

**SENSITIVITY ANALYSIS IN SOLIDIFICATION PROBLEMS –
VARIATION OF THERMAL CONDUCTIVITY OF MOULD**E. MAJCHRZAK¹, M. JASIŃSKI²Department for Strength of Materials and Computational Mechanics
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SUMMARY

In the paper the sensitivity analysis with respect to thermal conductivity of the mould is presented. The 1D problem is discussed and the direct method of sensitivity analysis is used. On the stage of numerical computations the boundary element method is applied. In the final part of the paper the example of computations is shown.

1. INTRODUCTION

The 1D problem is considered. Non-steady temperature field in casting and mould sub-domains is described by the system of partial differential equations (energy equations) in the form

$$c \frac{\partial T_1(x,t)}{\partial t} = \lambda \frac{\partial^2 T_1(x,t)}{\partial x^2} \quad (1)$$

and

$$c_2 \frac{\partial T_2(x,t)}{\partial t} = \lambda_2 \frac{\partial^2 T_2(x,t)}{\partial x^2} \quad (2)$$

where c is the substitute thermal capacity per unit of volume [1], λ is the thermal conductivity of the casting, while c_2 , λ_2 are the volumetric specific heat and thermal

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conductivity of the mould, respectively. The substitute thermal capacity can be defined as follows [1, 3]

$$c = \begin{cases} c_m, & T > T_L \\ c_p = 0.5(c_m + c_s) + \frac{L_V}{T_L - T_S}, & T_S \leq T \leq T_L \\ c_s, & T < T_S \end{cases} \quad (3)$$

where c_m, c_s are the specific heats of molten metal and solidified part of the casting, T_L, T_S are the liquidus and solidus temperatures, while L_V is the latent heat per unit of volume.

On the contact surface between casting and mould the continuity condition is given

$$x = L_1: \begin{cases} T_1(x, t) = T_2(x, t) = T(x, t) \\ q_1(x, t) = q_2(x, t) = q(x, t) \end{cases} \quad (4)$$

where $q_1(x, t) = -\lambda \partial T_1(x, t) / \partial x$, $q_2(x, t) = -\lambda_2 \partial T_2(x, t) / \partial x$, while for $x=0$ and $x=L$ the no-flux conditions can be accepted, this means

$$x = 0: q_1(x, t) = 0, \quad x = L: q_2(x, t) = 0 \quad (5)$$

For $t=0$ the initial temperature distribution is known

$$t = 0: T_1(x, 0) = T_{10}, \quad T_2(x, 0) = T_{20} \quad (6)$$

where T_{10} is the pouring temperature, while T_{20} is the initial temperature of the mould.

2. SENSITIVITY ANALYSIS - THE DIRECT APPROACH

In order to obtain the sensitivity field $\partial T / \partial \lambda_2$ we differentiate the equations (1) and (2) with respect to λ_2 [2]. So

$$c \frac{\partial}{\partial \lambda_2} \left(\frac{\partial T_1(x, t)}{\partial t} \right) = \lambda \frac{\partial}{\partial \lambda_2} \left(\frac{\partial^2 T_1(x, t)}{\partial x^2} \right) \quad (7)$$

and

$$c_2 \frac{\partial}{\partial \lambda_2} \left(\frac{\partial T_2(x, t)}{\partial t} \right) = \frac{\partial^2 T_2(x, t)}{\partial x^2} + \lambda_2 \frac{\partial}{\partial \lambda_2} \left(\frac{\partial^2 T_2(x, t)}{\partial x^2} \right) \quad (8)$$

Now, denoting

$$U_1 = \frac{\partial T_1(x,t)}{\partial \lambda_2}, \quad U_2 = \frac{\partial T_2(x,t)}{\partial \lambda_2} \quad (9)$$

and eliminating $\partial^2 T_2(x,t)/\partial x^2$ (using the equation (2)), we can write

$$c \frac{\partial U_1(x,t)}{\partial t} = \lambda \frac{\partial^2 U_1(x,t)}{\partial x^2} \quad (10)$$

and

$$c_2 \frac{\partial U_2(x,t)}{\partial t} = \lambda_2 \frac{\partial^2 U_2(x,t)}{\partial x^2} + \frac{c_2}{\lambda_2} \frac{\partial T_2(x,t)}{\partial t} \quad (11)$$

