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FUZZY BOUNDARY ELEMENT METHODS: A NEW MULTI-SCALE PERTURBATION APPROACH FOR SYSTEMS WITH FUZZY PARAMETERS

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Abstract. The aim of the paper is to present applications of the new algebraic system with specifically strictly defined new interval numbers. [8]. We present the Fuzzy Boundary Integral Equations where all operations are in the fuzzy perturbation sense. From now we assume that values of boundary conditions, material properties, internal prescribed fields and the shape of a boundary are uncertain and we'll model these uncertainties using the new methodology based on interval perturbation numbers. Illustrative examples from the potential theory are given to comment different aspects of the presented theory. Interval, triangle and trapezoidal - type fuzzy boundary conditions are considered. To complete the presentation the potential problem in a fuzzy domain is discussed. Presented methods give the complete methodology how to obtain good approximations of solutions of uncertain boundary problems with use of modern fuzzy analysis.

INTRODUCTION

The boundary problem may not be known exactly and some functions i.e. the shape of a structure, material properties, boundary conditions, external or internal excitations, solutions etc. may contain unknown parameters. Many different interpretations are possible for terminology of uncertain aspects of the Boundary Element Method (BEM) and we'll refer to these approaches as Fuzzy Boundary Element Method (FBEM). Applications of the FBEM have been initiated in the 1995 in papers by Skrzypczyk and Burczyński, cf. [4,10]. The earliest applications used the fuzzy independent numbers approach. From now we assume that values of boundary conditions, material properties, internal prescribed fields and the shape of a boundary are uncertain and we'll model these uncertainties using the new methodology based on interval perturbation numbers defined by Skrzypczyk, [5-8,11]. The new methodology can be applied to very complicated problems with different uncertain parameters. To present possibilities of the new method we consider the boundary potential problem with

- fuzzy boundary conditions $\tilde{u}_0, \tilde{q}_0$;
- fuzzy internal sources $\tilde{x}$;
- fuzzy fundamental solution $\tilde{u}^*(\cdot)$;
- fuzzy boundary $\tilde{G}$.

Let $\tilde{u}_0, \tilde{q}_0, \tilde{x}, \tilde{u}^*(\cdot)$ and $\tilde{W}$ with the fuzzy boundary $\tilde{G}$, be fuzzy functions. Boundary problems with such complicated conditions are not nosidered, cf. Witek [12].

1. $\bar{e}$ -FUZZY BOUNDARY ELEMENT METHOD

Formally we can write fuzzy boundary equation for the potential problem in the usual form and replace a boundary potential u_0, flux q_0,, internal sources ξ and a boundary Γ by corresponding fuzzy values $\tilde{u}_0, \tilde{q}_0, \tilde{x}$, $\tilde{G}$. Thus we obtain fuzzy boundary integral equation in the form

$$\tilde{c}(\mathbf{x})\tilde{u}(\mathbf{x}) + \int_{\tilde{G}} \tilde{q}^*(\mathbf{x},\mathbf{y})\tilde{u}(\mathbf{y})dG(\mathbf{y}) = \int_{\tilde{G}} \tilde{u}^*(\mathbf{x},\mathbf{y})\tilde{q}(\mathbf{y})d\Gamma(\mathbf{y}) + \int_{\tilde{W}} \tilde{u}^*(\mathbf{x},\mathbf{y})\tilde{x}(\mathbf{y})dW(\mathbf{y}), \quad \mathbf{x} \in \tilde{G} \text{ , } (1)$$

where $\tilde{u}^*(\cdot)$ denotes fuzzy fundamental solution for the potential problem and all operations are in the fuzzy sense, cf. [4,10]. Eq. (1) is called fuzzy boundary integral equation in the fuzzy domain with fuzzy parameters.
The Method of Fuzzy Boundary Elements with use of $\bar{e}$ -numbers is called further $\bar{e}$ ***-FBEM***.

