Vladimir SHEVELEV

Department of Mathematics
Ben-Gurion University of the Negev

COMBINATORIAL MINORS FOR MATRIX FUNCTIONS AND THEIR APPLICATIONS

Summary. As well known, permanent of a square (0,1)-matrix A of order n enumerates the permutations β of $1, 2, ..., n$ with the incidence matrices $B \leqslant A$. To obtain enumerative information on even and odd permutations with condition $B \leqslant A$, we should calculate two-fold vector (a_1, a_2) with $a_1 + a_2 = \operatorname{per} A$. More general, the introduced ω-permanent, where $\omega = e^{2\pi i/m}$, we calculate as m-fold vector. For these and other matrix functions we generalize the Laplace theorem of their expansion over elements of the first row, using the defined so-called "combinatorial minors". In particular, in this way, we calculate the cycle index of permutations with condition $B \leqslant A$.

MINORY KOMBINATORYCZNE DLA FUNKCJI MACIERZOWYCH I ICH ZASTOSOWANIA

Streszczenie. Jak wiadomo, permanent $(0,1)$-macierzy kwadratowej A stopnia n podaje liczbę permutacji β liczb $1, 2, ..., n$, mających macierz incydencji $B \leqslant A$. Aby otrzymać informację o liczbie parzystych i nieparzystych permutacji z warunkiem $B \leqslant A$, należy obliczyć dwuskładowy wektor (a_1, a_2), gdzie $a_1 + a_2 = \operatorname{per} A$. Ogólniej wprowadzamy po-

jęcie ω-permanentu, gdzie $\omega = e^{2\pi i/m}$, który obliczamy jako odpowiedni m-składowy wektor. Dla takich i innych funkcji macierzowych uogólniamy twierdzenie Laplace'a o ich rozwinięciu względem elementów pierwszego wiersza, wykorzystując zdefiniowane w tym celu tak zwane minory kombinatoryczne. W szczególności obliczamy w ten sposób indeks cyklowy permutacji spełniających warunek $B \leqslant A$.

1. Introduction

Let $\mathcal{S}^{(n)}$ be symmetric group of permutations of numbers $\{1, ..., n\}$. Let $A = \{a_{ij}\}$ be square matrix of order n. Recall that the permanent of A is defined by formula ([4]):

$$\operatorname{per} A = \sum_{s \in \mathcal{S}^{(n)}} \prod_{i=1}^{n} a_{i,s(i)}. \tag{1}$$

If A is a $(0,1)$ matrix, then it defines a class $\mathcal{B} = \mathcal{B}(A)$ of permutations with restricted positions, such that the positions of its zeros are prohibited. Such class could be equivalently defined by a simple inequality: a permutation $\pi \in \mathcal{B}$ if and only if for its incidence matrix P we have $P \leqslant A$. One of the most important application of $\operatorname{per} A$ consists of the equality $|\mathcal{B}| = \operatorname{per} A$. Thus $\operatorname{per} A$ enumerates permutations with restricted positions of the class $\mathcal{B}(A)$.

Let $\gamma(\pi)$ be number of independent cycles of π, including cycles of length 1. Then the difference $d(\pi) = n - \gamma(\pi)$ is called *decrement* of π ([3]). Permutation π is called even (odd), if $d(\pi)$ is even (odd). Note that the determinant of matrix A could be defined by formula

$$\det A = \sum_{even\ s \in \mathcal{S}^{(n)}} \prod_{i=1}^{n} a_{i,s(i)} - \sum_{odd\ s \in \mathcal{S}^{(n)}} \prod_{i=1}^{n} a_{i,s(i)}. \tag{2}$$

Since, evidently, we also have

$$\operatorname{per} A = \sum_{even\ s \in \mathcal{S}^{(n)}} \prod_{i=1}^{n} a_{i,s(i)} + \sum_{odd\ s \in \mathcal{S}^{(n)}} \prod_{i=1}^{n} a_{i,s(i)}, \tag{3}$$

then the numbers of even and odd permutations of class $\mathcal{B}(A)$ are given by vector

$$\left(\frac{1}{2}(\operatorname{per} A + \det A),\ \frac{1}{2}(\operatorname{per} A - \det A) \right). \tag{4}$$

Note that, in contrast to permanent, there exist methods of very fast calculation of $\det A$. Therefore, the enumerative information given by (4) one can obtain approximately for the same time as the number $|\mathcal{B}(A)|$ given by (1).

