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INTEGRABLE MIKUSIŃSKI OPERATORS AS ULTRADISTRIBUTIONS

Summary. In this paper we introduce a notion of integrable Mikusiński operators. The definition of Fourier transform of such operators is given. We also give a characterization of integrable operators which are ultradistributions.

CAŁKOWALNE OPERATORY MIKUSIŃSKIEGO JAKO ULTRADYSTRYBUCJE

Streszczenie. W artykule wprowadzamy pojęcie całkowalnych operatorów Mikusińskiego, definiujemy ich transformatę Fouriera i podajemy pewien warunek konieczny i wystarczający, aby operator tego typu był ultradystrybucją.

1. Introduction

The Mikusiński operational calculus has various applications and is also interesting from a theoretical point of view. The construction of Mikusiński operators

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is strictly algebraic. The definition of Mikusiński operators given in [7] started from the ring of continuous function on the half line $[0,\infty)$ with the usual addition and the multiplication defined as the convolution. From the famous theorem of Titchmarsh , this ring has no zero devisors, so the construction of the field of quotients is possible. Elements of this field are called Mikusiński operators. Every continuous function on $[0,\infty)$ can be treated in a natural way as a Mikusiński operator, but the field $\mathcal{F}$ of Mikusiński operators contains many other elements which are not functions. For example, the neutral element of multiplication as well as any shift operator cannot be identified with any function. Using shift operators and the integral operator we can identify any locally integrable function which vanishes to the left of some point with an operator. The space of all such functions with the convolution as multiplication is also a ring without zero devisors, so one can construct the field of quotients of this ring, but such field is isomorphic to the field of Mikusiński operators. We are using this fact and interpret Mikusiński operators as convolution quotients f/g where f and g are locally integrable functions vanishing to the left of some point.

Mikusiński operators are examples of the so-called "generalized functions". Among other examples of generalized functions are Schwartz distributions. Distributions which vanish to the left of some point can be identified with Mikusiński operators, but there are Mikusiński operators which cannot be identify with any distribution.

Ultradistributions are generalized functions which are more general than distributions. As in the case of distributions, ultradistributions vanishing to the left of some point are Mikusiński operators.

The definition of a space of ultradistributions depends on a sequence of numbers (M_n) which defines a space of test function. The aim of this paper is to give some conditions for Mikusiński operators which are ultradistributions not depending on the sequence (M_n). It is done for a class of Mikusiński operators which we call integrable operators. The main result of the paper is Theorem 10 which gives a description of integrable operators which are ultradistributions.

2. Preliminaries

In this paper by L and $\mathcal{L}$ we denote the space of integrable functions and the space of locally integrable functions on the real line, respectively. By L_0 we denote the subspaces of L which consists of all integrable functions with bounded support and by $\mathcal{L}_+$ the subspace of all locally integrable functions vanishing to the left of some point. Similarly, by L_+ we denote the subspace of all integrable functions vanishing to the left of some point. For functions $f, g \in \mathcal{L}$ the convolution is defined by

$$f * g(t) = \int_{-\infty}^{\infty} f(t-s)g(s)ds,$$

if the integral on the right-hand side is finite for almost all t. For example, the convolution is well defined if $f, g \in L$. In this case $f * g \in L$ and

$$\|f * g\| \leqslant \|f\|\|g\|.$$

The construction of Mikusiński operators is based on the fact that for $f, g \in \mathcal{L}_+$ the convolution always exists and $f * g \in \mathcal{L}_+$. This implies that $\mathcal{L}_+$ with the usual addition and the convolution as the product operation is a commutative ring. By the Titchmarsh theorem, $\mathcal{L}_+$ is a ring without zero divisors. The field of Mikusiński operators is defined as the field of quotients of this ring (see [7]).

Another important case when the convolution exist is when one of function is locally integrable and other is an integrable function with bounded support. For example, the construction of Boehmians (see [8]) is based on the existence of such convolutions.

By $\mathcal{D}$ we denote the space of all infinitely differentiable functions on the real line with compact support. This space is equipped with the natural inductive limit topology. Functionals on $\mathcal{D}$ are called distributions [11]. For each distribution x and a test function φ the convolution $x * \varphi$ can be defined in a natural way such that $x * \varphi$ is an infinitely differentiable function. For this reason, every distribution with the support bounded from the left can be considered a Mikusiński operator, namely the operator $\frac{x*\varphi}{\varphi}$, where φ is an arbitrary nonzero test function. Using the closed graph theorem it is easy to prove that an operator x is a distribution if and only if $x * \varphi$ is a continuous function for every test function φ whose support is contained in some interval P.

