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DARBOUX PROPERTY OF THE NONATOMIC σ-ADDITIVE POSITIVE AND FINITE DIMENSIONAL VECTOR MEASURES

Summary. In the paper some facts connected with Darboux property of the positive measures and the finite dimensional vector measures are discussed.

WŁASNOŚĆ DARBOUX σ-ADDYTYWNYCH NIEUJEMNYCH MIAR BEZATOMOWYCH I SKOŃCZENIE WYMIAROWYCH MIAR WEKTOROWYCH

Streszczenie. W artykule omawiane są pewne fakty związane z własnością Darboux miar nieujemnych i miar wektorowych skończenie wymiarowych.

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1. Introduction

The immediate cause of preparing this paper was the interest in the classic problem of cake cutting. We may consider Polish mathematicians, Steinhaus, Banach and Knaster, as the creators of this problem (see [11, 19]). This subject matter is still very inspiring which is evidenced by rich literature (we give only a selected set of respective papers [1, 10, 11, 15]).

Let E be a nonempty set. Let us denote by $\mathcal{P}(E)$ the family of all subsets of E. Let $\mathfrak{M}$ be a σ-algebra of subsets of nonempty set Ω. Let us denote by μ a positive, σ-additive measure on $\mathfrak{M}$.

Definition 1. *A set $A \in \mathfrak{M}$ is an atom of measure μ if $\mu(A) > 0$ and if $B \subset A$, $B \in \mathfrak{M}$, then either $\mu(B) = 0$ or $\mu(B) = \mu(A)$.*

Definition 2. *A set $A \in \mathfrak{M}$ is called to be atomless (with respect to measure μ) if neither A nor any of its μ-measurable subsets is the atom. A $\mathfrak{M}$-measurable set A, which is atomless with respect to μ, will be called the μ-atomless.*

By the last definition we get the following lemma.

Lemma 3. *If set $A \in \mathfrak{M}$ is μ-atomless, then any $\mathfrak{M}$-measurable subset of A is also μ-atomless.*

Lemma 4. *If set $A \in \mathfrak{M}$ is μ-atomless and $\mu(A) > 0$ then there exists a sequence of μ-measurable sets $\{B_n\}_{n\geqslant 1}$ such that $B_{n+1} \subset B_n \subset A$, $\mu(B_n) > 0$ and $\lim\limits_{n} \mu(B_n) = 0$.*

Proof. If $\mu(A) = +\infty$ then because set A is μ-atomless we get that there exists the μ-measurable set $A' \subset A$ such that $0 < \mu(A') < \mu(A)$. Obviously the set A' is also μ-atomless. Thus we may assume that $\mu(A) < +\infty$.

Since set A is atomless then there exists a μ-measurable set $B \subset A$ such that $\mu(A) > \mu(B) > 0$. A set $A \backslash B$ is also μ-measurable and $\mu(A \backslash B) = \mu(A) - \mu(B) > 0$. From equality $\mu(A) = \mu(B) + \mu(A \setminus B)$ we get that at least one of sets B or $A \setminus B$ possesses the measure no greater than $\frac{1}{2}\mu(A)$. We denote this set by B_1. So we know that $B_1 \in \mathfrak{M}$ and $\mu(B_1) \leqslant \frac{1}{2}\mu(A)$.

Assume that the μ-measurable set $B_n \subset A$ is already constructed for some $n \in \mathbb{N}$, where $0 < \mu(B_n) \leqslant \frac{1}{2^n}\mu(A)$. A set B_n, as a subset of atomless set, is atomless as well, so there exists a μ-measurable set $B'_n \subset B_n$ such that $\mu(B_n) > \mu(B'_n) > 0$. Also a set $B_n \setminus B'_n$ is measurable and $\mu(B_n \setminus B'_n) = \mu(B_n) - \mu(B'_n) > 0$. From equality $\mu(B_n) = \mu(B'_n) + \mu(B_n \setminus B'_n)$ we get that at least one of sets B'_n or $B_n \setminus B'_n$ possesses the measure no greater than $\frac{1}{2}\mu(B_n)$. We denote this set by B_{n+1}. Of course $\mathfrak{M} \ni\ B_{n+1} \subset B_n$ and $\mu(B_{n+1}) \leqslant \frac{1}{2}\mu(B_n) \leqslant \frac{1}{2^{n+1}}\mu(A)$. Applying the Axiom of Countable Dependent Choices we finish the proof. $\square$

Theorem 5. *If set $A \in \mathfrak{M}$ is μ-atomless and $\mu(A) > 0$, then for any $\alpha \in \big(0, \mu(A)\big)$ there exists $B \in \mathfrak{M},\ B \subset A$ such that $\mu(B) = \alpha$.*

Proof. Let $\alpha \in \big(0, \mu(A)\big)$. We create the auxiliary sequences – of $\mathfrak{M}$-measurable sets $\{B_n\}$ and of positive numbers $\{\beta_n\}$ in the following way.