Let $m \geqslant 3$ and $0 \leqslant k < m$ be given integers. We say that a permutation π belongs to *class k modulo m* ($\pi \in \mathcal{S}_{k,m}^{(n)}$), if $d(\pi) \equiv k \pmod{m}$. The first problem under our consideration in this paper is the enumeration of permutations $\pi \in \mathcal{B}(\mathcal{A})$ of class k modulo m. It is clear that this problem is a natural generalization of problem of enumeration of even and odd permutation with restricted positions which is solved by (4). In order to solve this more general problem, put $\omega = e^{\frac{2\pi i}{m}}$ and introduce a new matrix function which we call ω-permanent.

Definition 1. *Let A be a square matrix of order n. We call ω-permanent of matrix A the following matrix function*

$$\operatorname{per}_\omega A = \sum_{s \in \mathcal{S}_{0,m}^{(n)}} \prod_{i=1}^{n} a_{i,s(i)} + \omega \sum_{s \in \mathcal{S}_{1,m}^{(n)}} \prod_{i=1}^{n} a_{i,s(i)} +$$

$$+ \omega^2 \sum_{s \in \mathcal{S}_{2,m}^{(n)}} \prod_{i=1}^{n} a_{i,s(i)} + ... + \omega^{m-1} \sum_{s \in \mathcal{S}_{m-1,m}^{(n)}} \prod_{i=1}^{n} a_{i,s(i)}. \quad (5)$$

Note that, if $m = 1$, then $\operatorname{per}_\omega A = \operatorname{per} A$, and if $m = 2$, then $\operatorname{per}_\omega A = \det A$. In case $m \geqslant 3$, every sum in (5) essentially differs from permanent. Therefore, the known methods of evaluation of permanent ([4, ch. 7]) are not applicable. However, using so-called "combinatorial minors", below we find an expansion $\operatorname{per}_\omega A$ over the first row of matrix A. This allows to reduce a problem of order n to a few problems of order $n - 1$.

The second problem under our consideration is another important problem of enumeration of full cycles of length n with restricted positions. In connection with this problem, we introduce another new matrix function which we call *cyclic permanent*.

Definition 2. *Let A be square matrix of order n. The number*

$$Cycl(A) = \sum_{s} \prod_{i=1}^{n} a_{i,s(i)}, \quad (6)$$

where the summing is over all full cycles from $\mathcal{S}^{(n)}$, we call a cycle permanent of A.

The third problem is a problems of enumeration of permutations with restricted positions with a restrictions on their cycle structure.

Denote $\gamma(s)$ the number of all cycles of permutation s (including cycles of length 1, if they exist). Recall ([1]) that the absolute value of Stirling number $S(n,k)$ of the first kind equals to number of permutations $s \in \mathcal{S}^{(n)}$ with $\gamma(s) = k$ ([1, 5]). A natural generalization of Stirling numbers of the first kind is the following matrix function.

Definition 3. *The matrix function*

$$S(A; n, k) = \sum_{s \in \mathcal{S}^{(n)}, \gamma(s) = k} \prod_{i=1}^{n} a_{i,s(i)}, \tag{7}$$

where n is order of square matrix A, we call Stirling function of index k.

Finally, recall ([5]) that a permutation $s \in \mathcal{S}^{(n)}$ with k_1 cycles of length 1, k_2 cycles of length 2, and so on, is said to be of cycle structure $\overline{k} = (k_1, k_2, ..., k_n)$. Denote $\nu(k_1, k_2, ..., k_n)$ the number of permutations of class $\overline{k} = (k_1, k_2, ..., k_n)$. Then the polynomial

$$C(t_1, t_2, ..., t_n) = \sum \nu(k_1, k_2, ..., k_n) t_1^{k_1} t_2^{k_2} ... t_n^{k_n} \tag{8}$$

is called the cycle index of permutations of $\mathcal{S}^{(n)}$. A natural generalization of the cycle index on permutations with restricted positions is

$$C(A; t_1, t_2, ..., t_n) = \sum \nu(A; k_1, k_2, ..., k_n) t_1^{k_1} t_2^{k_2} ... t_n^{k_n} \tag{9}$$

where $\nu(A; k_1, k_2, ..., k_n)$ is the number of permutations of class $\mathcal{B}(A)$ with the cycle structure $\overline{k}$. In particular, if $A = J_n$, where J_n is $n \times n$ matrix from 1's only, we have $C(A; t_1, t_2, ..., t_n) = C(t_1, t_2, ..., t_n)$.

