APPLICATION OF FRACTIONAL CALCULUS IN THE THEORY OF VISCOELASTICITY

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The fractional derivative method has been used in studies of the complex moduli and impedances for various models of viscoelastic substances. Calculations have been performed for Maxwell and Barlow-Erginsav-Lamb models. Influence of the values of the fractional parameters on the frequency dependences of the complex moduli and impedances has been studied.

Keywords: viscoelasticity, fractional calculus

1. INTRODUCTION

The dynamic elastic and damping properties of viscoelastic materials depend on the frequency. In order to take the material properties into account in vibration calculations, the mathematical forms of the frequency dependences have to be known. In the last twenty years the concept of differentiation and integration to noninteger order [1-3], the so called fractional calculus has found use in studies of viscoelastic materials, as well as in many fields of science and engineering including fluid flow, rheology, diffusive transport, electrical networks, electromagnetic theory and probability [4,5,6-15]. The advantage of the method of fractional derivatives in theory of viscoelasticity is, that it affords possibilities for obtaining constitutive equations for elastic complex modulus of viscoelastic materials with only few experimentally determined parameters. But there is also theoretical reason for using fractional calculus in describing properties of viscoelastic materials as has shown Bagley [15] in the case of polymer solutions and polymer solids without crosslingking, to which the molecular theory of Rouse gives relationship between stress and strain with fractional derivative of strain. Rossikhin and Shitikova [7] have used fractional derivatives in studies of propagation of waves in viscoelastic media. Fellah and Depollier [10] have applied fractional calculus in studies of acoustic waves scattering from porous materials.

2. DEFINITIONS OF FRACTIONAL INTEGRALS AND FRACTIONAL DERIVATIVES

Fractional integration and fractional differentiation of functions are generalization of common integration and common differentiation [1-3]. The fractional integral results from generalization of the Cauchy formula for the n-fold integral:

$$J^n f(t) = \frac{1}{(n-1)!}\int_0^t (t-\tau)^{n-1} f(\tau)d\tau \tag{1}$$

(where n is an integer number), to an positive fractional number α . To this end the Euler gamma function for integer numbers: $\Gamma(n) = (n-1)!$ is generalized to real numbers, using the integral formula for the gamma function:

$$\Gamma(\alpha) = \int_0^\infty e^{-u} u^{\alpha-1} du \tag{2}$$

The fractional integral of order α is defined by the formula:

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau)d\tau \tag{3}$$

One can see from the above formula, that $J^\alpha f(t)$ is the convolution of the function $f(t)$ with function : $\Phi_\alpha(t) = t_+^{\alpha-1}/\Gamma(\alpha)$, where $t_+ = 0$ for $t < 0$, i. e.

$$J^\alpha f(t) = \Phi_\alpha(t) * f(t) \tag{4}$$

Fractional derivative operator D^α of order α is defined as the left-inverse operator to the J^α and is given by the formula:

$$D^\alpha f(t) = D^m J^{m-\alpha} f(t) = \frac{d^m}{dt^m}\left[\frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\tau)^{m-\alpha-1} f(\tau)d\tau\right] \tag{5}$$

where m denotes an integer number such that: $m-1 < \alpha < m$.

3. THE GENERALIZED MAXWELL MODEL OF VISCOELASTICITY

Using the fractional derivatives, the Maxwell model of viscoelasticity is described by the following equation :

$$\sigma + \tau^\alpha D^\alpha \sigma = G_\infty \tau^\alpha D^\alpha \varepsilon \tag{6}$$

where σ is the applied shear stress, ε - the shear strain, G_∞ – the limiting value of the shear modulus , τ - the relaxation time, while α denotes a real number from the range: $0 < \alpha \le 1$ Applying the Fourier transformation to the above equation , it takes the form:

$$\overline{\sigma} + (i\omega)^{\alpha}\tau^{\alpha}\overline{\sigma} = G_{\infty}(i\omega)^{\alpha}\tau^{\alpha}\overline{\varepsilon} \tag{7}$$

where $\overline{\sigma}$, $\overline{\varepsilon}$ are Fourier transforms of stress and strain, ω is the cyclic frequency and $i^2 = 1$. From equation (7) we obtain the following formula for the complex shear modulus of elasticity:

$$G^{*}(i\omega) = G_{\infty}\frac{(i\omega\tau)^{\alpha}}{1+(i\omega\tau)^{\alpha}} \tag{8}$$

where the relaxation time $\tau = \eta / G_{\infty}$. The real G' and imaginary G'' parts of the complex modulus are:

$$G' = G_{\infty}\frac{(\omega\tau)^{\alpha} + \cos(\alpha\pi/2)}{(\omega\tau)^{-\alpha} + (\omega\tau)^{\alpha} + 2\cos(\alpha\pi/2)} \tag{9}$$

$$G'' = G_{\infty}\frac{\sin(\alpha\pi/2)}{(\omega\tau)^{-\alpha} + (\omega\tau)^{\alpha} + 2\cos(\alpha\pi/2)} \tag{10}$$

In the Fig.1 G'/G_{∞} has been plotted against $\log(\omega\tau)$. In Fig. 2 the normalized shear mechanical impedance $R/(\rho G_{\infty})^{1/2}$ has been plotted against $\log(\omega\tau)$.

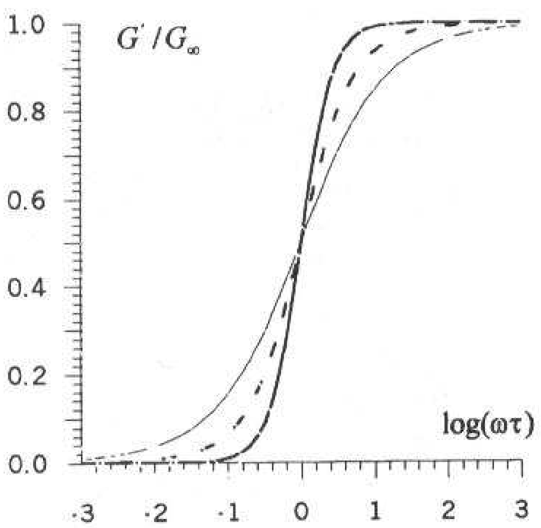

Fig.1 Dependence of G'/G_{∞} on $\omega\tau$ for various values of parameter α. The heavy line $\alpha = 1$, the dashed line $\alpha = 0.8$, the light line $\alpha = 0.6$.

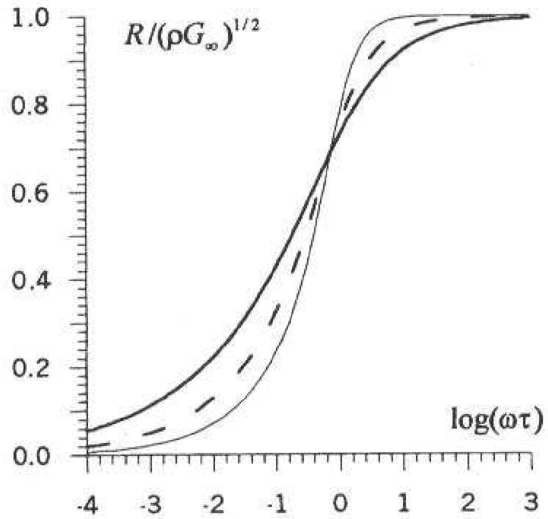

Fig. 2 Dependence of $R/(\rho G_{\infty})^{1/2}$ on $\omega\tau$ for various values of parameter α. The heavy line $\alpha = 1$, the dashed line $\alpha = 0.8$, the light line $\alpha = 0.6$.

Using the formulae (9) and (10), we have obtained for the complex impedance of the medium $Z = \sqrt{\rho G^{*}}$ the following equation:

$$Z = A^{1/2}\{[(1+B^2)^{1/2}+1]^{1/2} + i[(1+B^2)^{1/2}-1]^{1/2}\} \tag{11}$$

where:

$$A=(G_\infty\rho/2)\left(\frac{(\omega\tau)^\alpha+\cos(\alpha\pi/2)}{(\omega\tau)^{-\alpha}+(\omega\tau)^\alpha+2\cos(\alpha\pi/2)}\right)\ ,\ B=\frac{\sin(\alpha\pi/2)}{(\omega\tau)^\alpha+\cos(\alpha\pi?2)}$$

and ρ denotes the medium density.

