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Application of the Picard's iterative method for the solution of one-phase Stefan problem

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Abstract

In this paper, application of the Picard's iterative method for solving the one-phase Stefan problem is presented. In the proposed method, an iterative relation is formulated, which allows to determine the temperature distribution in the considered domain. The unknown function, describing the position of the moving interface, is approximated with the aid of the linear combination of some assumed base functions. The coefficients of this combination are determined by minimizing a properly constructed functional. Some examples, that illustrate the precision and speed of convergence of the considered iterative procedure, are also shown.

Keywords: Solidification process, Application of information technology to the foundry industry, Stefan problem, Moving boundary problem, Picard's iterative method

1. Introduction

In the paper we consider the one-phase Stefan problem, which consists of determining the temperature distribution in the given domain and the function describing position of the moving interface (the freezing front). The Stefan problem is a mathematical model of thermal processes, during which the changing of phase is taking place, connected with the heat absorption or emission. The examples of such kind of processes can be solidification of pure metals, melting of ice, freezing of water, deep freezing of foodstuffs and so on [1,2].

For some simple cases of the Stefan problem there are chances of finding the analytical solution [3,4,5], but for most of cases the approximated methods must be applied [1,6-10]. In paper [11], the authors have applied the Adomian decomposition method, combined with some minimization procedure, for finding the approximate solution of one-phase Stefan problem. Application of the variational iteration method [12] for calculating the approximate solution of the direct and inverse Stefan problem is considered in paper [13]. Besides, in papers [14,15] some new approach for solving the one-phase Stefan problem is presented. In this approach, the considered problem is first transformed for the domain of the unit square and after that, such transformed problem is solved by using the variational iteration method. Another applications of the variational iteration method for solving problems connected with the heat conductivity are presented in papers [16-18].

In the present paper, we propose to apply the Picard's iterative method for solving the one-phase Stefan problem. The Picard's iterative method [19] consists of formulating the iterative procedure, which enables to determine the form of the unknown function, describing the temperature distribution in the given domain, on the ground of the heat conduction equation and initial condition, which should be satisfied. Another unknown function, describing position of the moving interface, is approximated in the form of the linear combination of some assumed base functions. The coefficients of this linear combination are calculated by minimizing the properly constructed functional. Some examples, illustrating the accuracy of the obtained approximate solution (compared with the known analytical solution of the problem) and speed of convergence of the iterative procedure, are also shown.

2. One-phase Stefan problem

We deal with the one-phase Stefan problem, described in the domain $D = \{(x,t) : t \in [0,t^*],\ x \in [0,\xi(t)]\} \subset R^2$. On the boundary of domain D three following components are distributed (Fig. 1):

$$\Gamma_0 = \{(x,0) : x \in [0,s],\ s = \xi(0)\}, \tag{1}$$

$$\Gamma_1 = \{(0,t) : t \in [0,t^*)\}, \tag{2}$$

$$\Gamma_g = \{(x,t) : t \in [0,t^*),\ x = \xi(t)\}, \tag{3}$$

where the initial and boundary conditions are given. Function $\xi(t)$ is here un unknown function.

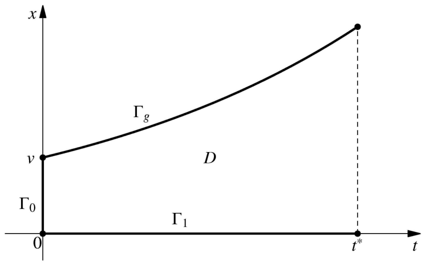

Fig. 1. Domain of the problem.

In domain D we consider the heat conduction equation:

$$\frac{\partial u(x,t)}{\partial t} = a \frac{\partial^2 u(x,t)}{\partial x^2}, \tag{4}$$

where a is the thermal diffusivity and u, t, x refer to the temperature, time and spatial location, respectively. On boundary Γ_0 the initial condition is given:

$$u(x,0) = \varphi(x), \tag{5}$$

on boundary Γ_1 the Dirichlet condition is defined:

$$u(0,t) = \vartheta(t), \tag{6}$$

and on the moving interface Γ_g the condition of temperature continuity and the Stefan condition are given:

$$u(\xi(t),t) = u^*, \tag{7}$$

$$-\lambda \left.\frac{\partial u(x,t)}{\partial x}\right|_{x=\xi(t)} = L \frac{d\xi(t)}{dt}, \tag{8}$$

where λ is the thermal conductivity, L is the latent heat of fusion per unit volume, u^* is the melting-point temperature and $\xi(t)$ is the function describing position of the moving interface Γ_g. The problem consists in finding the temperature distribution *u(x,t)* and the position of moving interface, represented by the function $\xi(t)$, which should satisfy equations (4)-(8).