2. Observations in case $m = 2$ of ω-permanent (determinant)

The case of determinant ($m = 2$) of ω-permanent is a unique case when it is easy to obtain a required enumerative information formally using formulas (4). For the passage to a general case, it is important for us to understand how one can obtain such information from the definition (2) of determinant only. Essentially, the required information is contained in vector

$$\overline{\det} A = \Big(\sum_{even\ s \in \mathcal{S}} \prod_{i=1}^{n} a_{i,s(i)}, \ - \sum_{odd\ s \in \mathcal{S}} \prod_{i=1}^{n} a_{i,s(i)} \Big), \tag{10}$$

and it immediately disappears, if to add its components, or formally to use an identity of the form $1 - 1 = 0$. Since $a_{i,j} = 0$ or 1, then in sums of (10) we do not use such an identity. However, none of algorithms of fast calculation of determinant exists without using of it. On the other hand, using the Laplace algorithm of expansion of determinant over (the first) row, it can only be used at the last step. Therefore, if *not to do* the last step, we can obtain the required enumerative information.

Example 4. By the Laplace expansion, we have

$$\det \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = -(1-1) + (1-1) = -1 + 1 + 1 - 1 = 2 - 2$$

and, if not to do the useless (with the enumerative point of view) last step, then we have

$$\overline{\det} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = (2, -2).$$

This means that there are two even and two odd permutations with the prohibited position $(1, 1)$.

Of course, in general, the Laplace expansion of determinant over the first row in its classic form

$$\det A = \sum_{j=1}^{n} (-1)^{j-1} a_{1,j} M_{1j}, \tag{11}$$

where M_{1j} is minor of element a_{1j}, i.e., determinant of the complementary to a_{1j} submatrix A_{1j}, does not work for ω-permanent.

Therefore, let us introduce for our aims a more suitable notion of so-called "combinatorial minor" of element a_{ij}. Let the complementary to a_{ij} submatrix A_{ij} have the following $n - 1$ columns

$$c_1, c_2, ..., c_{j-1}, c_{j+1}, ..., c_n. \tag{12}$$

The first $j - 1$ of these columns we change in the following cyclic order: $c_2, c_3, ...,$ c_{j-1}, c_1. Then we obtain a new matrix A'_{ij} with the columns

$$c_2, c_3, ..., c_{j-1}, c_1, c_{j+1}, ..., c_n. \tag{13}$$

Determinant of matrix A'_{ij} we call *combinatorial minor* $(CM)_{ij}$ of element a_{ij}. It is easy to see that

$$(CM)_{i1} = M_{i1}; \quad (CM)_{ij} = (-1)^{j-2} M_{ij}, \quad j = 2, ..., n. \tag{14}$$

Therefore, e.g., expansion (11) one can rewrite in the form

$$\det A = a_{1,1}(CM)_{11} - \sum_{j=2}^{n} a_{1,j}(CM)_{1j}. \tag{15}$$

In general, let us give a definition of combinatorial minors for arbitrary matrix function $X(A)$.

Definition 5. *Let X be matrix function defined on all square matrices of order $n \geqslant 3$. Let $A = \{a_{ij}\}$ be a square matrix of order n and A_{ij} be the complementary to a_{ij} submatrix with columns* (12). *Denote A'_{ij} a new square matrix of order $n-1$ with columns* (13). *Then the number $X(A'_{ij})$ is called a combinatorial minor of a_{ij}.*

It appears that our observation (15) has a general character. So, in Sections 4 we give a generalization of the Laplace expansion of type (15) for $per_\omega A$, $Cycl(A)$, $S(A; n, k)$ and $C_n(A; t_1, t_2, ..., t_n)$.