4. THE GENERALIZED BARLOW-ERGINSAV-LAMB MODEL OF VISCOELASTICITY

In order to derive the generalized B-E-L model of viscoelasticity, we evaluated values of the complex impedance (11) for small and large values of $\omega\tau$: Z_0 and Z_∞. The following expressions have been obtained:

$$Z_0=(G_\infty\rho)^{1/2}2^{-1/2}(\omega\tau)^{\alpha/2}(\sqrt{1+\cos(\alpha\pi/2)}+i\sqrt{1-\cos(\alpha\pi/2)}) \tag{12}$$

$$Z_\infty=(G_\infty\rho/2)^{1/2} \tag{13}$$

The impedance Z_1 of the generalized B-E-L model was calculated from the equation:

$$\frac{1}{Z_1}=\frac{1}{Z_0}+\frac{1}{Z_\infty} \tag{14}$$

and the following formula has been obtained:

$$Z_1=(G_\infty\rho)^{1/2}\left[\frac{(\omega\tau)^{\alpha/2}\cos(\alpha\pi/4)+(\omega\tau)^\alpha+i(\omega\tau)^{\alpha/2}\sin(\alpha\pi/4)}{1+2(\omega\tau)^{\alpha/2}\cos(\alpha\pi/4)+(\omega\tau)^\alpha}\right] \tag{15}$$

The real part R and the imaginary part X of the impedance Z_1 are respectively:

$$\frac{R}{(G_\infty\rho)^{1/2}}=\frac{(\omega\tau)^{\alpha/2}\cos(\alpha\pi/4)+(\omega\tau)^\alpha}{1+2(\omega\tau)^{\alpha/2}\cos(\alpha\pi/4)+(\omega\tau)^\alpha} \tag{16}$$

$$\frac{X}{(G_\infty\rho)^{1/2}}=\left[\frac{(\omega\tau)^{\alpha/2}\sin(\alpha\pi/4)}{1+2(\omega\tau)^{\alpha/2}\cos(\alpha\pi/4)+(\omega\tau)^\alpha}\right] \tag{17}$$

In the Fig. 3 $R/(\rho G_\infty)^{1/2}$, and in Fig. 4 $X/(\rho G_\infty)^{1/2}$ have been plotted against $\log(\omega\tau)$ for various values of the parameter α . Using the relations between elastic moduli and impedances:

$$G'=\frac{R^2-X^2}{\rho}\ ,\ G''=\frac{2RX}{\rho}$$

we have obtained the following formulae for the real and imaginary parts of the complex shear modulus of elasticity in the generalized B-E-L model:

$$\frac{G'}{G_\infty}=\frac{\cos(\alpha\pi/2)+(\omega\tau)^\alpha+2(\omega\tau)^{\alpha/2}\cos(\alpha\pi/4)}{((\omega\tau)^{-\alpha/2}+(\omega\tau)^{\alpha/2}+2\cos(\alpha\pi/4))^2} \tag{18}$$

$$\frac{G''}{G^{\alpha}} = \frac{2(\cos(\alpha\pi/4) + (\omega\tau)^{\alpha/2})\sin(\alpha\pi/4)}{((\omega\tau)^{-\alpha/2} + (\omega\tau)^{\alpha/2} + 2\cos(\alpha\pi/4))^2} \qquad (19)$$

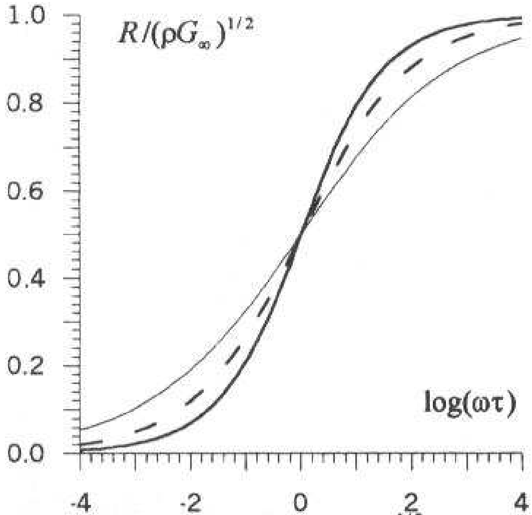

Fig. 3 Dependence of $R/(\rho G_\infty)^{1/2}$ on $\omega\tau$ for various values of parameter α. The heavy line $\alpha = 1$, the dashed line $\alpha = 0.8$, the light line $\alpha = 0.6$.