According to the discussed Picard's method, we transform the heat conduction equation (4) into the following integral form:

$$u(x,t) = u(x,0) + a\int_0^t \frac{\partial^2 u(x,\tau)}{\partial x^2}\, d\tau, \quad t \in [0,t^*], \quad x \in [0,\xi(t)], \tag{9}$$

from which we receive the iterative formula:

$$u_k(x,t) = \varphi(x) + a\int_0^t \frac{\partial^2 u_{k-1}(x,\tau)}{\partial x^2}\, d\tau, \tag{10}$$

for *k=1,2,...*, where $u_0(x,t)$ is the initial approximation of the sought solution, introduced so, that it satisfies the initial condition (5) and the boundary condition (6):

$$u_0(x,t) = e^x(\vartheta(t) - \vartheta(0)) + \varphi(x). \tag{11}$$

In this way we receive the sequence $\{u_k\}_{k=0}^{\infty}$, which is convergent (under the proper assumptions - see [19]) to the exact solution of equation (4). In the paper [19] the sufficient conditions for the convergence of the Picard's iterative method are formulated. However, checking whether or not the given equation satisfies those conditions is difficult in many cases (for example in case of the problem considered in the present paper). That is why the problem of formulating (and proving) such conditions, sufficient and necessary, which would be easy to verify for any given equation, is still open.

The unknown function $\xi(t)$ we derive in the form of a linear combination:

$$\xi(t) = \sum_{i=1}^{m} p_i\, \psi_i(t), \tag{12}$$

where $p_i \in R$ and the base functions $\psi_i(t)$ are linearly independent. The coefficients p_i are selected to obtain a minimal deviation of function $u_n(x,t)$ from the condition of temperature continuity (7) and the Stefan condition (8). Thus, we are looking for the minimum of the following functional:

$$J(p_1,\ldots,p_m)=\int_0^{t^*}\left(u_n(\xi(t),t)-u^*\right)^2dt+ \\ +\int_0^{t^*}\left(\lambda\left.\frac{\partial u_n(x,t)}{\partial x}\right|_{x=\zeta(t)}+L\frac{d\xi(t)}{d\,t}\right)^2dt. \qquad (13)$$

In the course of minimizing this functional (by using the gradient method), coefficients p_i are determined, and thereby, the approximate distribution of temperature *u(x,t)* in the domain *D* and position of the moving interface $\xi(t)$ are obtained.

3. Examples

The theoretical consideration, introduced in the previous section, will be now illustrated with examples, for which the calculated approximate solutions will be compared with the known exact solutions. The values of the absolute errors will be calculated from the formulas:

$$\delta_\xi=\left(\frac{1}{t^*}\int_0^{t^*}\left(\xi_e(t)-\xi_r(t)\right)^2dt\right)^{1/2}, \qquad (14)$$

$$\delta_u=\left(\frac{1}{|D|}\iint_D\left(u_e(x,t)-u_n(x,t)\right)^2dxdt\right)^{1/2}, \qquad (15)$$

where $\xi_e(t)$ is the exact position and $\xi_r(t)$ is the reconstructed position of the moving interface, $u_e(x,t)$ is the exact distribution and $u_n(x,t)$ is the approximate distribution of temperature in the domain *D*, and $|D|=\iint_D 1\,dxdt$. The percentage relative errors will be calculated from:

$$\Delta_\xi=\delta_\xi\cdot\left(\frac{1}{t^*}\int_0^{t^*}\left(\xi_e(t)\right)^2dt\right)^{-1/2}\cdot 100\%, \qquad (16)$$

$$\Delta_u=\delta_u\cdot\left(\frac{1}{|D|}\iint_D\left(u_e(x,t)\right)^2dxdt\right)^{-1/2}\cdot 100\%. \qquad (17)$$