3. Main lemma

Lemma 6. *Let $\pi \in \mathcal{S}^{(n)}$ with $\pi(j) = 1$ and $\sigma = \sigma_j(\pi)$, $j \geqslant 2$, such that*

$$\sigma(1) = \pi(2), \ \ \sigma(2) = \pi(3), \ \ ..., \ \ \sigma(j-2) = \pi(j-1), \ \ \sigma(j-1) = \pi(1),$$

$$\sigma(j) = \pi(j) = 1, \ \ \sigma(j+1) = \pi(j+1), \ \ ..., \ \ \sigma(n) = \pi(n). \tag{16}$$

Let, further, $\pi^* \in \mathcal{S}^{(n-1)}$ defined by the formula

$$\pi^*(i) = \begin{cases} \sigma(i) - 1, & \text{if } 1 \leqslant i \leqslant j-1, \\ \sigma(i+1) - 1, & \text{if } j \leqslant i \leqslant n-1. \end{cases} \tag{17}$$

Then permutations π and π^* have the same number of cycles:

$$\gamma(\pi) = \gamma(\pi^*). \tag{18}$$

Proof. From (16)–(17) we find

$$\pi^*(i) = \begin{cases} \pi(i+1) - 1, & \text{if } i \neq j-1, \\ \pi(1) - 1, & \text{if } i = j-1. \end{cases} \tag{19}$$

Consider a cycle of π containing element j. Let it has length $l \geqslant 2$, such that

$$\pi(j) = 1, \ \ \pi(1) = k_1, \ \ \pi(k_1) = k_2, \ \ ..., \ \ \pi(k_{l-3}) = k_{l-2}, \ \ \pi(k_{l-2}) = j.$$

Beginning with the equality $\pi(1) = k_1$, by (19), this means that

$$\pi^*(j - 1) = k_1 - 1, \ \ \pi^*(k_1 - 1) = k_2 - 1, \ \ ...,$$

$$\pi^*(k_{l-3} - 1) = k_{l-2} - 1, \ \ \pi^*(k_{l-2} - 1) = j - 1.$$

Thus to a cycle of π of length $l \geqslant 2$ containing element j corresponds a cycle of length $l-1$ of π^*. Quite analogously, we verify that to cycle of π of length $l \geqslant 2$ not containing element j corresponds a cycle of the same length of π^*. For example, to cycle of length $l \geqslant 2$ of the form

$$\pi(j - 1) = k_1, \ \ \pi(k_1) = k_2, \ \ \pi(k_2) = k_3, \ \ ..., \ \ \pi(k_{l-2}) = k_{l-1}, \ \ \pi(k_{l-1}) = j - 1$$

(beginning with $\pi(k_1) = k_2$), corresponds the cycle of the same length

$$\pi^*(k_1 - 1) = k_2 - 1, \ \ \pi^*(k_2 - 1) = k_3 - 1, \ \ ...,$$

$$\pi^*(k_{l-2} - 1) = k_{l-1} - 1, \ \ \pi^*(k_{l-1} - 1) = j - 2$$

such that $\pi^*(j - 2) = \pi(j - 1) - 1 = k_1 - 1$. $\qquad\qquad\square$

Note that the structure of Lemma 6 completely corresponds to the procedure of creating the combinatorial minors.

4. Laplace expansions of type (15) of $\mathrm{per}_\omega A$, $Cycl(A)$, $S(A; n, k)$ and $C(A; t_1, t_2, ..., t_n)$

1) $\mathrm{per}_\omega A$. Consider all permutations π with the condition $\pi(j) = 1$. Let j corresponds to j-th column of matrix A. Then the considered permutations correspond to diagonals of matrix A having the common position $(1, j)$. If $j = 1$, then, removing the first row and column, we diminish on 1 the number of cycles of every such permutation, but also we diminish on 1 the number of elements of permutations. Therefore, the decrement of permutations does not change. If $j \geqslant 2$, consider continuation of these diagonals to the matrix of combinatorial minor $A'_{1,j}$. Then, by Lemma 6, the number of cycles of every its diagonal does

not change and, consequently, the decrement is diminished by 1. This means that we have the following expansion of $\text{per}_\omega A$ over the first row

$$\det_\omega A = a_{1,1}(CM)_{11} + \omega \sum_{j=2}^{n} a_{1,j}(CM)_{1j}, \tag{20}$$

where $(CM)_{1j}, \ j \geqslant 1$, are combinatorial minors for $\text{per}_\omega A$.