By Lemma 4 there exists the μ-measurable set $B \subset A$ such that $0 < \mu(B) < \alpha$. Let us define

$$B_1 := B,\ \beta_1 := \sup\{\mu(D) :\ D \in \mathfrak{M},\ B_1 \subseteq D \subseteq A,\ \mu(D) \leqslant \alpha\}.$$

We choose a set $B_2 \in \mathfrak{M}$ such that $B_1 \subset B_2 \subset A$ and $\beta_1 - \frac{1}{2} \leqslant \mu(B_2) \leqslant \beta_1$. Having a specified set $B_n \in \mathfrak{M}$, for some $n \in \mathbb{N}$, we define

$$\beta_n := \sup\{\mu(D) :\ D \in \mathfrak{M},\ B_n \subseteq D \subseteq A,\ \mu(D) \leqslant \alpha\},$$

and we choose set $B_{n+1} \in \mathfrak{M}$ such that the following two conditions are satisfied: $B_n \subset B_{n+1} \subset A$ and $\beta_n - \frac{1}{2^n} \leqslant \mu(B_{n+1}) \leqslant \beta_n$. Since sequence $\{B_n\}$ is increasing, therefore $\lim\limits_n \mu(B_n) = \mu\Big(\bigcup\limits_{n\in\mathbb{N}} B_n\Big)$. We get also $\lim\limits_n \beta_n = \mu\Big(\bigcup\limits_{n\in\mathbb{N}} B_n\Big)$ (indeed, from inequality $\beta_n - \frac{1}{2^n} \leqslant \mu(B_{n+1}) \leqslant \beta_n$, $n \in \mathbb{N}$, we get that

$$\limsup_n \beta_n \leqslant \lim_n \mu(B_n) \leqslant \liminf_n \beta_n$$

which implies $\limsup\limits_n \beta_n = \liminf\limits_n \beta_n$, what means that the sequence $\{\beta_n\}$ is convergent and $\lim\limits_n \beta_n = \lim\limits_n \mu(B_n)$). Therefore, since there is always $\beta_n \leqslant \alpha$ we get that $\mu\Big(\bigcup\limits_{n\in\mathbb{N}} B_n\Big) \leqslant \alpha$.

Suppose that $\mu\Big(\bigcup\limits_{n\in\mathbb{N}} B_n\Big) < \alpha$. Let us define $C := A \setminus \bigcup\limits_{n\in\mathbb{N}} B_n$. Then $\mu(C) > 0$ and C, as a subset of the μ-atomless set is μ-atomless as well.

According to Lemma 4 there exists the μ-measurable set $C_0 \subset C$ such that $\mu(C_0) > 0$ and $\alpha > \mu(C_0) + \mu\Big(\bigcup\limits_{n\in\mathbb{N}} B_n\Big)$.

Let us note that the following inclusions $B_m \subseteq C_0 \cup \bigcup_{n\in\mathbb{N}} B_n \subseteq A$ are satisfied for $m \in \mathbb{N}$ and because $\mu\Big(C_0 \cup \bigcup_{n\in\mathbb{N}} B_n\Big) < \alpha$, then from definition of numbers β_n we have $\mu\Big(C_0 \cup \bigcup_{n\in\mathbb{N}} B_n\Big) \leqslant \beta_n$, $n \in \mathbb{N}$ which implies $\mu\Big(C_0 \cup \bigcup_{n\in\mathbb{N}} B_n\Big) \leqslant \lim_n \beta_n$ and we obtain the contradiction. Therefore $\mu\Big(\bigcup_{n\in\mathbb{N}} B_n\Big) = \alpha$. □

Historical remark. Theorem 5 was proved in the first independently by Fichtenholz and Sierpiński (see [18, remark to problem 12]).