1.1 $\bar{e}$ -Fundamental solutions

By the fundamental solution $\tilde{u}^*(\cdot)$ of the fuzzy Laplace equation for isotropic media we call a fuzzy solution of the equation

$$\tilde{l}\nabla^2\tilde{u}^* = -\tilde{d}\left(|\mathbf{x}-\mathbf{y}|\right), \quad \tilde{l} \in F(R), \mathbf{x},\mathbf{y} \in R^n, \tag{2}$$

where $\tilde{d}(\mathbf{x}), \mathbf{x} \in R^n$ denotes the fuzzy Dirac δ-distribution. For simplicity n=2 and y=(ξ,η)$\in R^2$ is an arbitrary point in the plane.
In the present we have not uniqe, sufficiently advanced theory of fuzzy partial differential equations for generalized function formulation, such as eq. (2). Further we use the theory of $\bar{e}$ intervals.
Transform the Laplace operator to polar coordinates (r,φ), then we get

$$\nabla\left(\bar{l}_{\bar{e}}\nabla\bar{u}\right) = \bar{l}_{\bar{e}} \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\bar{u}}{\partial r}\right) + \bar{l}_{\bar{e}} \frac{1}{r^2}\frac{\partial^2\bar{u}}{\partial j^2} \tag{3}$$

The generalized fuzzy ε-δ-Dirac function in polar coordinates takes the form

$$\bar{d}_{\bar{e}}\left(|(x-x, y-h)|\right) = \frac{1}{2pr}\bar{d}_{\bar{e}}(r) \tag{4}$$

where symbol r denotes the distance between point (x,y) and (ξ,η). The Dirac impulse excitation is radial symmetric and since we have the problem in infinite domain, we have not disturbaces from the boundary. After neglecting terms which are zero due to symmetry of the solution, i.e.

$$\frac{\bar{l}_{\bar{e}}}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\bar{u}}{\partial r}\right) = -\frac{1}{2pr}\bar{d}_{\bar{e}}(r) \tag{5}$$

where $\bar{l}_{\bar{e}}$ is the fuzzy constant. Futher we get

$$r\frac{\partial\bar{u}}{\partial r} = -\int\frac{1}{2p\bar{l}_{\bar{e}}}\bar{d}_{\bar{e}}(r)dr = -\frac{1}{2p\bar{l}_{\bar{e}}}H(r) + C_1 \tag{6}$$

Since r>0 then

$$\frac{\partial \bar{u}}{\partial r} = -\frac{1}{2\pi\bar{\lambda}_{\bar{\varepsilon}} r} + \frac{1}{r}C_1 \tag{7}$$

$$\bar{u}^*_{\bar{\varepsilon}}(r) = -\frac{1}{2\pi\bar{\lambda}_{\bar{\varepsilon}}}\ln_{\bar{\varepsilon}}(r) + C_1 \ln_{\bar{\varepsilon}}(r) + C_2 \tag{8}$$

where C_1 i C_2 are integration constans. We can prove that $C_1=0$ and C_2 is some arbitrary potential so we can assume that $C_2=0$.

Tab. 1. $\bar{\varepsilon}$ -Fundamental solutions of 2D and 3D Laplace equation

	2D	3D
$\bar{u}^*_{\bar{\varepsilon}}(r)$	$\bar{u}^*_{\bar{\varepsilon}}(r) = -\frac{1}{2\pi\bar{\lambda}_{\bar{\varepsilon}}}\ln_{\bar{\varepsilon}}(r)$	$\bar{u}^*_{\bar{\varepsilon}}(r) = \frac{1}{4\pi\bar{\lambda}_{\bar{\varepsilon}} r}$
$\frac{\partial \bar{u}^*_{\bar{\varepsilon}}}{\partial \mathbf{n}}$	$\frac{\partial \bar{u}^*_{\bar{\varepsilon}}}{\partial r} = -\frac{1}{2\pi\bar{\lambda}_{\varepsilon} r}$	$\frac{\partial \bar{u}^*_{\bar{\varepsilon}}}{\partial r} = -\frac{1}{4\pi\bar{\lambda}_{\varepsilon} r^2}$

1.2 $\bar{\varepsilon}$ -FBEM in perturbation formulation

The fuzzy solution of Eq. (1) is formulated in the conditional sense, i.e. we assume that the fuzzy boundary $\tilde{\Gamma} \in F(M)$. For arbitrary, nonfuzzy Γ eq. (1) can be formulated as the family of λ-cuts of the form

$$\begin{aligned}
&\tilde{c}_\lambda(\mathbf{x})\tilde{u}_\lambda(\mathbf{x}) + \int_\Gamma \tilde{q}_\lambda^*(\mathbf{x},\mathbf{y})\tilde{u}_\lambda(\mathbf{y})d\Gamma(\mathbf{y}) = \\
&= \int_\Gamma \tilde{u}_\lambda^*(\mathbf{x},\mathbf{y})\tilde{q}_\lambda(\mathbf{y})d\Gamma(\mathbf{y}) + \int_\Omega \tilde{u}_\lambda^*(\mathbf{x},\mathbf{y})\tilde{\xi}_\lambda(\mathbf{y})d\Omega(\mathbf{y}), \quad \mathbf{x}\in\Gamma,\ \lambda\in[0,1],
\end{aligned} \tag{9}$$

