MULTI-SCALE PERTURBATION METHODS IN MECHANICS

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<u>Abstract.</u> The aim of the paper is to present a modern algebraic system with specifically defined addition and multiplication operations. The new numbers called multi-scale perturbation numbers are introduced. It's proved that the system of real numbers ($R,+,\bullet$) is imbedded into the new algebraic system ($R_{\epsilon n},+_{\epsilon},\bullet_{\epsilon}$). Some additional properties as subtraction, inversion and division can be analyzed too. Elementary formulations of function extensions are introduced. Some basic properties of perturbation matrices, inverse matrices and solution properties of linear equations and eigenvalue problems in the new algebraic system can be defined. Classical multi-scale perturbation problems can be solved in the new algebraic system as easy as usual problems of applied mathematics, theoretical physics and techniques. Additional analytical transformations are not required.

INTRODUCTION

Theory of perturbations is a part of science of the great theoretical and practical meaning. It begins in 1926/27 with papers of Rayleigh [10] and Schrödinger [12]. The first papers treat about eigenvalues and applications in physics, namely in acoustic. First papers were from mathematical point of view formal and incomplete. The first one who consider the convergence of expansion series of perturbation theory was Rellich [11]. Papers of Rellich are fundamental for perturbation theory. Now perturbation theory has a bibliography which has thousands positions and is still in use, cf. monograph by Kato [6].

Finding the exact solution values for many computational problems is not an easy task. Sometimes it is easier to calculate the solutions of a nearby problem and then use the knowledge from perturbation theory to locate approximately solutions of the original problem. In some problems, the underlying physical system may be subjected to changes (perturbations) and we may want to determine the consequent change in solutions. On the other occasions, we may know an problem only approximately due to errors of observation, or we may have to feed an approximation of it to a computing device. In each case we would like to know how much this error or approximation would affect the solutions of the problem cf. [1],[5],[7].

Classical perturbation methods can be formulated in the following sense. Consider how perturbations (small) of nominal parameter values can change solutions of the considered problem. Assume that a solution of the considered problem, say \mathbf{x}_0 corresponds to the matrix of coefficients **A**. The basic problem of perturbation theory is to answer: how much the solution changes if matrix **A** takes new value $\mathbf{A}+\epsilon\mathbf{B}$, where ϵ is called a one-scale small parameter and **B** is the perturbation. It is often convenient to seek the solution in the form of a

series of homogeneous terms in the coefficients of the perturbation matrix B, that is, solutions of the form

$$\mathbf{x} := \mathbf{x}_0 + \mathbf{\varepsilon} \mathbf{x}_1 + \mathbf{\varepsilon}^2 \mathbf{x}_2 + \mathbf{\varepsilon}^3 \mathbf{x}_3 + \dots \dots \dots$$
(1)

are sought. If we restrict our considerations to first two terms in (1) we have one-scale perturbation method of the 1st-order. In perturbation method applications a serious difficulty is a necessity of a large amount of analytical calculations. As a result we obtain a set of classical problems which are usually simpler to solve numerically, cf. [1],[3],[6],[7],[10-16].

1. ALGEBRAIC SYSTEM OF MULTI-SCALE PERTURBATION NUMBERS

DEFINITION 1. Define a new number called further n-scale perturbations numbers (n-PN's) as ordered (n+1)-couples of real numbers $(x_0,x_1,x_2,...,x_n) \in \mathbb{R}^{n+1}$. The set of perturbation numbers is denoted by $\mathbb{R}_{\epsilon n}$. The first element x_0 of the (n+1)-couple is called a main value and the following are the perturbation values or simply the perturbations.[13]-[16]

Let $\zeta, \zeta_1, \zeta_2, \zeta_3 \in \mathbb{R}_{\epsilon n}$ denote any perturbation numbers and $\zeta:=(x_0, x_1, x_2, ..., x_n)$, $\zeta_1:=(y_0, y_1, y_2, ..., y_n)$, $\zeta_2:=(z_0, z_1, z_2, ..., z_n)$, $\zeta_3:=(v_0, v_1, v_2, ..., v_n)$, $x_i, y_i, z_i, v_i \in \mathbb{R}$, i=0,1,2,...,n. It is called that two perturbation numbers are equal: $\zeta_1 \equiv \zeta_2$ if and only if $y_i = z_i$ for any i=0,1,2,...,n.