The notion of support cannot be defined for all operators. It can be done for the so-called regular operators introduced by T. K. Boehme in [2] (see also [3]).

Next we sketch the construction of ultradistributions. For more details see for example [1, 9] or [4].

Let $\mathbf{M} = (M_n)$ be a sequence of positive numbers such that

$$M_0 = 1, \quad M_n^2 \leqslant M_{n-1}M_{n+1} \qquad (n = 1, 2, ...). \tag{1}$$

Define

$$\mathcal{D}(\mathbf{M}) := \{\varphi \in \mathcal{D} : \quad \forall \lambda > 0 \;\; p_{\mathbf{M},\lambda}(\varphi) < \infty\},$$

where

$$p_{\mathbf{M},\lambda}(\varphi) := \sup\{\lambda^{-k} M_k^{-1} \|\varphi^{(k)}\| : \ k \in \mathbb{N}_0\}$$

and $\mathbb{N}_0 = \{0, 1, 2, \dots\}$. It is known (see for example [10, p. 376]) that $\mathcal{D}(\mathbf{M})$ contains a non-zero function if and only if

$$\sum_{k=1}^{\infty} \frac{M_{k-1}}{M_k} < \infty \tag{2}$$

and in this case $\mathcal{D}(\mathbf{M})$ is a dense subspace of $\mathcal{D}$.

Let us denote by $\mathcal{M}$ the family of all sequences $\mathbf{M}$ satisfying (1) and (2). For any $\mathbf{M} \in \mathcal{M}$ and any positive number r, by $\mathcal{D}_r(\mathbf{M})$ we denote the subspace of $\mathcal{D}(\mathbf{M})$ consisting of functions whose support is a subset of the interval $[-r, r]$. The space $\mathcal{D}_r(\mathbf{M})$ can be considered as a locally convex metric space with the topology generated by the pseudonorms $p_{\mathbf{M},\lambda}$ ($\lambda \in \mathbb{N}$). Next we can equip the space $\mathcal{D}(\mathbf{M})$ with the inductive limit topology of the spaces $\mathcal{D}_n(\mathbf{M})$. Linear and continuous functionals on this space are called **M**-ultradistributions. The notions of support of ultradistributions can be defined in the same way as for distributions. Also the convolution of an ultradistribution with a test function can be defined and the result is a continuous function. By $\mathcal{D}(\mathbf{M})'_+$ we denote the space of all **M**-ultradistributions whose support is bounded from the left.

If x is an ultradistribution and $\varphi, \psi \in \mathcal{D}(\mathbf{M})$ then

$$(x * \varphi) * \psi = (x * \psi) * \phi.$$

From this reason, if $x \in \mathcal{D}(\mathbf{M})'_+$, then for any $\varphi, \psi \in \mathcal{D}(\mathbf{M})$ we have

$$\frac{x * \varphi}{\varphi} = \frac{x * \psi}{\psi}$$

in the sens of Mikusiński operators. Thus any ultradistribution $x \in \mathcal{D}(\mathbf{M})'_+$ can be identified with the Mikusiński operator $\frac{x*\varphi}{\varphi}$.

For any function φ on the real line and any positive number λ let

$$\varphi_\lambda(t) = \lambda\varphi(\lambda t).$$

In all cases considered in this paper, when the convolution exists, we have

$$(\varphi * \psi)_\lambda = \varphi_\lambda * \psi_\lambda. \tag{3}$$

This equality allows us to define x_λ in the case when x is a Mikusiński operator. Namely, for $x = \varphi/\psi$ we define $x_\lambda = \varphi_\lambda/\psi_\lambda$. Clearly, x_λ can be defined in a natural way for all distributions and ultradistributions.