Note that, as for determinant (see Section 2), in order to receive the required enumerative information, we should prohibit to use the identities of type $1 + \omega + ... + \omega^{m-1} = 0$.

2) $Cycl(A)$. For $n > 1$, here we should ignore element a_{11}. Consider all full cycles π with the condition $\pi(j) = 1$. Let j corresponds to j-th column of matrix A. Then the considered full cycles correspond to diagonals of matrix A having the common position $(1,j), \ j \geqslant 2$. Consider continuation of these diagonals to the matrix of combinatorial minor $A'_{1,j}$. Then, by Lemma 6, the number of cycles of every its diagonal does not change, i.e., they are full cycles of of order $n-1$. Therefore, we have the following expansion of $Cycle(A)$ over the first row of A

$$Cycle(A) = \sum_{j=2}^{n} a_{1,j}(CM)_{1j}, \tag{21}$$

where $(CM)_{1j}, \ j \geqslant 1$, are combinatorial minors for $Cycle(A)$.

3) $S(A; n, k)$. From very close to 1) arguments, we have the following expansion of $S(A; n, k)$ over the first row of A

$$S(A; n, k) = a_{1,1}(CM)_{11}^{(k-1)} + \sum_{j=2}^{n} a_{1,j}(CM)_{1j}^{(k)}, \tag{22}$$

where $(CM)_{1j}^{(k)}, \ j \geqslant 1$, are combinatorial minors of $S(A; n, k)$.

Note that a close to (22) formula was found by the author in [7] but using much more complicated way.

4) $C(t_1, t_2, ..., t_n)$. We need lemma.

Lemma 7. *Let*

$$\{a_{1j}, a_{k_1 1}, a_{k_2 k_1}, ..., a_{k_r, k_{r-1}}, a_{j,k_r}\} \tag{23}$$

be a cycle. Then

$$\{a_{k_1 1}, a_{k_2 k_1}, ..., a_{k_r, k_{r-1}}, a_{j,k_r}\} \tag{24}$$

is a cycle with respect to the main diagonal of the matrix of combinatorial minor $A'_{1,j}$.

Proof. According to the construction of $A'_{1,j}$, the main its diagonal is

$$\{a_{22}, a_{33}, ..., a_{j-1j-1}, a_{j1}, a_{j+1j+1}, ..., a_{nn}\}. \tag{25}$$

With respect to this diagonal we have the following contour which shows that (24) is, indeed, a cycle.

$$\{a_{k_1 1} \to a_{j1} \to a_{jk_r} \to a_{k_r k_{r-1}} \to a_{kr-1k_{r-2}} \to ... \to a_{k_2 k_1}(\to a_{k_1 1})\}. \tag{26}$$

$\square$

Quite analogously we can prove that to every another cycle of a diagonal containing element a_{1j} correspond the same cycle with respect to the main diagonal of the matrix $A'_{1,j}$.

Let A be $(0, 1)$ square matrix of order n. Denote by $C^{(r)}(A; t_1, t_2, ..., t_n)$ a partial cycle index of index (9) of permutations $\pi \in \mathcal{B}(A)$ for which $\{1, \pi(1), \pi(2), ..., \pi(r-1)\}$ is a cycle of length r. Then we have

$$\sum_{r=1}^{n} C^{(r)}(A; t_1, t_2, ..., t_n) = C(A; t_1, t_2, ..., t_n). \tag{27}$$

Therefore, it is sufficient to give an expansion of $C^{(r)}(A; t_1, t_2, ..., t_n)$, $r = 1, ..., n$. First of all, note that

$$C^{(1)}(A; t_1, t_2, ..., t_n) = a_{11} t_1 C(A; t_1, t_2, ..., t_{n-1}). \tag{28}$$

Furthermore, using Lemmas 6-7, we have

$$C^{(r)}(A; t_1, t_2, ..., t_n) = \frac{t_r}{t_{r-1}} \sum_{j=2}^{n} a_{1,j}(CM)_{1,j}, \tag{29}$$

where

$$(CM)_{1,j} = C^{(r-1)}(A'_{1,j}; t_1, t_2, ..., t_{n-1}), \quad j = 2, ..., n, \tag{30}$$

are the combinatorial minors for cycle index of permutations with restricted positions. Note that factor $\frac{t_r}{t_{r-1}}$ in (29) corresponds to the diminution of the length of cycle (24) with respect to length of cycle (23).