In the set $R_{\epsilon n}$ we introduce the addition $(+_{\epsilon})$ and multiplication (\bullet_{ϵ}) as follows:

$$\zeta_1 +_{\varepsilon} \zeta_2 = (y_0, y_1, y_2, \dots, y_n) +_{\varepsilon} (z_0, z_1, z_2, \dots, z_n) := (y_0 + z_0, y_1 + z_1, y_2 + z_2, \dots, y_n + z_n)$$
(2)

$$\zeta_1 \bullet_{\varepsilon} \zeta_2 = (y_0, y_1, y_2, \dots, y_n) \bullet_{\varepsilon} (z_0, z_1, z_2, \dots, z_n) := (y_0 z_0, y_0 z_1 + y_1 z_0, y_0 z_2 + y_2 z_0, \dots, y_0 z_n + y_n z_0)$$
(3)

THEOREM 1. The set $R_{\epsilon n}$ with addition $(+_{\epsilon})$ and multiplication (\bullet_{ϵ}) defined by Eqs. (2) and (3) with selected neutral addition element $0_{\epsilon n}:=(0,0,\ldots,0)$ and neutral multiplication element $1_{\epsilon n}:=(1,0,0,\ldots,0)$ is a field. Defined in such a way field is called a field of n-PN's.

The field $R_{\epsilon n}$ as defined in Def. 1 doesn't contain the field of real numbers R. We can show that real numbers can be considered as some elements of field $R_{\epsilon n}$ with all classical addition and multiplication formulas and neutral elements of addition and multiplication, cf. [2],[3],[9].

THEOREM 2. The map $j_n: R \to R_{\epsilon n}$, $j_n(x):=(x,0,0,...,0)$ for each $x \in R$, is called the injection of the algebraic system of real numbers R into the algebraic system $R_{\epsilon n}$. It's the single-valued mapping and preserves corresponding algebraic operations and neutral elements of addition and multiplications.

Further details see [13]-[16].

2. SIMPLIFIED NOTION FOR PERTURBATION CALCULUS

Notice, that since $j_n(.)$ is the injection then each perturbation number of the form $(a,0,0,0,...,0) \in \mathbb{R}_{\epsilon n}$, $a \in \mathbb{R}$, we can identify with a real number a. We use this notice to simplify a notion for perturbation operations. Denote by ϵ_1 the n-PN (0,1,0,0,...,0), by ϵ_2 n-PN (0,0,1,0,0,...,0), ..., and by ϵ_n n-PN (0,0,0,...,0,1) respectively. Then for every n-PN $\zeta = (x_0,x_1,x_2,...,x_n) \in \mathbb{R}_{\epsilon n}$ we can write

$$\begin{aligned} (x_0, x_1, x_2, \dots, x_n) &= (x_0, 0, 0, \dots, 0) +_{\epsilon} (0, x_1, 0, 0, \dots, 0) +_{\epsilon} (0, 0, x_2, 0, 0, \dots, 0) +_{\epsilon} \dots +_{\epsilon} (0, 0, \dots, 0, x_n) = \\ &= (x_0, 0, 0, \dots, 0) +_{\epsilon} (x_1, 0, 0, \dots, 0) \bullet_{\epsilon} (0, 1, 0, 0, \dots, 0) +_{\epsilon} (x_2, 0, 0, \dots, 0) \bullet_{\epsilon} (0, 0, 1, 0, 0, \dots, 0) +_{\epsilon} \dots +_{\epsilon} (x_n, 0, 0, \dots, 0, 0) \bullet_{\epsilon} (0, 0, \dots, 0, 0) \bullet_{\epsilon} (0, 0, \dots, 0, 0) +_{\epsilon} \dots +_{\epsilon} (x_n, 0, 0, \dots, 0, 0) \bullet_{\epsilon} (0, 0, \dots, 0, 0) = \\ &= j_n(x_0) +_{\epsilon} \epsilon_1 \bullet_{\epsilon} j_n(x_1) +_{\epsilon} \epsilon_2 \bullet_{\epsilon} j_n(x_2) +_{\epsilon} \dots +_{\epsilon} \epsilon_n \bullet_{\epsilon} j_n(x_n) = x_0 +_{\epsilon} \epsilon_1 \bullet_{\epsilon} x_1 +_{\epsilon} \epsilon_2 \bullet_{\epsilon} x_2 +_{\epsilon} \dots +_{\epsilon} \epsilon_n \bullet_{\epsilon} x_n \end{aligned}$$

if we assume for simplicity that any n-PN (x_k ,0,0,...,0) is identical with the real number x_k , for every $x_k \in R$, k=0,1,2,...,n.