For a function $\varphi \in L$ by $\widehat{\varphi}$ we denote the Fourier transform of φ, that is

$$\widehat{\varphi}(t) = \int_{-\infty}^{\infty} e^{ist}\varphi(s)ds.$$

If f is a measurable function on $[0,\infty)$, by $\widetilde{f}$ we denote the function on $[0,\infty)$ defined by formula

$$\widetilde{f}(t) = \operatorname{ess\,sup}\{|f(s)| : s \in [0,t]\}.$$

In the case of a measurable function on $\mathbb{R}$ we define

$$\widetilde{f}(t) = \operatorname{ess\,sup}\{|f(s)| : s \in [-t,t]\},$$

for all $t \geqslant 0$.

3. Ultradistributions as Mikusiński operators

The following theorem plays an important role in the proof of Theorem 10. It can be easily obtained from [9] (Lemma 2, page 67).

Theorem 1. *Suppose that* $f \in \mathcal{L}$ *and*

$$\int_0^{\infty} \frac{\log \widetilde{f}(t)}{1+t^2} dt < \infty.$$

Then there exists a sequence $\mathbf{M} \in \mathcal{M}$ *such that*

$$f \cdot \widehat{\varphi} \in L \text{ for each } \varphi \in \mathcal{D}(\boldsymbol{M}).$$

As a consequence of the above theorem and Theorem 1 in [3] we obtain the following result.

Theorem 2. *A Mikusiński operator* x *is an ultradistribution with bounded support if and only if it admits a representation*

$$x = \frac{\varphi}{\psi}$$

such that $f(z) = \frac{\widehat{\varphi}(z)}{\widehat{\psi}(z)}$ *is an entire function satisfying*

$$\int_0^{\infty} \frac{\log |\widetilde{f}(t)|}{1+t^2} dx < \infty.$$

Definition 3. *We say that a Mikusiński operator x is integrable if there exists a dense subset A of L_+ such that $x * \psi \in L$ for all $\psi \in A$.*

Definition 4. *For an integrable operator x we define the Fourier transform of x as a function $\widehat{x}$ on the real line such that for every $t \in \mathbb{R}$:*

$$\widehat{x}(t) = \frac{\widehat{\varphi}(t)}{\widehat{\psi}(t)}, \tag{4}$$

where φ/ψ is an arbitrary representation of x such that $\widehat{\psi}(t) \neq 0$.

Since the equality

$$\frac{\varphi_1}{\psi_1} = \frac{\varphi_2}{\psi_2}$$

in the sens of Mikusiński operators means that $\varphi_1 * \psi_2 = \varphi_2 * \psi_1$, properties of the Fourier transform imply

$$\widehat{\varphi_1}(t)\widehat{\psi_2}(t) = \widehat{\varphi_2}(t)\widehat{\psi_1}(t),$$

so the Fourier transform of integrable operators is well-defined. Moreover, since the Fourier transform of an integrable function is a continuous function, the Fourier transform of any integrable operator is a continuous function.

From elementary properties of the Fourier transform in L we easily obtain the following important result.

Theorem 5. *If Mikusiński operators x and y are transformable then the operator $x * y$ is transformable and*

$$\widehat{x * y} = \widehat{x} \cdot \widehat{y}.$$

Here are some examples of integrable operators:

(a) An arbitrary function $\varphi \in L_+$ is an integrable Mikusiński operator.

(b) A Mikusiński operator with bounded support is an integrable operator.

(c) If $\mathbf{M} = (M_n) \in \mathcal{M}$, then the distribution

$$\sum_{n=1}^{\infty} \frac{1}{M_n} \delta^{(n)}(t - n)$$

is an example of distribution of infinite order which is an integrable operator.

(d) From the Wiener theorem about shifting of integrable functions it follows that, if $\psi \in L_+$ and $\widehat{\psi}(t) \neq 0$ for all $t \in \mathbb{R}$, then the operator $1/\varphi$ is integrable. Consequently, any operator φ/ψ with $\varphi, \psi \in L_+$ and $\widehat{\psi}(t) \neq 0$ for all $t \in \mathbb{R}$ is an integral operator.

Before we proceed to prove the main theorem we need to recall some definitions and some known facts and formulate some technical lemmas.

In the field of Mikusiński operators three types of convergence are considered. For the purpose of this paper it will be convenient to consider the so-called type II convergence (see [7]). We say that a sequence (x_n) of Mikusiński operators is convergent to an operator x if there exists sequences (φ_n) and (ψ_n) in $\mathcal{L}_+$ and functions $\varphi, \psi \in \mathcal{L}_+$ such that $x = \varphi/\psi$, all functions ψ_n have the supports bounded from the left by some common point, and

$$x_n = \frac{\varphi_n}{\psi_n},$$

where $\varphi_n \to \varphi$ and $\psi_n \to \psi$ in $\mathcal{L}$.