Because

$$\frac{\partial q_1(x,t)}{\partial \lambda_2} = -\lambda \frac{\partial}{\partial \lambda_2} \left(\frac{\partial T_1(x,t)}{\partial x} \right) = -\lambda \frac{\partial U_1(x,t)}{\partial t} \quad (12)$$

and

$$\frac{\partial q_2(x,t)}{\partial \lambda_2} = -\frac{\partial T_2(x,t)}{\partial x} - \lambda_2 \frac{\partial}{\partial \lambda_2} \left(\frac{\partial T_2(x,t)}{\partial x} \right) = \frac{1}{\lambda_2} q_2(x,t) - \lambda_2 \frac{\partial U_2(x,t)}{\partial t} \quad (13)$$

therefore the boundary conditions (4), (5) take a form

$$x = L_1: \begin{cases} U_1(x,t) = U_2(x,t) \\ Q_1(x,t) = Q_2(x,t) + \frac{1}{\lambda_2} q_2(x,t) \end{cases} \quad (14)$$

and

$$x = 0: Q_1(x,t) = 0, \quad x = L: Q_2(x,t) = 0 \quad (15)$$

where

$$Q_1(x,t) = -\lambda \frac{\partial U_1(x,t)}{\partial x}, \quad Q_2(x,t) = -\lambda_2 \frac{\partial U_2(x,t)}{\partial x} \quad (16)$$

The initial condition is the following

$$t = 0: U_1(x,0) = 0, \quad U_2(x,0) = 0 \quad (17)$$

Summing up, in order to obtain the sensitivity field $\partial T/\partial\lambda_2$, the additional problem described by equations (10), (11), (14), (15), (17) must be solved.

3. BOUNDARY ELEMENT METHOD

At first the time grid

$$0 = t^0 < t^1 < K < t^{f-1} < t^f < K < t^F < \infty \quad (18)$$

with constant step $\Delta t = t^f - t^{f-1}$ is introduced.

The computations concerning the casting sub-domain are realized under the assumption that substitute thermal capacity in equation (1) corresponds to c_S . Next the obtained temporary temperature field is rebuilt using the numerical procedure called the temperature field correction method [1, 3]. In this way one can avoid the difficulties connected with the BEM application in numerical modelling of non-linear solidification problem. Application of the boundary element method for transition $t^{f-1} \rightarrow t^f$ requires the solution of the following systems of equations [3]

$$\begin{bmatrix} -h_{11}^1 & -h_{12}^1 & g_{12}^1 & 0 \\ -h_{21}^1 & -h_{22}^1 & g_{22}^1 & 0 \\ 0 & -h_{11}^2 & g_{11}^2 & -h_{12}^2 \\ 0 & -h_{21}^2 & g_{21}^2 & -h_{22}^2 \end{bmatrix} \begin{bmatrix} T_1(0, t^f) \\ T(L_1, t^f) \\ q(L_1, t^f) \\ T_2(L, t^f) \end{bmatrix} = \begin{bmatrix} p_1(0) \\ p_1(L_1) \\ p_2(L_1) \\ p_2(L) \end{bmatrix} \quad (19)$$

and

$$\begin{bmatrix} -h_{11}^1 & -h_{12}^1 & g_{12}^1 & 0 \\ -h_{21}^1 & -h_{22}^1 & g_{22}^1 & 0 \\ 0 & -h_{11}^2 & g_{11}^2 & -h_{12}^2 \\ 0 & -h_{21}^2 & g_{21}^2 & -h_{22}^2 \end{bmatrix} \begin{bmatrix} U_1(0, t^f) \\ U(L_1, t^f) \\ Q_1(L_1, t^f) \\ U_2(L, t^f) \end{bmatrix} = \begin{bmatrix} p_1(0) \\ p_1(L_1) \\ \frac{1}{\lambda_2} g_{11}^2 q(L_1, t^f) + p_2(L_1) + z(L_1) \\ \frac{1}{\lambda_2} g_{21}^2 q(L_1, t^f) + p_2(L) + z(L) \end{bmatrix} \quad (20)$$

next

$$Q_2(L_1, t^f) = Q_1(L_1, t^f) - \frac{1}{\lambda_2} q(L_1, t^f) \quad (21)$$

In equations (19), (20)

$$g_{12}^1 = \frac{1}{2} \sqrt{\frac{\Delta t}{\lambda c_s}} \exp\left(-\frac{L_1}{\sqrt{a_1 \Delta t}}\right), \quad g_{22}^1 = \frac{1}{2} \sqrt{\frac{\Delta t}{\lambda c_s}} \quad (22)$$

$$g_{11}^2 = -\frac{1}{2} \sqrt{\frac{\Delta t}{\lambda_2 c_2}}, \quad g_{21}^2 = -\frac{1}{2} \sqrt{\frac{\Delta t}{\lambda_2 c_2}} \exp\left(-\frac{L-L_1}{\sqrt{a_2 \Delta t}}\right) \quad (23)$$