From multiplicity formulas it follows that

3. ORDER RELATION IN THE SET OF PERTURBATION NUMBERS

In the set of n-PN's, similarly as in the sets: R^2 , set of complex numbers C^1 , R^n , n>1 etc it's not possible to introduce the complete order relation. Followig that fact we define the relation of partial order in the following matter.

3.1 Partial ordering in the strong form

DEFINITION 2. For $\zeta_1, \zeta_2 \in \mathbb{R}_{\varepsilon n}$, we say that $\zeta_1 \ge_{\varepsilon} \zeta_2$ if

 $y_0 \geq z_0 \quad \text{and} \quad y_1 \geq z_1, \ y_2 \geq z_2, \dots, \ y_n \geq z_n.$

DEFINITION 3. For $\zeta_1, \zeta_2 \in \mathbb{R}_{\epsilon n}$, we say that $\zeta_1 >_{\epsilon} \zeta_2$ if

 $y_0 > z_0$ and $y_1 > z_1, y_2 > z_2,..., y_n > z_n$.

DEFINITION 4. For $\zeta_1, \zeta_2 \in \mathbb{R}_{\epsilon n}$, we say that $\zeta_1 =_{\epsilon} \zeta_2$ if $\zeta_1 \ge_{\epsilon} \zeta_2$ and $\zeta_2 \ge_{\epsilon} \zeta_1$.

In an analogous way we define relations " \leq_{ϵ} " and " $<_{\epsilon}$ ".

DEFINITION 5. A perturbation number $\zeta \in R_{\epsilon n}$, is said to be positive (nonnegative) if $\zeta >_{\epsilon} 0_{\epsilon 2}$ $(\zeta \ge_{\epsilon} 0_{\epsilon 2})$.

DEFINITION 6. A perturbation number $\zeta \in R_{\epsilon n}$, is said to be negative (nonpositive) if $\zeta <_{\epsilon} 0_{\epsilon 2}$ ($\zeta \leq_{\epsilon} 0_{\epsilon}$).

REMARK 1. In further considerations we notice that, $\dot{z}e \zeta \neq_{\epsilon} 0_{\epsilon n}$ (strongly), if

$$x_0 \neq 0$$
 and $x_1 \neq 0, x_2 \neq 0, ..., x_n \neq 0$.

REMARK 2. In further considerations we change the symbol $,=_{\varepsilon}$ with the simplified ,=.

3.2 Partial ordering in the weak form

In further considerations we often use the partial order relation in the simpler (weaker) form. DEFINITION 4.7. For $\zeta_1, \zeta_2 \in \mathbb{R}_{\epsilon n}$, we say that $\zeta_1 \overset{\bullet}{=} \zeta_2$ if $y_0 \ge z_0$ and y_1, y_2, \dots, y_n , z_1, z_2, \dots, z_n are arbitrary.

DEFINITION 4.8. For $\zeta_1, \zeta_2 \in \mathbb{R}_{\varepsilon n}$, we say that $\zeta_1 \bigotimes_e \zeta_2$ if $y_0 > z_0$ and y_1, y_2, \dots, y_n , z_1, z_2, \dots, z_n are arbitrary.

DEFINITION 4.9. For $\zeta_1, \zeta_2 \in \mathbb{R}_{\varepsilon n}$, we say that $\zeta_1 \not\in_e \zeta_2$ if $\zeta_1 \not\leq_e \zeta_2$ and $\zeta_2 \not\leq_e \zeta_1$ (or equivalently $y_0 = z_0$.

In the analogous way we define the relation " \mathbf{a}_{e} " and " \mathbf{a}_{e} ".

DEFINITION 4.10. A perturbation number $\zeta \in R_{\epsilon n}$, is said to be weakly positive (nonnegative) if $\zeta \bigotimes_{e} 0_{\epsilon n}$ ($\zeta \bigotimes_{e} 0_{\epsilon n}$) (notice that perturbation parts $x_1, x_2, ..., x_n$ can be arbitrary).

DEFINITION 4.11. A perturbation number $\zeta \in R_{\epsilon n}$, is said to be weakly negative (nonpositive) if $\zeta \bigotimes_{e} 0_{\epsilon n}$ ($\zeta \bigotimes_{e} 0_{\epsilon n}$) (notice that perturbation parts $x_1, x_2, ..., x_n$ can be arbitrary).

Notice that relations between perturbation numbers of the "strong" type as \leq_{ϵ} , \geq_{ϵ} , \geq_{ϵ} , \geq_{ϵ} , $<_{\epsilon}$ " implies the "weak" relations: " $\&_{e}$, $\&_{e}$, $\&_{e}$, $\&_{e}$, $\&_{e}$ ", respectively.