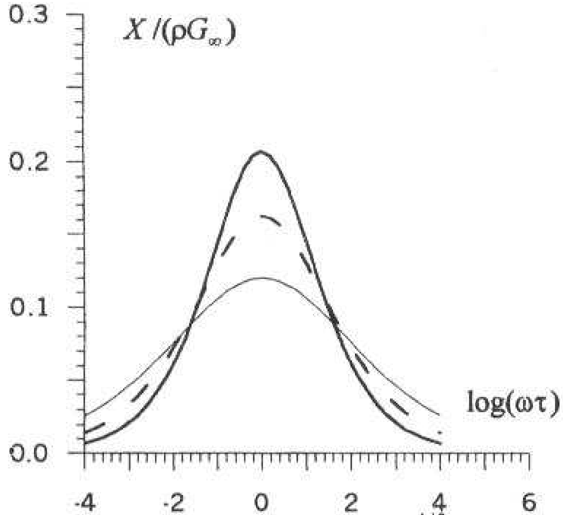

Fig. 4 Dependence of $X/(\rho G_\infty)^{1/2}$ on $\omega\tau$ for various values of parameter α. The heavy line $\alpha = 1$, the dashed line $\alpha = 0.8$, the light line $\alpha = 0.6$.

In Fig.5 values of G'/G_∞ and in Fig. 6 G''/G_∞ for generalized B-E-L model have been plotted against $\log(\omega\tau)$

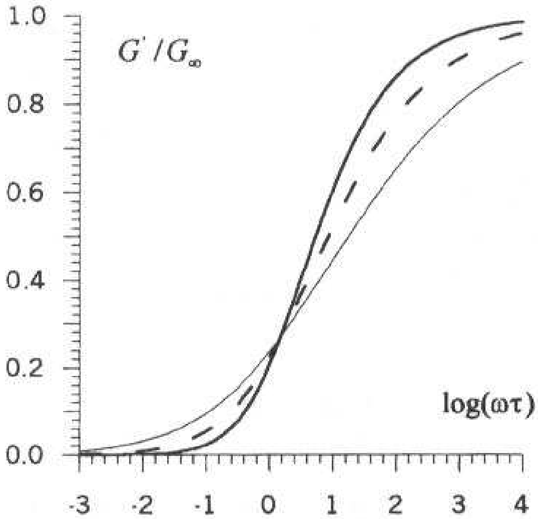

Fig.5 Dependence of G'/G_∞ on $\omega\tau$ for various values of the parameter α. The heavy line $\alpha = 1$, the dashed line $\alpha = 0.8$, the light line $\alpha = 0.6$.

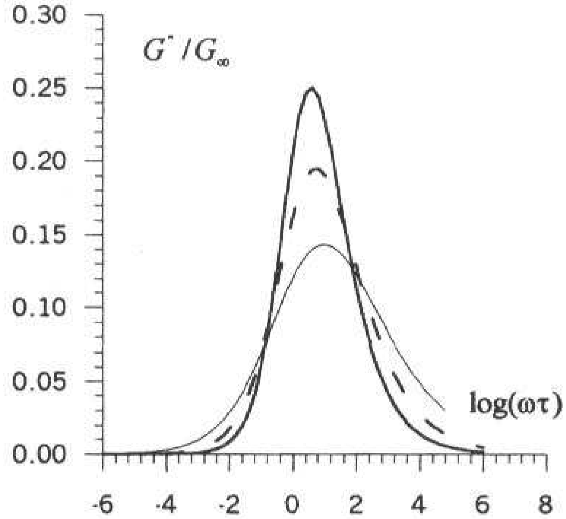

Fig.6 Dependence of G''/G_∞ on $\omega\tau$ for various values of the parameter α. The heavy line $\alpha = 1$, the dashed line $\alpha = 0.8$, the light line $\alpha = 0.6$.