Corollary 6. *Let $A \in \mathfrak{M}$ be the same as in assumptions of the above theorem. Then there exists a $\mathfrak{M}$-measurable partition $\{A_n\}_{n\in\mathbb{N}}$ of set A such that*

$$\forall\alpha \in \big(0, \mu(A)\big)\ \exists\{A_{n_i}\}_{i\in\mathbb{N}}\ : \mu\Big(\bigcup_{i\in\mathbb{N}} A_{n_i}\Big) = \alpha.$$

Proof. It is sufficient to note that (see Lemma 7 below) if $\mu(A) < +\infty$ then there exists the $\mathfrak{M}$-measurable partition $\{A_n\}_{n\in\mathbb{N}}$ of set A such that

$$\mu(A_n) \leqslant \sum_{i\geqslant n+1} \mu(A_i),\ n \in \mathbb{N}.$$

Indeed, by Theorem 5 there exists $A_1 \in \mathfrak{M},\ A_1 \subset A$ such that $\mu(A_1) = \frac{1}{2}\mu(A)$. The remaining sets are defined by the Axiom of Dependent Choices and on the basis of Theorem 5 such that

$$A_n \in \mathfrak{M},\ A_n \subset A \setminus \bigcup_{i=1}^{n-1} A_i,\ \mu(A_n) = \frac{1}{2}\mu\big(A \setminus \bigcup_{i=1}^{n-1} A_i\big),\ n \in \mathbb{N},\ n > 1.$$

□

Lemma 7 ([16]). *Assume that $\sum a_n$ is a convergent series with nonnegative terms such that $a_n \leqslant \sum_{i\geqslant n+1} a_i$, $n \in \mathbb{N}$. Then for every $\alpha \in \big(0, \sum a_n\big)$ the subseries $\sum a_{n_i}$ exist, sum of which is equal to α.*

Remark 8. Lemma 7 was also used in discussion of some facts in paper [23]. It is worth to note that this result is an important part of contemporary discussed problem concerning the description of subsums of given convergent series with positive terms [3].

Remark 9. Moreover, let us note that not only the discussed here atomless measures have the interesting applications. In contrast, it is proven in papers [13, 17, 21] that the following statements are equivalent.

(i) $L^p(\mu) \subseteq L^q(\mu)$ for some pair $p, q \in (0, \infty)$ with $p < q$.

(ii) There exists a constant $m > 0$ such that $\mu(E) \geqslant m$ for every μ-non-null set $E \in \mathfrak{M}$.

(iii) $L^p(\mu) \subseteq L^q(\mu)$ for every pair $p, q \in (0, \infty)$ with $p \leqslant q$.

We note that condition (ii) is equivalent to the statement saying that there exists $m > 0$ such that each μ-non-null set $E \in \mathfrak{M}$ contains some μ-atom $E' \in \mathfrak{M}$ with $\mu(E') \geqslant m$.

Next theorem is our main result and it seems that it may have many different applications (also technical).

Theorem 10. *Let $(\Omega, \mathfrak{M})$ be a measurable set and let $\mu_1, \ldots, \mu_n$ be the nonnegative and σ-additive measures on $\mathfrak{M}$. Suppose that the following condition is satisfied:*

$$\begin{aligned}&\textit{If } E \in \mathfrak{M} \textit{ and } 0 < \mu_1(E) = \ldots = \mu_n(E) < +\infty \\ &\textit{then for every } \alpha \in \big(0, \mu_1(E)\big) \textit{ there exists } F \in \mathfrak{M} \cap \mathcal{P}(E) \\ &\textit{such that } \mu_1(F) = \ldots = \mu_n(F) = \alpha.\end{aligned} \tag{1}$$

Then there exists a family of sets $V_r \in \mathfrak{M}$, $r \in [0, \mu_1(E)]$ with the following properties:

$$\begin{cases} V_0 = \emptyset, \; V_{\mu_1(E)} = E, \\ \mu_1(V_r) = \ldots = \mu_n(V_r) = r, \\ V_r \subset V_{r'} \iff r \leqslant r', \\ V_{r'} = \bigcup\limits_{r<r'} V_r. \end{cases} \tag{2}$$

Besides, for each nonnegative and σ-additive measure μ on $\mathfrak{M}$ the function

$$[0, \mu_1(E)] \ni r \overset{f}{\longmapsto} \mu(V_r)$$

is left-continuous. If additionally $\mu(V_{\mu_1(E)}) < \infty$ and μ is absolutely continuous with respect to one of the measures μ_i, then function f is continuous on interval $[0, \mu_1(E)]$.