REMARK 3. In further considerations in place of the symbol $,=_{\varepsilon}$ " the simplified notion is used ,=", since it is not going to misunderstanding.

REMARK .4. In further considerations we say that $\zeta \mathbf{a}_{e} \mathbf{0}_{\varepsilon}$ (weakly), if

 $x_0 \neq 0$ and x_1, x_2, \dots, x_n are arbitrary.

4. EXTENDED *en-FUNCTIONS*

n-PN value functions are defined for n-PN arguments as extensions of classical elementary and trigonometric functions. Properties of ɛn-functions are analyzed in details, cf. [13]-[16].

Let $D \subset R_{\epsilon n}$ be an arbitrary subset. Suppose that we have a rule $f_{\epsilon n}$ which assigns to each element $\zeta \in D$ exactly one element w of $R_{\epsilon n}$. Then we say that $f_{\epsilon n}$ is an extended function defined on D with values in $R_{\epsilon n}$. We will denote that function as $f_{\epsilon n} : D \rightarrow R_{\epsilon n}$ or $w = f_{\epsilon n}(\zeta)$ or simplified $w = \epsilon n - f(\zeta)$.

To illustrate how we can construct generalizations of usual real functions we use a simple function. We discuss now an extension of a simple exponential function $\exp(x)$, $x \in \mathbb{R}$. With polynomials and rational functions it is one of the simplest elementary functions. How can we understand the notion $\exp(\zeta)$, where $\zeta = (x_0, x_1, x_2, ..., x_n) \in \mathbb{R}_{\epsilon n}$?

Notice that we can expand exp(x), $x \in R$ into a classical series

$$\exp(\mathbf{x}) = 1 + \frac{\mathbf{x}}{1!} + \frac{\mathbf{x}^2}{2!} + \frac{\mathbf{x}^3}{3!} + \dots = \sum_{k=0}^{\infty} \frac{\mathbf{x}^k}{k!}, \qquad \mathbf{x} \in \mathbf{R}$$
(4)

which is convergent for all $x \in \mathbb{R}$. Define the new function $\exp_{\varepsilon}(\zeta)$, for $\zeta \in \mathbb{R}_{\varepsilon n}$ as

$$\exp_{e}(z) := 1 + \frac{z}{1!} + \frac{z^{2}}{2!} + \frac{z^{3}}{3!} + \dots = \sum_{k=0}^{\infty} \frac{z^{k}}{k!}, \qquad z \in \mathbb{R}_{en}$$
(5)

Following equations (4) and (5) we write

$$\exp_{e_{n}}(z) = 1 + \frac{x_{0}}{1!} + \frac{x_{0}}{2!} + \dots + \frac{x_{0}}{n!} + \dots + \frac{x_{0}}{n!} + \dots + e_{n}x_{n}\left(1 + x_{0} + \dots + \frac{x_{0}}{(n-1)!}\right) + \dots = (6)$$
$$= \exp(x_{0}) + (e_{1}x_{1} + \dots + e_{n}x_{n})\exp(x_{0})$$

We can prove the generalized convergence of the Seq. (6) for every $\zeta \in R_{\epsilon n}$. We have additionally that

$$j_n(\exp(x)) = (\exp(x), 0, 0, \dots, 0) = \exp_{\varepsilon n}(x),$$

which proves that the new function $\exp_{\varepsilon}(.)$ is the extension of the real function $\exp(x)$.

5. EXAMPLE

In the simple example, see Fig. 1. multiscale perturbations (two-scale) are introduced into boundary conditions and into boundary, see Fig.3. Perturbations of the boundary are of the first kind (ϵ_1), details about the contour see Fig.1. The boundary temperature is perturbed in nodes 13-16 (ϵ_2). The results the temperature and the flux on the boundary with corresponding perturbations are calculated for all nodes, see Figs. 2 and 3.



Fig. 1. Scheme of the boundary, boundary elements and boundary conditions



6. CONCLUSIONS

Calculations with use of new multi-scale perturbation numbers lead to applications which are mathematically equivalent with I-order approximations in classical perturbation methods. Advantages of the new algebraic system are as follows:

- we can omit all complex analytical calculations which are typical for expanding approximated values of solutions in infinite series. It works for expanding unknown values solutions as well as for perturbed coefficients of the problem;
- we get a great simplification of all arithmetic calculations which appear in analytical formulation and analysis of the problem;
- most of known numerical algorithms can be simply adapted for the new algebraic system without any serious difficulties.

With the new algebraic system we get a set of very simple and useful mathematical tools which can be easy used in analytical and computational parts of analysis of complex multi-scale perturbation problems.

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