We define the conditional solution set as

$$(R_\lambda(\mathbf{x}|\Gamma) := \left\{ (u,q):\ \begin{array}{l}
c(\mathbf{x})u(\mathbf{x}) + \int_\Gamma q^*(\mathbf{x},\mathbf{y})u(\mathbf{y})d\Gamma(\mathbf{y}) + \int_\Omega u^*(\mathbf{x},\mathbf{y})\xi(\mathbf{y})d\Omega(\mathbf{y}) = \\
= \int_\Gamma u^*(\mathbf{x},\mathbf{y})q(\mathbf{y})d\Gamma(\mathbf{y}), \\
u_0(\mathbf{z}) \in \bar{u}_{0\lambda}(\mathbf{z})\big|_{\mathbf{z}\in\Gamma_1}, q_0(\mathbf{z}) \in \bar{q}_{0\lambda}(\mathbf{z})\big|_{\mathbf{z}\in\Gamma_2}, \xi(\mathbf{z}) \in \bar{\xi}_\lambda(\mathbf{z})\big|_{\mathbf{z}\in\Omega} \\
u^*(\mathbf{x},\mathbf{y}) \in \bar{u}_\lambda^*(\mathbf{x},\mathbf{y})\big|_{\mathbf{x},\mathbf{y}\in R^2\times R^2}, q^*(\mathbf{x},\mathbf{y}) \in \bar{q}_\lambda^*(\mathbf{x},\mathbf{y})\big|_{\mathbf{x},\mathbf{y}\in R^2\times R^2}
\end{array} \right\} \tag{10}$$

for Γ∈ M. Methodology of fuzzy equations is based on the theory of $\bar{\varepsilon}$ intervals and methods described in [8].

1.3 ε-Interval Bounday Equations - methodology of calculations

Remember that eq. (1) is analysed in the conditional sense, with the assumption that the fuzzy contour $\tilde{\Gamma} \in F(M)$. For arbitrary Γ eqs. (9) must be solved with sufficient quality. Mathematical problems with fuzzy differential equations for generalized functions force us to use the new algebraic methodology based on ε-intervals. We use that theory to solve the family of equations (9) for Γ∈M, $0 \le \lambda \le 1$ and to obtain upper approximations for (10).

Assume that we looking for ε-type intervals for λ-ε-cuts in the form

$$\tilde{u}_\lambda(\mathbf{x}) = \breve{u}_\lambda(\mathbf{x}) + \mathrm{rad}(\tilde{u}_\lambda(\mathbf{x}))\bar{\varepsilon}, \quad \mathbf{x}\in\Gamma, \Gamma\in M, \tag{11}$$

where $0 \le \lambda \le 1$. Let

$$\tilde{u}_{0l}(\mathbf{x}) = \overset{\circ}{u}_{0l}(\mathbf{x}) + \mathrm{rad}(\tilde{u}_{0l}(\mathbf{x}))\,\bar{e}\;, \quad \mathbf{x}\in G_1, \tilde{q}_{0l}(\mathbf{x}) = \overset{\circ}{q}_{0l}(\mathbf{x}) + \mathrm{rad}(\tilde{q}_{0l}(\mathbf{x}))\,\bar{e}\;, \quad \mathbf{x}\in G_2 \quad (12)$$

$$\tilde{x}_l(\mathbf{x}) = \overset{\circ}{x}_l(\mathbf{x}) + \mathrm{rad}(\tilde{x}_l(\mathbf{x}))\,\bar{e}\;, \quad \mathbf{x}\in W, \tilde{c}_l(\mathbf{x}) = \overset{\circ}{c}_l(\mathbf{x}) + \mathrm{rad}(\tilde{c}_l(\mathbf{x}))\,\bar{e}\;, \quad \mathbf{x}\in W \quad (13)$$

and for λ-ε-cuts of $\bar{e}$-fundamental solutions

$$\tilde{u}^*_l(\mathbf{x}) = \overset{\circ}{u}^*_l(\mathbf{x}) + \mathrm{rad}(\tilde{u}^*_l(\mathbf{x}))\,\bar{e}\;,\; \tilde{q}^*_l(\mathbf{x}) = \overset{\circ}{q}^*_l(\mathbf{x}) + \mathrm{rad}(\tilde{q}^*_l(\mathbf{x}))\,\bar{e}\;, \quad \mathbf{x}\in \mathrm{R}^k, \mathrm{k}=2 \quad (14)$$