Proof. Replacing μ_j with $\frac{\mu_j}{\mu_j(E)}$ we can assume, without loss of generality, that $\mu_j(E) = 1$ for every $j = 1, \ldots, n$. First we define the sets $V^*_{i2^{-n}}$, for $i = 0, 1, \ldots, 2^n$ and $n \in \mathbb{N}$, by induction on $n \in \mathbb{N}_0$.

Let us set $V^*_0 = \emptyset$ and $V^*_1 = E$. Next, let us suppose that sets $V^*_{i2^{-n}}$ have been defined for all $n = 0, 1, \ldots, k$ and $i = 0, 1, \ldots, 2^n$ such that

$$\mu_1(V^*_{i2^{-n}}) = \ldots = \mu_n(V^*_{i2^{-n}}) = i2^{-n},$$

and

$$V^*_{i2^{-m}} \subset V^*_{j2^{-n}} \iff i2^{-m} \leqslant j2^{-n}.$$

Then we have

$$\mu_1(E_i) = \ldots = \mu_n(E_i) = 2^{1-k}$$

for every odd index i and sets

$$E_i := V^*_{\frac{1}{2}(i+1)2^{1-k}} \setminus V^*_{\frac{1}{2}(i-1)2^{1-k}}.$$

By (1) there exists the $\mathfrak{M}$-measurable set $F_i \subset E_i$ such that

$$\mu_1(F_i) = \ldots = \mu_n(F_i) = 2^{-k}.$$

Let us put $V^*_{i2^{-k}} = F_i \cup V^*_{\frac{1}{2}(i-1)2^{1-k}}$ for all odd i, $0 < i < 2^k$. Then

$$\mu_j(V^*_{i2^{-k}}) = \mu_j(F_i) + \mu_j(V^*_{\frac{1}{2}(i-1)2^{1-k}}) = 2^{-k} + \frac{1}{2}(i-1)2^{1-k} = i2^{-k}$$

for every $j = 1, \ldots, n$ and $V^*_{i2^{-k}} \subset V^*_{(i+1)2^{-k}}$ for each $i = 0, 1, \ldots, 2^k - 1$. At last, by the Principle of Mathematical Induction the sets $V^*_{i2^{-k}}$ are defined for each $k \in \mathbb{N}$ and $i = 0, 1, \ldots, 2^k$.

Now we set

$$V_r = \bigcup_{i2^{-n} \leqslant r} V^*_{i2^{-n}}$$

for every $r \in (0, 1]$. Moreover let $V_0 = \emptyset$. We can easily verify that

$$\mu_j(V_r) = \sup_{i2^{-n} \leqslant r} \{\mu_j(V^*_{i2^{-n}})\} = \sup_{i2^{-n} \leqslant r} \{i2^{-n}\} = r$$

and the sets V_r, $r \in [0, 1]$, possess all other properties from (2). Now let μ be a positive measure on $\mathfrak{M}$. We want to prove that the function $[0, \mu_1(E)] \ni r \overset{f}{\longmapsto} \mu(V_r)$ is continuous. Let us take $r_n, r' \in [0, \mu_1(E)]$, $n \in \mathbb{N}$, such that $r_n \nearrow r'$. Then $\mu(V_{r'}) = \mu\Big(\bigcup_{n\in\mathbb{N}} V_{r_n}\Big) = \lim_{n\to\infty} \mu(V_{r_n})$ which implies that $\mu(V_{r'}) = \sup_{r<r'} \mu(V_r)$, i.e.

f is left-continuous. If $\mu(V_{\mu_1(E)}) < \infty$ then for each sequence $\{r_n\} \subset [0, \mu_1(E)]$, $r_n \searrow r'$ we obtain

$$\lim_{n\to\infty} \mu(V_{r_n}) = \mu\Big(\bigcap_{n\in\mathbb{N}} V_{r_n}\Big) = \mu\Big(V_{r'} \cup \Big(\bigcap_{n\in\mathbb{N}} V_{r_n} \setminus V_{r'}\Big)\Big). \tag{3}$$