In the presented calculations we will use as the base functions of combination (12) the monomials:

$$\psi_i(t)=t^{i-1}, \qquad i=1,\ldots,m. \qquad (18)$$

3.1. Example 1

First example concerns the one-phase Stefan problem, in which $\lambda=1$, $s=0$, $L=\lambda/a$, $u^*=1$, $\varphi(x)=\exp(-x)$, $\vartheta(t)=\exp(a\,t)$. The calculations are made for the thermal diffusivity $a=0.1$ and $a=1$, and for the final time $t^*=0.5$ and $t^*=1$. Thus, the exact solution of considered problem is given by the functions:

$$u(x,t)=\exp(a\,t-x), \qquad (x,t)\in D, \qquad (19)$$

$$\xi(t)=a\,t, \qquad t\in[0,t^*]. \qquad (20)$$

In course of calculation (for $a=0.1$) we receive the sequence $\{u_n\}_{n=0}^{\infty}$ of approximate functions describing the distribution of temperature, with the general term of the form:

$$u_n(x,t)=\exp\left(\frac{t}{10}+x\right)-2\sinh(x)\sum_{k=0}^{n}\frac{t^k}{10^k\,k!}, \qquad n\ge 2, \qquad (21)$$

converging to the function:

$$u(x,t)=\lim_{n\to\infty}u_n(x,t)=\exp\left(\frac{t}{10}-x\right), \qquad (22)$$

which satisfies the boundary conditions (5)-(6).

Function describing the position of the moving interface we derive as the linear combination (12) of the base functions, which coefficients are determined by minimizing the functional (13). For example, for *m=4* and *n=4* we get the following approximation $\xi_m(t)$ of position of the moving interface:

$$\xi_4(t)=\text{-}0.0000542565+0.100027\,\mathrm{t}-0.0000770071\,\mathrm{t}^2+ \\ +0.0000755801\,\mathrm{t}^3.$$

Table 1. Values of error of the reconstructed position of moving interface $\xi(t)$ and temperature distribution $u(x,t)$.

	m=2	*m=4*	*m=2*	*m=4*
	n=1		*n=2*	
δ_ξ	0.03840	0.03073	0.00566	0.00419
Δ_ξ[%]	13.30252	10.64611	1.96111	1.45279
δ_u	0.04131	0.04088	0.00615	0.00614
Δ_u[%]	3.44703	3.41088	0.51290	0.51202
	n=3		*n=4*	
δ_ξ	0.00059	0.00038	0.00005	0.00003
Δ_ξ[%]	0.20598	0.13042	0.01754	0.00897
δ_u	0.00069	0.00069	0.00006	0.00006
Δ_u[%]	0.05752	0.05749	0.00525	0.00525

The errors, with which the approximate solutions (calculated for $a=1$, $t^*=0.5$, *n=1,2,3,4* and *m=2,4*) reconstruct the exact solution are compiled in Table 1.

Presented results show, that after already few iterations we get the solution with very small errors of approximation. For the longer time interval ($t^*=1$) the errors are bigger, but not significantly, but again, the errors are getting smaller with the bigger number of iterations. For example, for $t^*=1$, *n=4* and *m=4* the values of error are equal to: $\delta_\xi = 0.00210$, $\Delta_\xi = 0.36308\,[\%]$, $\delta_u = 0.00467$, $\Delta_u = 0.31499\,[\%]$, whereas for *n=5* and *m=4* the values of error reduce to: $\delta_\xi = 0.00025$, $\Delta_\xi = 0.04246\,[\%]$, $\delta_u = 0.00071$, $\Delta_u = 0.04818\,[\%]$.

Similar conclusion, about the approximation error in reconstruction of position of the moving interface, can be made basing on Figures 2 and 3. Those figures display errors in reconstructing the function $\xi(t)$, corresponding to the case of *n=5, m=4,* $\alpha=1$, $t^*=0.5$ and $t^*=1$, respectively.

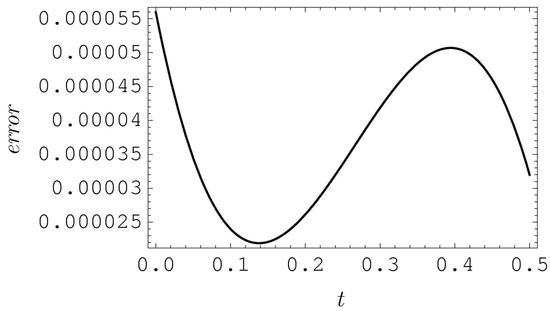

Fig. 2. Error of reconstructed position of the moving interface $\xi(t)$ (for *n=5, m=4,* $a=1$ and $t^*=0.5$).

