$$

or

$$L^{-1}\left[\frac{2}{(s^2+1)^2}\right]=\lim_{\substack{\varepsilon,\delta\to 1\\ \varepsilon\neq\delta}}\frac{1}{\delta-\varepsilon}\left(\frac{\sin(\varepsilon t)}{\varepsilon}-\frac{\sin(\delta t)}{\delta}\right).$$

Simultaneously, applied Lemma 4.1, we get [14,15]:

$$\sin t-t\cos t=\lim_{\substack{\varepsilon,\delta\to 1\\ \varepsilon\neq\delta}}\frac{1}{\delta-\varepsilon}\left(\frac{\sin(\varepsilon t)}{\varepsilon}-\frac{\sin(\delta t)}{\delta}\right).$$

Example 4.6. We have:

$$\frac{1}{(s^2+1)^3}=\lim_{\varepsilon,\delta,\gamma\to 1}\frac{1}{(s^2+\varepsilon^2)(s^2+\delta^2)(s^2+\gamma^2)}=$$

$$=\lim_{\substack{\varepsilon,\delta,\gamma\to 1\\ \varepsilon\neq\delta\quad\varepsilon\neq\gamma\quad\delta\neq\gamma}}\Big(\frac{1}{(\delta^2-\varepsilon^2)(\gamma^2-\varepsilon^2)(s^2+\varepsilon^2)}+\frac{1}{(\varepsilon^2-\delta^2)(\gamma^2-\delta^2)(s^2+\delta^2)}+$$

$$+\frac{1}{(\varepsilon^2-\gamma^2)(\delta^2-\gamma^2)(s^2+\gamma^2)}\Big).$$

Hence, by (28) and Lemma 4.1 again we get:

$$\frac{1}{8}((3-t^2)\sin t-3t\cos t)=L^{-1}\left[\frac{1}{(s^2+1)^3}\right]=$$

$$=\lim_{\substack{\varepsilon,\delta,\gamma\to 1\\ \varepsilon\neq\delta\quad\varepsilon\neq\gamma\quad\delta\neq\gamma}}\Big(\frac{1}{(\delta^2-\varepsilon^2)(\gamma^2-\varepsilon^2)}L^{-1}\left[\frac{1}{s^2+\varepsilon^2}\right]+$$

$$+\frac{1}{(\varepsilon^2-\delta^2)(\gamma^2-\delta^2)}L^{-1}\left[\frac{1}{s^2+\delta^2}\right]+\frac{1}{(\varepsilon^2-\gamma^2)(\delta^2-\gamma^2)}L^{-1}\left[\frac{1}{s^2+\gamma^2}\right]\Big)=$$

$$=\lim_{\substack{\varepsilon,\delta,\gamma\to 1\\ \varepsilon\neq\delta\quad\varepsilon\neq\gamma\quad\delta\neq\gamma}}\left(\frac{\sin(\varepsilon t)}{\varepsilon(\delta^2-\varepsilon^2)(\gamma^2-\varepsilon^2)}+\frac{\sin(\delta t)}{\delta(\varepsilon^2-\delta^2)(\gamma^2-\delta^2)}+\frac{\sin(\gamma t)}{\gamma(\varepsilon^2-\gamma^2)(\delta^2-\gamma^2)}\right).$$

One special, two-parametric case of this formula should be highlighted (for $\gamma=1$):

$$\frac{1}{8}((3-t^2)\sin t-3t\cos t)=$$

$$=\lim_{\substack{\varepsilon,\delta\to 1\\ \varepsilon\neq\delta}}\left(\frac{\sin(t)}{(\varepsilon^2-1)(\delta^2-1)}+\frac{1}{\varepsilon^2-\delta^2}\left(\frac{\sin(\varepsilon t)}{\varepsilon(\varepsilon^2-1)}-\frac{\sin(\delta t)}{\delta(\delta^2-1)}\right)\right)$$