Since λ-cuts of any fuzzy number are intervals we write eqs. (9) in the ε-interval form

$$\begin{aligned} &\bar{c}_l(\mathbf{x})\bar{u}_l(\mathbf{x}) + \int_G \bar{q}^*_l(\mathbf{x},\mathbf{y})\bar{u}_l(\mathbf{y})\mathrm{d}G(\mathbf{y}) = \\ &= \int_G \bar{u}^*_l(\mathbf{x},\mathbf{y})\bar{q}_l(\mathbf{y})\mathrm{d}G(\mathbf{y}) + \int_W \bar{u}^*_l(\mathbf{x},\mathbf{y})\bar{x}_l(\mathbf{y})\mathrm{d}W(\mathbf{y}), \quad \mathbf{x}\in G,\; l\in[0,1] \end{aligned}, \quad (15)$$

where all operations are in the ε-interval sense, cf. [8,12] and integrals are $\bar{e}$-extensions of surface integrals.

Consider now how we can discretize eq. (15) to obtain ε-interval algebraic equations for boundary values. For simplicity we assume that the domain is 2D, the boundary is divided into N elements. Let $M \supset G \cong \bigcup_{\mathrm{j}=1}^{\mathrm{N}} G_{\mathrm{j}}$, where Γ_{j} is the boundary of j-element. ε-interval values $\bar{u}_l$ and $\bar{q}_l$ are considered as $\bar{e}$-constant/linear/ quadratic over each element.

2. ALGEBRAIC METHODOLOGY OF ε-CALCULATIONS

If we assume that boundary points are numbered between 1 to N we get from eq. (15) the system of N ε-interval algebraic equations in the $\bar{e}$-matrix form

$$\bar{\mathbf{H}}_l \bar{\mathbf{U}}_l = \bar{\mathbf{G}}_l \bar{\mathbf{Q}}_l + \bar{\mathbf{V}}_l\;, \quad (16)$$

where $\bar{\mathbf{H}}_l$ and $\bar{\mathbf{G}}_l$ are two NxN $\bar{e}$-matrices and $\bar{\mathbf{U}}_l, \bar{\mathbf{Q}}_l, \bar{\mathbf{V}}_l$ are $\bar{e}$-vectors of lenth N, $\forall\,\lambda\in\,]0,1]$. Notice that some N_1 $\bar{e}$-fuzzy values of $\bar{u}_l$ and N_2 $\bar{e}$-values of $\bar{q}_l$ are known on the boundaries Γ_1 i Γ_2 respectively, so we have only $\bar{e}$-unknowns in the system (16). We have to rearrange system (16) to obtain the standard system of ε-interval algebraic equations

$$\bar{\mathbf{A}}_l \bar{\mathbf{X}}_l = \bar{\mathbf{F}}_l\;, \quad \forall\; 0\le l\le 1, \quad (17)$$

where $\bar{\mathbf{X}}_l$ is ε-interval vector of unknown λ-cuts $\bar{u}_l$ and $\bar{q}_l$ .Eqs. (17) are very similar to the classic linear equations over the field of real numbers and we can easy obtain unique ε-interval solution $\bar{S}_e(\bar{\mathbf{A}}_l, \bar{\mathbf{F}}_l)$. That family of ε-interval solutions is called *e-ALGEBRAIC INTERVAL SOLUTION - e-AIS.*

REMARK. ε-interval solution is an abstract object for any $\lambda\in\,]0,1]$. To obtain real interval solution we have to substitute for ε any real "small" number and make necessary interval operations. We get approximate solution which for every $\lambda\in\,]0,1]$ is the I-order approximation of interval equation, in the same notion $\bar{S}_e(\bar{\mathbf{A}}_l, \bar{\mathbf{F}}_l)$. We can prove that accuracy of that approximation is of order $o(\|\bar{e}\|^{2-\delta})$, where $\delta>0$ is some arbitrary small constant.