Since

$$\mu_j\Big(\bigcap_{n\in\mathbb{N}} V_{r_n} \setminus V_{r'}\Big) = \mu_j\Big(\bigcap_{n\in\mathbb{N}} V_{r_n}\Big) - \mu_j(V_{r'}) =$$
$$= \lim_{n\to\infty} \mu_j(V_{r_n}) - \mu_j(V_{r'}) = r' - \mu_j(V_{r'}) = 0,$$

for every $j = 1, \dots, n$, so if μ is absolutely continuous with respect to one of measures μ_j, then by (3) we get

$$\lim_{n\to\infty} \mu(V_{r_n}) = \mu(V_{r'})$$

which implies that $\mu(V_{r'}) = \inf_{r'<r} \mu(V_r)$ and f is also the right-continuous function. The proof is finished. $\square$

Theorem 11. *Let $(\Omega, \mathfrak{M})$ be a measurable space and $\mu_1, \dots, \mu_n$ be the nonnegative σ-additive measures on $\mathfrak{M}$. Assume that $0 < \mu_1(E) = \ldots = \mu_n(E) < \infty$ for some $E \in \mathfrak{M}$. Then if E is a set which is atomless with respect to any of measures $\mu_1, \dots, \mu_n$, then for every $\alpha \in \big(0, \mu_1(E)\big)$ there exists $F \in \mathfrak{M} \cap \mathcal{P}(E)$ such that $\mu_1(F) = \ldots \mu_n(F) = \alpha$.*

Proof. Case for $n = 1$ is proved by Theorem 5. Suppose now that theorem is true for every n nonnegative measures on $\mathfrak{M}$ and let $\mu_1, \dots, \mu_n, \mu$ be the nonnegative measures on $\mathfrak{M}$ such that $\mu_1(E) = \ldots = \mu_n(E) = \mu(E) > 0$ for some $E \in \mathfrak{M}$, whereby $\mu_1, \dots, \mu_n, \mu$ are atomless on E.

Replacing, if necessary, measure μ_1 by measure $\frac{1}{2}(\mu_1 + \mu)$ we may assume that μ is absolutely continuous with respect to measure μ_1.

We prove that for every $r \in \big(0, \mu_1(E)\big)$ there exists $F \in \mathfrak{M} \cap \mathcal{P}(E)$ such that $\mu_1(F) = \ldots \mu_n(F) = \mu(F) = r$. First, we consider the case for $r = \frac{1}{m}\mu_1(E)$, $m \in \mathbb{N}$. By the induction hypothesis we can divide E into $\mathfrak{M}$-measurable subsets $E_1, \dots, E_m$ such that $\mu_j(E_i) = r$, $j = 1, \dots, n$ and $i = 1, \dots, m$.

Suppose that $\mu(E_i) \neq r$ for all indices i. After the possible renumbering we may assume that $\mu(E_1) < r < \mu(E_2)$. By Theorem 10 we can construct sets $V_t \in \mathfrak{M} \cap \mathcal{P}(E_1)$ and $W_t \in \mathfrak{M} \cap \mathcal{P}(E_2)$, $t \in [0, r]$, with the following properties

$$\begin{cases} V_0 = W_0 = \emptyset, \quad V_r = E_1, W_r = E_2, \\ V_t \subset V_{t'} \ \wedge \ W_k \subset W_{k'} \iff t \leqslant t' \ \wedge \ k \leqslant k', \\ \mu_j(V_t) = \mu_j(W_t) = t, \ j = 1, \dots, n. \end{cases}$$

Let us put $G(t) = \mu(V_t \cup W_{r-t})$, $t \in [0, r]$. From Theorem 10 we get that G is a continuous function and because $G(0) = \mu(W_r) = \mu(E_2) > r$ and $G(r) = \mu(V_r) = \mu(E_1) < r$ then there exists $t_0 \in (0, r)$ such that $G(t_0) = r$. Thus

$$\mu(V_{t_0} \cup W_{r-t_0}) = r = t_0 + (r - t_0) = \mu_j(V_{t_0} \cup W_{r-t_0}).$$