The results for $R/(\rho G_\infty)^{1/2}$ given by the generalized B-E-L formula (16) have been compared with results obtained by Płowiec [16] for various oils, by means of the formula:

$$R/(\rho G_\infty)^{1/2} = \mathrm{Re}[1 + 1/(i\omega\tau) + 2K/(i\omega\tau)^\beta]^{1/2} \tag{20}$$

where values of K and β were fitted empirically . It has been stated that it is possible to match shapes of the curves given by formulae (16) and (20), but the curves given by equation (20) are parallel translated of about $\omega\tau = 3.9$ along x-axis and 0.01 along y-axis. In the case of liquids for which the simple B-E-L formula can be used : simple derivatives of benzene, phosphate, silicate and phthalate esters, long-chain hydrocarbons and phenyl ethers [24], it is possible to improve the agreement between experimental and calculated values of $R/(\rho G_\infty)^{1/2}$ using the formula (16) with suitably chosen values of the parameter α .

5. ANOTHER FRACTIONAL MODELS OF VISCOELASTICITY

Some fractional models of viscoelastic materials are cited by Rossikhin in the work [8]. The fractional Maxwell model with two independent fractional parameters:

$$\sigma + \tau_\varepsilon^\alpha D^\alpha \sigma = E_\infty \tau_\varepsilon^\beta D^\beta \varepsilon \tag{21}$$

where τ_ε is the relaxation time, E_∞ - the nonrelaxed elastic modulus, α, β are the fractional parameters. The complex modulus resulting from the above equation is:

$$E^*(i\omega) = E_\infty \frac{\tau_\varepsilon^\beta (i\omega)^\beta}{1 + \tau_\varepsilon^\alpha (i\omega)^\alpha} \tag{22}$$

Makris and Constantinou [17] have obtained very good agreement between prediction of model (21) with $\alpha = 0.6$, $\beta = 1$ and recorded mechanical properties of the storage modulus and damping coefficient and force-displacement loops in tests with vertical motion of viscous dampers consisting of a piston mooving in highly viscoelastic gel, in the frequency range from 0 to 50 Hz. Friedrich and Braun [18] used model (21) to fit experimental data for the frequency dependence of the storage and loss moduli for a monodisperse polybutadiene at $\alpha = 0.88$, $\beta = 0.98$ and for modified polybutadiene at $\alpha = 0.54$, $\beta = 0.59$ in the frequency range $-1.5 \le \log\omega \le 2$. Palade et al. [19] have shown that model (21) allows to describe the entire viscoelastic behaviour of polybutadienes from flow to glassy regime.

Another model of frequent use is the generalized standard linear solid model with two independent fractional parameters:

$$\sigma + \tau_\varepsilon^\alpha D^\alpha \sigma = E_0(\varepsilon + \tau_\sigma^\beta D^\beta \varepsilon) \tag{23}$$

$$E^*(i\omega) = E_0 \frac{1+\tau_\sigma^\beta (i\omega)^\beta}{1+\tau_\varepsilon^\alpha (i\omega)^\alpha} \tag{24}$$

where τ_σ is the retardation time and E_0 - the relaxed magnitude of the elastic modulus. Bagley and Torvik [20] found an excellent agreement between the above model and the mechanical properties at $550^0 C$ of a Corning glass doped with oxides of aluminium, sodium, and cobalt . They have used in the model $\alpha = 0.631$, $\beta = 0.641$. Morgenthaler [21] has found good agreement between the model (23) at $\alpha = 0.4885$, $\beta = 0.7049$ and frequency dependence of the shear modulus and loss factor of the studied by him viscoelastic material DYAD-606 in the frequency range from 0 to 46 Hz. Cupial [22] has found good agreement between predictions of the above model at $\alpha = 0.39$, $\beta = 0.64$ and the experimental data for a high damping polymer ISD112 in the frequency range from 1 to 10000 Hz.

Friedrich [23] suggested a model:

$$\sigma + \tau_\varepsilon^\alpha D^\alpha \sigma = E_0[\varepsilon + \tau_\sigma^\alpha D^\alpha \varepsilon + \tau_\sigma^\beta D^\beta \varepsilon] \tag{25}$$

$$E^*(i\omega) = E_0 \frac{1+\tau_\sigma^\alpha (i\omega)^\alpha + \tau_\sigma^\beta (i\omega)^\beta}{1+\tau_\varepsilon^\alpha (i\omega)^\alpha} \tag{26}$$

Friedrich and Braun [18] have shown that the above model provides good agreement with experimental dependences of the storage and loss moduli from the frequency for the polyisobutylene at $\alpha = 0.552$, $\beta = 0.626$ in the interval $-2 \le \log\omega \le 9$.

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