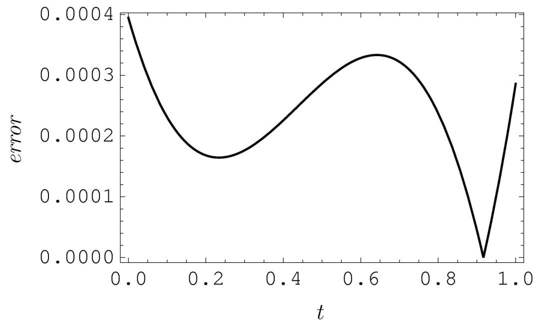

Fig. 3. Error of reconstructed position of the moving interface $\xi(t)$ (for *n=5, m=4,* $a=1$ and $t^*=1$).

Moreover, in the next figures the errors of satisfying the condition of temperature continuity (7) (Figures 4 and 5) and the Stefan condition (8) (Figures 6 and 7) by the selected approximate solutions are presented. Again, we can find the received errors as acceptable.

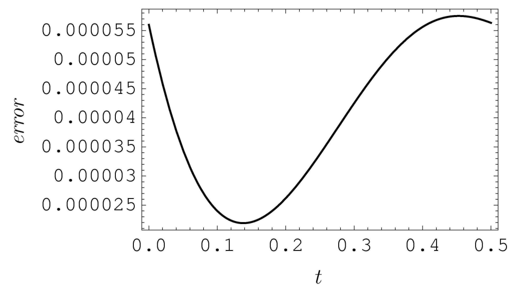

Fig. 4. Error of satisfying the condition of temperature continuity (for *n=5, m=4,* $a=1$ and $t^*=0.5$).

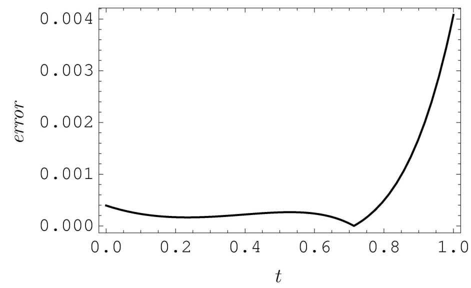

Fig. 5. Error of satisfying the condition of temperature continuity (for *n=5, m=4,* $a=1$ and $t^*=1$).

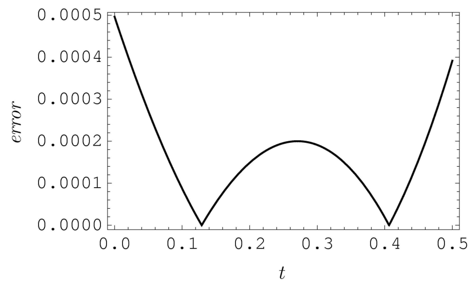

Fig. 6. Error of satisfying the Stefan condition (for *n=5, m=4,* $a=1$ and $t^*=0.5$).

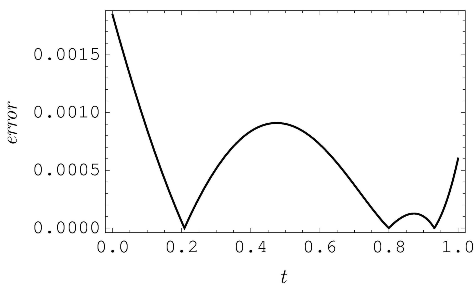

Fig. 7. Error of satisfying the Stefan condition (for *n=5, m=4,* $a=1$ and $t^*=1$).