Let us now consider the more general case with number $r \in \big(0, \mu_1(E)\big)$. Let n_1 be the smallest natural number such that $n_1^{-1}\mu(E) < r$. As shown above, there exists $X_1 \in \mathfrak{M} \cap \mathcal{P}(E)$ such that $\mu(X_1) = \mu_j(X_1) = n_1^{-1}\mu(E)$ for $j = 1, \dots, n$. Then it will be also $\mu(E \setminus X_1) = \mu_j(E \setminus X_1)$ for every j. Let n_2 be the smallest natural number such that $n_2^{-1}\mu(E \setminus X_1) < r - \mu(X_1)$. As above, there exists $X_2 \in \mathfrak{M} \cap \mathcal{P}(E \setminus X_1)$ such that $\mu(X_2) = \mu_j(X_2) = n_2^{-1}\mu(E \setminus X_1)$, $j = 1, \dots, n$. In addition let us note that $n_1^{-1}\mu(E) \geqslant \frac{r}{2}$ and $n_2^{-1}\mu(E \setminus X_1) \geqslant \frac{1}{2}\big(r - \mu(X_1)\big)$. Continuing the algorithm of selecting the sets X_i we get in result the sequence of sets $\{X_i\}$, $\mathfrak{M}$-measurable, pairwise disjoint and such that

$$\mu(X_i) = \mu_j(X_i) = n_i^{-1}\mu\Big(E \setminus \bigcup_{k=1}^{i-1} X_k\Big) \geqslant \frac{1}{2}\big(r - \mu\Big(\bigcup_{k=1}^{i-1} X_k\Big)\big)$$

for $j = 1, \dots, n$, which implies that for $F = \bigcup\limits_{i \in \mathbb{N}} X_i$ we get $\mu(F) = \mu_j(F) = r$ for every $j = 1, \dots, n$. □

Theorem 12. *Let $\mathfrak{M}$ be a σ-algebra of subsets of set $\Omega \neq \emptyset$ and $\mu_i : \mathfrak{M} \to \mathbb{R}$, $i = 1, \dots, n$, be the σ-additive measures. If $0 < \mu_1(E) = \mu_2(E) = \ldots = \mu_n(E)$ for some $E \in \mathfrak{M}$ and set E is atomless with respect to measures μ_i, then for every $r \in \big(0, \mu_1(E)\big)$ there exists the $\mathfrak{M}$-measurable subset $F \subset E$ such that $\mu_1(F) = \ldots = \mu_n(F) = r$.*

Proof. Let us put

$$\mu(A) = 2\sum_{i=1}^{n} |\mu_i|(A), \quad \nu_i(A) = \mu_i(A) + \mu(A), \ i = 1, \dots, n,$$

for $A \in \mathfrak{M}$, where $|\mu_i|$ is the total variation of measure μ_i. It is easily to check that μ and ν_i, $i = 1, \ldots, n$, are simultaneously the nonnegative, finite, σ-additive and atomless measures. Since $\nu_1(E) = \ldots = \nu_n(E) > 0$ then by Theorem 11 and Theorem 10 there exist the sets $V_t \in \mathfrak{M} \cap \mathcal{P}(E)$ for $t \in [0, \nu_1(E)]$ such that

$$V_0 = \emptyset, V_{\nu_1(E)} = E,$$
$$\nu_1(V_t) = \ldots = \nu_n(V_t) = t,$$
$$V_{t'} \subset V_t \iff t' \leqslant t.$$

From inequality $\nu_j \geqslant \sum_{i=1}^{n} |\mu_i|$, for $j = 1, \ldots, n$, we get that measure μ is absolutely continuous with respect to any measure ν_j. Hence, by Theorem 10 the function $t \longmapsto \mu(V_t)$ is continuous in interval $[0, \nu_1(E)]$. Now, from equalities $\mu_i(V_t) = \nu_i(V_t) - \mu(V_t) = t - \mu(V_t)$ and

$$\mu_i(V_{\nu_1(E)}) = \nu_1(E) - \mu(V_{\nu_1(E)}) = \nu_1(E) - \mu(E) = \mu_1(E)$$

for $i = 1, \ldots, n$, and from the Darboux property for continuous functions we may conclude that for every $r \in \big(0, \mu_1(E)\big)$ there exists $t(r) \in \big(0, \nu_1(E)\big)$ such that

$$\mu(V_{t(r)}) = r, \ i = 1, \ldots, n.$$

$\square$

Corollary 13 ([5]). *Let $(\Omega, \mathfrak{M})$ be a measurable space, let μ be a nonnegative σ-additive measure on set $\mathfrak{M}$ and let $f_1, \ldots, f_n \in L_1(\Omega, \mathfrak{M}, \mu)$ be the nonnegative functions. Suppose also that $E \in \mathfrak{M}$ and μ is atomless on E. If $\int\limits_E f_1\, d\mu = \ldots = \int\limits_E f_n\, d\mu > 0$, then for every number $r \in \big(0, \int\limits_E f_j\, d\mu\big)$ there exists a set $F \in \mathfrak{M}$, $F \subset E$, such that $r = \int\limits_F f_1\, d\mu = \ldots = \int\limits_F f_n\, d\mu$.*