The following lemma is obvious.

Lemma 6. *For each Mikusiński operators x, y the function*

$$\lambda \ni (0, \infty) \mapsto x * y_\lambda \in \mathcal{F}$$

is continuous.

A function $f \in L$ can be considered in a natural way as a linear and continuous functional on the Banach space $\mathcal{C}_0$ of continuous functions on the real line vanishing at infinity. The usual norm of a function $f \in L$ is the same as the norm of f as a functional on $\mathcal{C}_0$. Because the space $\mathcal{C}_0$ is separable, each bounded sequence (f_n) in L contains a subsequence convergent in the space $\mathcal{C}_0'$ to some measure μ. It is clear that, if the measure μ is determined be an integrable function f, than $\|f\| \leqslant \liminf_{n\to\infty} \|f_n\|$.

In the next part of the paper by week convergence in L we mean the convergence in the space $\mathcal{C}_0'$ (note, that the defined week convergence in L implies week convergence in $\mathcal{C}_0'$, but they are not equivalent). All functionals on $\mathcal{C}_0$ are distributions and convergence in $\mathcal{C}_0$ implies convergence in the distributional sense and hence also in the field of Mikusiński opertators.

Lemma 7. *Let x be a Mikusiński operator. If $\varphi \in \mathcal{L}_+$ is such that $x * \varphi_\lambda \in L$ for all numbers λ in some interval $P \subset (0, \infty)$, then there exists an interval $Q \subset P$ such that the map*

$$Q \ni \lambda \mapsto x * \varphi_\lambda \in L \tag{5}$$

is weekly continuous and bounded.

Proof. Consider the set

$$A_m = \{\lambda \in P : \ \|x * \varphi_\lambda\| \leqslant m\}.$$

Suppose that (λ_n) is a sequence of elements of A_m and $\lambda_n \to \lambda$ for some $\lambda \in P$. Denote $f_n = x * \varphi_{\lambda_n}$. Since the sequence (f_n) is bounded in L, it has subsequence convergent in $\mathcal{C}_0'$ to some measure μ. From Lemma 6 and the remarks following that lemma, we have $\mu = x * \varphi_\lambda$. This shows that (f_n) has a subsequence weakly convergent to $x * \varphi_\lambda$. The same can done for each subsequence of (f_n), from which we conclude that $x * \varphi_{\lambda_n} \to x * \varphi_\lambda$. Therefore, A_m is a closed subset of P. From the assumptions of the lemma, we have $\bigcup_{n=1}^\infty A_m = P$. Consequently, A_m contains an interval Q, by the Baire category theorem. □

Corollary 8. *Under the assumptions of Lemma 6 we can conclude that there exists a nonzero function $\psi \in \mathcal{L}_+$ vanishing to the left of* 0 *and some real number $b \in [1, \infty)$ such that for each $\lambda \in [1, b]$ and some M:*

$$x * \psi_\lambda \in L \text{ and } \|x * \psi_\lambda\| \leqslant M.$$

Lemma 9. *Suppose that φ is a measurable nonnegative function on the interval $[0, \infty)$ such that*

$$\int_0^\infty \frac{\varphi(t)}{1+t^2} dt < \infty.$$

Let $\varepsilon \in (0,1)$ and define

$$\psi(t) = \operatorname{ess\,inf}\{\varphi(s) : \ s \in [\varepsilon t, t]\}.$$

Then

$$\int_0^\infty \frac{\widetilde{\psi}(t)}{1+t^2} dt < \infty.$$

Proof. It is sufficient to prove that the integral $\displaystyle\int_1^\infty \widetilde{\psi}(t) t^{-2} dt$ is convergent. Since for each positive numbers t we have

$$\widetilde{\psi}(t) \leqslant \varphi(t) + \widetilde{\psi}(\varepsilon t),$$

for any $T > 0$ we have

$$\begin{aligned}\int_1^{\varepsilon T} \widetilde{\psi}(t)t^{-2}dt &\leqslant \int_1^{T} \widetilde{\psi}(t)t^{-2}dt \leqslant \int_1^{\varepsilon T} \varphi(t)t^{-2}dt + \int_1^{T} \widetilde{\psi}(\varepsilon t)t^{-2}dt \\ &\leqslant \int_1^{\infty} \varphi(t)t^{-2}dt + \varepsilon \int_{\varepsilon}^{\varepsilon T} \widetilde{\psi}(t)t^{-2}dt.\end{aligned}$$