3.2. Example 2

In the second considered example of one-phase Stefan problem we take the following values of parameters: $\lambda = 1$, $a = 1$, $L = 1$, $u^* = 0$, $t^* = 3/2$, $s = \sqrt{2} - 1$, $\varphi(x) = \exp\left(1 - 2^{-1/2}(1 + x)\right) - 1$, $\vartheta(t) = \exp\left(1 - 2^{-1/2} + t/2\right) - 1$. In such case the exact solution takes the form:

$$u(x,t) = \exp\left(1 - 2^{-1/2}(1 + x) + \frac{t}{2}\right) - 1, \qquad (x,t) \in D, \tag{23}$$

$$\xi(t) = 2^{-1/2}\left(t + 2 - \sqrt{2}\right), \qquad t \in [0, t^*]. \tag{24}$$

By following the Picard's iterative method we received the sequence of approximate functions, describing the temperature distribution $u(x,t)$, with the general term of the form:

$$u_n(x,t) = -1 + \exp\left(1 - \frac{1}{\sqrt{2}}(1+x)\right) + 64\exp\left(1 + \frac{t}{2} + x - \frac{1}{\sqrt{2}}\right) + \\ + \exp\left(1 - \frac{1}{\sqrt{2}}(1+x)\right)\sum_{k=1}^{n}\frac{t^k}{2^k k!} - 64\exp\left(1 + x - \frac{1}{\sqrt{2}}\right)\sum_{k=1}^{n}\frac{t^k}{2^k k!}.$$

One can prove, that the above function sequence converges to the function (23), representing the exact solution of the considered problem.

Reconstruction of the function $\xi(t)$, presenting position of the moving interface, was provided by minimising the functional (13) and assuming the form (12) of the sought function. Taking the monomials (18) as the base functions, we obtained, for example for *m=4* and *n=6*, the following approximation:

$$\xi_4(t) = 0.41588 + 0.70422\,t + 0.00598\,t^2 - 0.00329\,t^3.$$

Table 2 compiles the values of errors in reconstruction of the temperature distribution and position of the moving interface received for *m=4* and *n=4,5,6*. Obviously, errors are getting smaller with bigger number of iterations, but even for small number of iterations the results are satisfying.

Table 2.
Values of error of the reconstructed position of moving interface $\xi(t)$ and temperature distribution $u(x,t)$ (*m=4*).

	n=4	*n=5*	*n=6*
δ_ξ	0.02754	0.00639	0.00128
$\Delta_\xi[\%]$	2.77380	0.64307	0.12879
δ_u	0.03986	0.00928	0.00186
$\Delta_u[\%]$	6.38383	1.48653	0.29710

Figures 8 and 9 present distribution of errors, with which the approximate solution, obtained for *m=4* and *n=6*, reconstructs position of the moving interface $\xi(t)$ (Figure 8) and satisfies the condition of temperature continuity (7) (Figure 9).

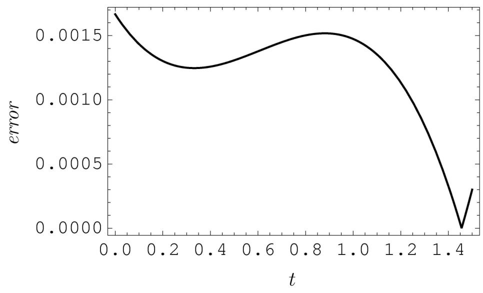

Fig. 8. Error of reconstructed position of the moving interface $\xi(t)$ (for *n=6, m=4*).

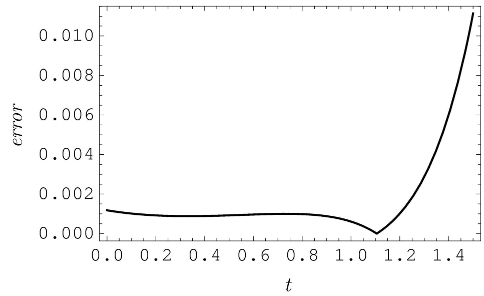

Fig. 9. Error of satisfying the condition of temperature continuity (for *n=6, m=4*).

4. Conclusions

The paper presents application of the Picard's iterative method for finding the approximate solution of one-phase Stefan problem. The proposed approach consists in determining the temperature distribution with the aid of the proper iterative formulas and calculating the coefficients of the linear combination of some base functions, approximating the position of the moving interface, in course of minimizing the properly constructed functional. Presented examples show, that the approximate solution, obtained even for small number of iterations, in satisfactory way reconstructs the sought solution, and the sequence of successive approximations, we receive in this method, is convergent to the exact solution, if it exists. In [19] the sufficient conditions of this convergence are formulated, however, they are difficult to check in most of cases (also in examples considered in the current paper). That is why the problem of formulating and proving the convergence conditions of the Picard's method, easy to verify for any equation, is still open.

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