Proof. If $f \in L_1(\Omega, \mathfrak{M}, \mu)$ then integral $\int\limits_F f \mathrm{d}\mu$, treated as a function of set, is a countably additive measure defined on $\mathfrak{M}$. Because measure μ is atomless on E, we get that measure $\int\limits_F f \mathrm{d}\mu$ is also atomless on E (see [12]). Then we may apply Theorem 12. $\square$

Remark 14. Theorem 5 may be generalized, with the reduced proof, by applying the Lyapunov Theorem (1940) [14] formulated below.

Theorem 15. *Let $\mathfrak{M}$ be a σ-algebra of subsets of set $\Omega \neq \emptyset$ and suppose that $(X, \|\cdot\|)$ is the finite dimensional normed vector space (and therefore complete) over $\mathbb{K} = \mathbb{R} \vee \mathbb{C}$. Then for every atomless and countably additive measure $m: \mathfrak{M} \longrightarrow X$ the set $m(\mathfrak{M})$ is compact and convex.*

The Lyapunov Theorem is a special case of the more general Knowles Theorem (see [7] and [4] for generalizations). Moreover, the Lyapunov Theorem allows to generalize Theorem 5 to the following form.

Theorem 16. *Let $\mathfrak{M}$ be a σ-algebra of subsets of set $\Omega \neq \emptyset$ and let $\mu_i: \mathfrak{M} \longrightarrow \mathbb{K}$ ($\mathbb{K} = \mathbb{R} \vee \mathbb{C}$, $1 \leqslant i \leqslant n$) be the countably additive measures. Then for every $E \in \mathfrak{M}$, with respect to which all measures μ_i are atomless, and for each $t \in [0,1]$ there exists a set $F \in \mathfrak{M}$, $F \subset E$ such that*

$$\mu_i(F) = t\mu_i(E),\ 1 \leqslant i \leqslant n.$$

Proof. Assume that $\mu(A) := \big(\mu_1(A \cap E), \dots, \mu_n(A \cap E)\big)$ for $A \in \mathfrak{M}$. Obviously $\mu: \mathfrak{M} \longrightarrow \mathbb{K}^n$ is countably additive, atomless measure, therefore set $\mu(\mathfrak{M})$ is convex by the Lyapunov Theorem where, in particular, we obtain that for every number $t \in (0,1)$ there exists a set $B \in \mathfrak{M}$ such that $\mu(B) = (1-t)\mu(\emptyset) + t\mu(E) = t\mu(E)$, i.e. $\mu_i(B \cap E) = t\mu_i(E)$, $1 \leqslant i \leqslant n$. □

Remark 17. Theorem 16 can be generalized in many ways. One of such generalizations can be obtained by applying the Dvoretsky, Wald and Wolfovitz Theorem (see [2, 8]). Also the new extension of the Lyapunov Theorem to subranges given by Dai and Feinberg in [6] can be consider here.

Remark 18. Stromquist and Woodall proved in [20] that for a given positive integer n, the non-atomic probability measures $\mu_1, \dots, \mu_n$ on $I = [0,1]$ and a number $\alpha \in (0,1)$ there exists a subset K of I such that $\mu_i(K) = \alpha$ for every $i = 1, \dots, n$. Moreover, K may be chosen to be a union of at most n intervals. If I is replaced by S^1 then for each $\alpha \in [0,1]$ there exists a set $K \subseteq S^1$ such that K is a union of at mots $n-1$ intervals and $\mu_i(K) = \alpha$ for each $i = 1, \dots, n$. Furthermore, if α is irrational or $\alpha = \frac{r}{s}$, $r, s \in \mathbb{N}$, $(r,s) = 1$, $s \geqslant n$, then the number of intervals is optimal.

Remark 19. We note that Theorem 16 is not true for the case of infinitely many measures $\mu_i: \mathfrak{M} \longrightarrow \mathbb{K}$ (see [5]).