A one-parametric version of the above equations is derived in the following way

$$\frac{1}{8}((3-t^2)\sin t-3t\cos t)=L^{-1}\Big[\lim_{\varepsilon\to 1}\frac{1}{(s^2+1)(s^2+\varepsilon^2)^2}\Big]=$$

$$=L^{-1}\Big[\lim_{\varepsilon\to 1}\Big(\frac{A}{s^2+1}+\frac{B}{s^2+\varepsilon^2}+\frac{C}{(s^2+\varepsilon^2)^2}\Big)\Big]$$

(where $A=-B=1/(\varepsilon-1)^2$, $C=1/(1-\varepsilon)$)

$$=\lim_{\varepsilon\to 1}\left(A\sin(t)+\frac{B}{\varepsilon}\sin(\varepsilon t)+C\left(\frac{\sin(\varepsilon t)}{\varepsilon}-t\cos(\varepsilon t)\right)\right)=$$

$$=\lim_{\varepsilon\to 1}\frac{1}{(\varepsilon-1)^2}\big(\sin(t)-\sin(\varepsilon t)+(\varepsilon-1)t\cos(\varepsilon t)\big).$$

Example 4.7. We have:

$$\frac{1}{8}(t\sin t-t^2\cos t)=L^{-1}\Big[\frac{s}{(s^2+1)^3}\Big]=$$

$$=\lim_{\substack{\varepsilon,\delta,\gamma\to 1\\ \varepsilon\neq\delta\quad\varepsilon\neq\gamma\quad\delta\neq\gamma}}\Big(\frac{1}{(\delta^2-\varepsilon^2)(\gamma^2-\varepsilon^2)}L^{-1}\Big[\frac{s}{s^2+\varepsilon^2}\Big]+$$

$$+\frac{1}{(\varepsilon^2-\delta^2)(\gamma^2-\delta^2)}L^{-1}\Big[\frac{s}{s^2+\delta^2}\Big]+\frac{1}{(\varepsilon^2-\gamma^2)(\delta^2-\gamma^2)}L^{-1}\Big[\frac{s}{s^2+\gamma^2}\Big]\Big)=$$

$$=\lim_{\substack{\varepsilon,\delta,\gamma\to 1\\ \varepsilon\neq\delta\quad\varepsilon\neq\gamma\quad\delta\neq\gamma}}\left(\frac{\cos(\varepsilon t)}{(\delta^2-\varepsilon^2)(\gamma^2-\varepsilon^2)}+\frac{\cos(\delta t)}{(\varepsilon^2-\delta^2)(\gamma^2-\delta^2)}+\frac{\cos(\gamma t)}{(\varepsilon^2-\gamma^2)(\delta^2-\gamma^2)}\right)=$$

$$=\lim_{\varepsilon\to 1}\frac{1}{(\varepsilon-1)^2}\left(\cos(t)-\cos(\varepsilon t)-\frac{\varepsilon-1}{2\varepsilon}t\sin(\varepsilon t)\right)$$

Example 4.8. (General formulae) We have:

$$\frac{1}{(2n-2)!!}\big(P_{s,n}(t)\sin(t)+Q_{s,n}(t)\cos(t)\big)=$$

$$=L^{-1}\Big[\frac{s}{(s^2+1)^n}\Big]=\lim_{\substack{\varepsilon_k\to 1\\ k=1,2,\ldots,n}}\sum_{k=1}^{n}\frac{\cos(\varepsilon_k t)}{\prod_{\substack{1\le l\le n\\ l\neq k}}(\varepsilon_l^2-\varepsilon_k^2)},\tag{29}$$

and

$$\frac{1}{(2n-2)!!}\big(P_{1,n}(t)\sin(t)+Q_{1,n}(t)\cos(t)\big)=$$

$$= L^{-1}\Big[\frac{1}{(s^2+1)^n}\Big] = \lim_{\substack{\varepsilon_k\to 1\\ k=1,2,\ldots,n}} \sum_{k=1}^{n} \frac{\sin(\varepsilon_k t)}{\varepsilon_k \prod\limits_{\substack{1\le l\le n\\ l\ne k}} (\varepsilon_l^2-\varepsilon_k^2)}, \tag{30}$$

where ε_k, $k=1,2,\ldots,n$, *are distinct elements of* $\mathbb{R}$.