We say that the family $\bar{S}_e(\bar{\mathbf{A}}_l, \bar{\mathbf{F}}_l)$ of ε-AIS, dla $\Gamma\in M$, generates *e-CONDITIONAL ALGEBRAIC FUZZY SOLUTION (e-CAFS)* $\tilde{\mathbf{r}}_e(\mathbf{x}|G), \mathbf{x}\in G\in M$,with the membership function

$$m(\mathbf{y};\tilde{\mathbf{r}}_e(\mathbf{x}|G)) := \bigcup_{l\in[0,1]} l\cdot \bar{S}_e(\bar{\mathbf{A}}_l, \bar{\mathbf{F}}_l)\,,\; \mathbf{x}\in G\in M,\; \mathbf{y}\in \mathrm{R}^{\mathrm{N}}. \quad (18)$$

Thus *e-FUZZY ALGEBRAIC SOLUTION (e-FAS)* $\tilde{\mathbf{r}}_e(\tilde{\mathbf{x}}), \tilde{\mathbf{x}} \in \tilde{G}$ is defined as follows

$$m(\mathbf{y};\tilde{\mathbf{r}}_e(\tilde{\mathbf{x}})) := \sup_{G\in M}\left(m(G;\tilde{G}) \wedge m(\mathbf{y};\tilde{\mathbf{r}}_e(\mathbf{x}|G))\right), \qquad \mathbf{y}\in \mathrm{R}^N, \tilde{\mathbf{x}} \in \tilde{G}, \tag{19}$$

3. EXAMPLE

In the simple example, details about the contour see Fig. 1. in Skrzypczyk J., Multi-Scale Perturbation Methods In Mechanics, (this journal). Uncertainties are introduced into boundary conditions and into boundary, see Figs.2 and 3. Uncertainties are of the fuzzy - triangle type (290,300,310) and of the fuzzy-trapezoidal type, see Fig.3 for nodes 13-16. Fuzzy results for temperature of nodes 1 and 12 are illustrated by α-cuts of their membershipfunctions at Fig.1. The deformation of the triange shape of membership function to the trapezoidal one is forced by the fuzzy character of the boundary, see Fig. 4.

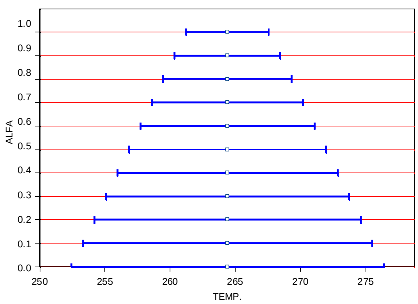

Fig. 1. α-cuts of temperature in 1 and 12 nodes

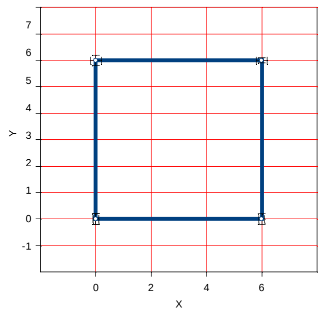

Fig. 2. Uncertainty of the boundary

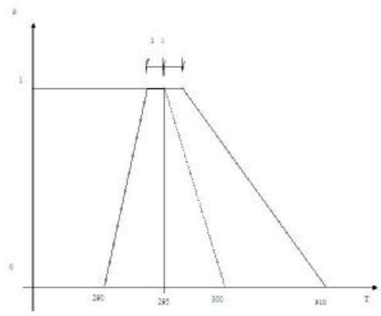

Fig. 3. Membership function of fuzzy temperature in 13 - 16 nodes

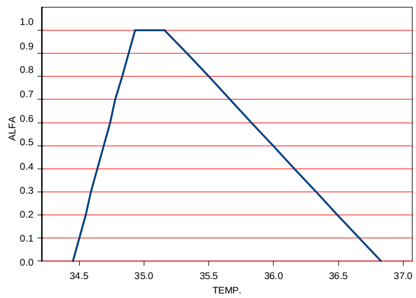

Fig. 4. Membership function of fuzzy temperature in 4 and 9 nodes

4. CONCLUSIONS

With the new $\bar{e}$ -Fuzzy Boundary Element Method we get a set of very simple and useful mathematical tools which can be easy used in analytical and computational parts of analysis of complex technical problems with uncertain parameters.

Advantages of the new algebraic system are as follows:

- we can omit all complex analytical calculations, which are typical for expanding approximated values of solutions in infinite series. It works for expanding unknown values - solutions as well as for perturbed coefficients of the problem;
- we get a great simplification of all arithmetic calculations which appear in analytical formulation and analysis of the problem;
- most of known numerical algorithms can be simply adapted for the new algebraic system without any serious difficulties.

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