where $a_1 = \lambda/c_s$, $a_2 = \lambda_2/c_2$, and

$$h_{11}^1 = h_{22}^1 = h_{11}^2 = h_{22}^2 = -0.5 \quad (24)$$

$$h_{12}^1 = h_{21}^1 = \frac{1}{2} \exp\left(-\frac{L_1}{\sqrt{a_1 \Delta t}}\right), \quad h_{12}^2 = h_{21}^2 = \frac{1}{2} \exp\left(-\frac{L-L_1}{\sqrt{a_2 \Delta t}}\right)$$

while

$$p_1(\xi) = \frac{1}{2\sqrt{a_1 \Delta t}} \int_0^{L_1} \exp\left(-\frac{|x-\xi|}{\sqrt{a_1 \Delta t}}\right) T_1(x, t^{f-1}) dx \quad (25)$$

$$p_2(\xi) = \frac{1}{2\sqrt{a_2 \Delta t}} \int_{L_1}^L \exp\left(-\frac{|x-\xi|}{\sqrt{a_2 \Delta t}}\right) T_2(x, t^{f-1}) dx$$

and

$$z(\xi) = \frac{c_2}{2\lambda_2} \sqrt{\frac{\Delta t}{\lambda_2 c_2}} \int_{L_1}^L \frac{\partial T_2(x, t)}{\partial t} \exp\left(-\frac{|x-\xi|}{\sqrt{a_2 \Delta t}}\right) dx \quad (26)$$

After solving the systems (19), (20) the temperatures and the functions U_1 and U_2 at internal points can be found [2, 3].

4. EXAMPLE OF COMPUTATIONS

The steel plate of thickness $2L_1 = 0.05$ [m] solidifying in typical sand mix ($L-L_1 = 0.05$ [m]) is considered. The following values of thermophysical parameters have been assumed: $\lambda = 35$ [W/mK], $c_m = 5.74 \cdot 10^6$ [J/m³·K], $c_s = 5.175 \cdot 10^6$ [J/m³·K], $L = 1.9575 \cdot 10^9$ [J/m³], $\lambda_2 = 2.6$ [W/mK], $c_2 = 1.75 \cdot 10^6$ [J/m³·K]. Pouring temperature: $T_{10} = 1550^\circ\text{C}$, initial temperature of mould: $T_{20} = 20^\circ\text{C}$, liquidus temperature: $T_L = 1505^\circ\text{C}$, solidus temperature: $T_S = 1470^\circ\text{C}$. Time step: $\Delta t = 2$ [s].

In Figure 1 the results of computations are shown. This sensitivity information can be used to optimize the experiment design in order to obtain the best estimate of λ_2 .

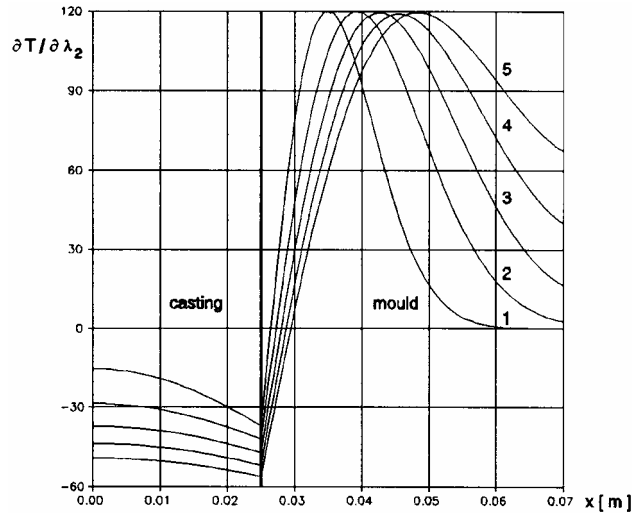


Fig. 1. Function $\partial T/\partial \lambda_2$: 30, 60, 90, 120, 150 [s]

Rys. 1. Funkcja $\partial T/\partial \lambda_2$: 30, 60, 90, 120, 150 [s]

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ANALIZA WRAŻLIWOŚCI W PROBLEMACH KRZEPNIĘCIA - WARIACJA WSPÓLCZYNNIKA PRZEWODZENIA FORMY

STRESZCZENIE

W pracy przedstawiono analizę wrażliwości ze względu na współczynnik przewodzenia ciepła masy formierskiej. Rozważano zadanie 1D, zastosowano metodę bezpośrednią. Na etapie obliczeń numerycznych wykorzystano metodę elementów brzegowych. W końcowej części artykułu przedstawiono przykład obliczeń.

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