Remark 4.9. *For the sake of comparison, a direct verification of (30) (for* $n=3$, $\varepsilon_1=1$, $\varepsilon_2=\varepsilon$, $\varepsilon_3=\delta$*) is shown below. We have (*$\varepsilon\ne\delta$*):*

$$V_k(\varepsilon,\delta) := 1-\frac{\varepsilon^k-\delta^k-\varepsilon\delta(\varepsilon^{k-1}-\delta^{k-1})}{\varepsilon-\delta} = (1-\varepsilon)(1-\delta)\sum_{\substack{l+m\le k-1\\ l,m\ge 0}} \varepsilon^l\delta^m,$$

which easily by induction on k *(*$k\ge 2$*) follows. Hence, we get:*

$$\lim_{\substack{\varepsilon,\delta\to 1\\ \varepsilon\ne\delta}} \frac{V_k(\varepsilon,\delta)}{(1-\varepsilon)(1-\delta)} = \lim_{\varepsilon,\delta\to 1} \sum_{\substack{l+m\le k-1\\ l,m\ge 0}} \varepsilon^l\delta^m = \binom{k}{2}.$$

Moreover, we have $V_0(\varepsilon,\delta)=V_1(\varepsilon,\delta)=0$. *At last we obtain:*

$$\lim_{\substack{\varepsilon,\delta\to 1\\ \varepsilon\ne\delta}} \left(\frac{\sin(t)}{(\varepsilon^2-1)(\delta^2-1)} + \frac{1}{\delta^2-\varepsilon^2}\left(\frac{\sin(\varepsilon t)}{\varepsilon(1-\varepsilon^2)} - \frac{\sin(\delta t)}{\delta(1-\delta^2)} \right)\right) =$$

$$= \lim_{\substack{\varepsilon,\delta\to 1\\ \varepsilon\ne\delta}} \sum_{k=1}^{\infty} (-1)^{k-1}\frac{t^{2k-1}}{(2k-1)!}\Big(\frac{1}{(\varepsilon^2-1)(\delta^2-1)} +$$

$$+ \frac{1}{\delta^2-\varepsilon^2}\left(\frac{\varepsilon^{2k-1}}{\varepsilon(1-\varepsilon^2)} - \frac{\delta^{2k-1}}{\delta(1-\delta^2)}\right)\Big) =$$

$$= \lim_{\substack{\varepsilon,\delta\to 1\\ \varepsilon\ne\delta}} \sum_{k=1}^{\infty} (-1)^{k-1}\frac{t^{2k-1}}{(2k-1)!}\frac{V_{k-1}(\varepsilon^2,\delta^2)}{(1-\varepsilon^2)(1-\delta^2)} =$$

$$= \sum_{k=3}^{\infty} (-1)^{k-1}\binom{k-1}{2}\frac{t^{2k-1}}{(2k-1)!} = \sum_{k=2}^{\infty} (-1)^{k}\binom{k}{2}\frac{t^{2k+1}}{(2k+1)!}.$$

On the other hand we generate

$$8L^{-1}\Big[\frac{1}{(s^2+1)^3}\Big](t) = (3-t^2)\sin(t) - 3t\cos(t) =$$

$$= (3-t^2)\sum_{k=1}^{\infty}(-1)^{k-1}\frac{t^{2k-1}}{(2k-1)!} - 3t\sum_{k=0}^{\infty}(-1)^k\frac{t^{2k}}{(2k)!} =$$

$$= \sum_{k=1}^{\infty}(-1)^k\frac{3+4k(k-1)-3(2k+1)}{(2k+1)!}t^{2k+1} = 8\sum_{k=2}^{\infty}(-1)^k\binom{k}{2}\frac{t^{2k+1}}{(2k+1)!},$$

which was our aim.

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