Final remark. Cater's paper [5] gave the inspiration for some results obtained by the young co-author and presented in this paper.

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References

1. Barbanel J.B.: *On the possibilities for partitioning a cake*. Proc. Amer. Math. Soc. **124** (1996), 3443–3451.
2. Barbanel J.B., Zwicker W.S.: *Two applications of a theorem of Dvortsky, Wald, and Wolfovitz to cake division*. Theory and Decision **43** (1997), 203–207.
3. Bartoszewicz A., Filipczak M., Prus-Wiśniowski F.: *Topological and algebraic aspects of subsums of series*. In monograph: *Traditional and Present-Day Topics in Real Analysis*, Łódź Univ. Press, Łódź 2013.
4. Brook C.H., Graves W.H.: *The range of a vector measure*. J. Math. Anal. Appl. **73** (1980), 219–237.
5. Cater F.S.: *Equal integrals of functions*. Canad. Math. Bull. **28** (1985), 200–204.
6. Dai P., Feinberg E.A.: *Extension of Lyapunov's convexity theorem to subranges*. Manuscript (prepared to be published in Proc. Amer. Math. Soc.).
7. Diestel J., Uhl Jr. J.J.: *Vector Measures*. Math. Surveys **15**, Amer. Math. Soc., Providence 1977.
8. Dvoretsky A., Wald A., Wolfovitz J.: *Relations among certain ranges of vector measures*. Pacific J. Math. **1** (1951), 59–74.
9. Halmos P.R.: *Measure Theory*. D. van Nostrand Company, Princeton 1964.
10. Hill T.P., Morrison K.E.: *Cutting cakes carefully*. College Math. J. **41** (2010), 281–288.
11. Jones M.L.: *A note on a cake cutting algorithm of Banach and Knaster*. Amer. Math. Monthly **104** (1997), 353–355.
12. Kołodziej W.: *Mathematical Analysis*. PWN, Warsaw 1979 (in Polish).

13. Miamee A.G.: *The inclusion* $L^p(\mu) \subseteq L^q(\mu)$. Amer. Math. Monthly **98** (1991), 342–345.
14. Lyapounov A.: *Sur les fonctions-vecteurs completement additives.* Bull. Acad. Sci. URSS **6** (1940), 465–478.
15. Pikhurko O.: *On envy-free cake division.* Amer. Math. Monthly **107** (2000), 736–738.
16. Polya G., Szegö G.: *Aufgaben und Lehrsätze aus der Analysis.* Erster Band, Springer, Berlin 1964.
17. Romero J.I.: *When is* $L^p(\mu)$ *contained in* $L^q(\mu)$? Amer. Math. Monthly **90** (1983), 203–206.
18. Sikorski R.: *Real Functions*, vol. 1. PWN, Warsaw 1958 (in Polish).
19. Steinhaus H.: *Sur la division progmatique.* Econometrika (suplement) **17** (1949), 315–319.
20. Stromquist W., Woodall D.R.: *Sets on which several measures agree.* J. Math. Anal. and Appl. **108** (1985), 241–248.
21. Subramanian B.: *On the inclusion* $L^p(\mu) \subseteq L^q(\mu)$. Amer. Math. Monthly **85** (1978), 479–481.
22. Wang Z., Klir G.J.: *Generalized Measure Theory.* Springer, New York 2009.
23. Wituła R.: *Continuity and the Darboux property of nonatomic finitely additive measures.* In *Generalized Functions and Convergence – Memorial Volume for Professor Jan Mikusiński*, World Scientific, Singapore 1990, 377–386.

Omówienie

W artykule omawiane jest uogólnienie klasycznego wyniku Fichtenholza-Sierpińskiego o własności Darboux σ-addytywnej nieujemnej miary bezatomowej na skończenie wymiarowe miary wektorowe. Przedstawiono dwa różne dowody. Jeden, ważny od strony technicznej, nawiązuje do słynnego lematu Uryshona z topologii. Drugi dowód otrzymujemy łatwo z twierdzenia Lapunowa o zwartości i wypukłości μ-obrazu σ-przestrzeni dla skończenie wymiarowej miary wektorowej μ. Prezentowane są różne powiązania i uogólnienia wykorzystywanych w artykule narzędzi technicznych, co wypływało głównie z pobudek poznawczych.