From this we obtain

$$\int_1^{\varepsilon T} \widetilde{\psi}(t)t^{-2}dt \leqslant \frac{1}{1-\varepsilon}\int_1^{\infty} \varphi(t)t^{-2}dt,$$

which implies convergence of the considered integral. □

Now we are ready to prove the main result of the paper.

Theorem 10. *Let x be an integrable Mikusiński operator. Then x is an ultradistribution if and only if there exists a nonzero function $\varphi \in L_+$ and some interval P such that $x * \varphi_\lambda \in L$ for each $\lambda \in P$.*

Proof. By Corollary 8 we can assume that $P = [1, b]$ for some $b > 1$ and that there exists $M > 0$ such that $\|x * \varphi_\lambda\| \leqslant M$ for each $\lambda \in P$. In particular, $\widehat{x * \varphi_\lambda}(t) \leqslant M$ for all $t \in \mathbb{R}$.

Denote by Φ the Fourier transform of x. Because $\widehat{x * \varphi_\lambda} = \Phi \cdot \widehat{\varphi_\lambda}$, we have

$$|\Phi(t)| \leqslant \frac{M}{|\widehat{\varphi_\lambda}(t)|}. \tag{6}$$

for each $\lambda \in P$ and $t \in \mathbb{R}$.

It is well known that the Fourier transform of an integrable function vanishing to the left of 0 can be extended to the closed upper half of the complex plane. The extension is a continuous bounded function $F(z)$ that is analytic on the open upper half plane and such that

$$\int_{-\infty}^{\infty} \frac{|\log|F(t)||}{1+t^2}dt < \infty.$$

Hence, for each $\lambda \in [1, b]$ the function f_λ defined on the interval $[0, \infty)$ by the formula

$$f_\lambda(t) = \frac{M}{|\widehat{\varphi_\lambda}(t)|} + \frac{M}{|\widehat{\varphi_\lambda}(-t)|}$$

satisfies the inequality

$$\int_0^{\infty} \frac{|\log|f_\lambda(t)||}{1+t^2} < \infty. \tag{7}$$

Let f be a function defined on $[0,\infty)$ by the formula

$$f(t)=\inf\{f_\lambda(t):\lambda\in[1,b]\}. \tag{8}$$

Since $\widehat{\varphi_\lambda}(t)=\widehat{\varphi}(\lambda^{-1}t)$ for each $\lambda>0$, we have $\widehat{\varphi_\lambda}(t)=\widehat{\varphi}(\lambda^{-1}t)$. Thus, for any $s\in[1/b,1]$ we have

$$f_1(st)=\frac{M}{|\widehat{\varphi}(st)|}+\frac{M}{|\widehat{\varphi}(-st)|}=f_{\frac{1}{s}}(t). \tag{9}$$

By (8) and (9) we can write

$$f(t)=\inf\{f_1(st):s\in[1/b,1]\}.$$

By Lemma 9 we have

$$\int_0^\infty\frac{\widetilde{f}(t)}{1+t^2}dt<\infty. \tag{10}$$

From (6) and the definitions of f and f_λ we conclude that

$$\Phi(t)\leqslant\widetilde{f}(|t|)$$

for all $t\in\mathbb{R}$. This completes the proof in view of Theorem 1. □

Remark 11. A condition similar to that in the above theorem appears in the definition of strong Boehmians (see [5]). An extension of the method used in this paper to Boehmians will be considered in a forthcoming paper by the author.

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Omówienie

W artykule omawia się związki pomiędzy operatorami Mikusińskiego i innego rodzaju funkcjami uogólnionymi. Wprowadzono pojęcie całkowalnego operatora Mikusińskiego, zdefiniowano transformatę Fouriera operatorów całkowalnych i omówiono jej podstawowe własności. Główny wynik artykułu dotyczy pewnego warunku wystarczającego na to, aby operator całkowalny był ultradystrybucją.