Thus formulas (27)-(30) reduce the calculation of cycle index of n-permutations to the calculation of cycle index of $n - 1$-permutations with a rather simple computer realization of this procedure. Note that similar but much more complicated procedure was indicated by the author in [8].

5. An example of enumerating the permutations of classes 0,1,2 modulo 3 with restricted positions

Let

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Consider class $\mathcal{B}(A)$ of permutations with restricted positions and find the distribution of them over classes 0,1,2 modulo 3. We use ω-permanent with $\omega = e^{\frac{2\pi i}{3}}$ and its expansion over elements of the first row, given by (20). Recall that, for the receiving the required enumerative information, we should not use identities of type $1 + \omega + \omega^2 = 0$.

We have

$$\text{per}_\omega A = \text{per}_\omega \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} +$$

$$+ \omega \left(\text{per}_\omega \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} + \text{per}_\omega \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} + \text{per}_\omega \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \right) =$$

$$= \text{per}_\omega \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} + \omega \left(\text{per}_\omega \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} + \text{per}_\omega \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \right) +$$

$$+ \omega^2 \left(\text{per}_\omega \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} + \text{per}_\omega \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \right) + \omega \, \text{per}_\omega \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} +$$

$$+ \omega^2 \left(\text{per}_\omega \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} + \text{per}_\omega \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \right) + \omega \, \text{per}_\omega \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} +$$

$$+ \omega^2 \, \text{per}_\omega \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} =$$

$$= (\omega + \omega^2) + (\omega + \omega^2 + 1) + (2\omega^2 + 1) +$$
$$+ (\omega^2 + 2 + \omega) + (2 + 2\omega) + (\omega^2 + 1) + (\omega^2 + 1 + \omega) + (2 + \omega) +$$
$$+ (\omega + 2\omega^2 + 1) + (\omega^2 + 2 + \omega) = 13 + 9\omega + 10\omega^2.$$

Thus in $\mathcal{B}(A)$ we have 13 permutation of class 0 modulo 3; 9 permutations of class 1 modulo 3 and 10 permutations of class 2 modulo 3. □

6. On two sequences connected with $Cycle(A)$

In summer of 2010, the author published two sequences A179926 and A180026 in OEIS [10]. $a(n) := A179926(n)$ is defined as the number of permutations of all $\tau(n)$ divisors of n of the form: $d_1 = n$, d_2, d_3, ..., $d_{\tau(n)}$ such that $\frac{d_{i+1}}{d_i}$ is a prime or 1/prime for $i = 1, ...\tau(n)$. Note that $a(n)$ is a function of exponents of prime power factorization of n only; moreover, it is invariant with respect to permutations of them. This sequence is equivalently defined as the number of ways, for a given finite multiset E, to get, beginning with E, all submultisets of E, if in every step we remove or join one element of E. Sequence $b(n) := A180026(n)$ differs from A179926 by an additional condition: $\frac{d_{\tau(n)}}{d_1}$ is a prime. In the equivalent formulation it corresponds to the condition that in the last step E is obtained from a submultiset by joining one element.

Note that, it is easy to prove that, knowing any permissible permutation of divisors, say, $\delta_1 = n$, δ_2, ..., $\delta_{\tau(n)}$ (such that $\frac{\delta_{i+1}}{\delta_i}$ is a prime or 1/prime), we can calculate $b(n)$ using the following construction. Consider square (0,1) matrix $B = \{b_{ij}\}$ of order $\tau(n)$ in which $b_{ij} = 1$, if $\frac{\delta_i}{\delta_j}$ is prime or 1/prime, and $b_{ij} = 0$, otherwise. Then $b(n) = Cycle(B)$. In case of A179926, the construction is a little more complicated: $a(n) = Cycle(A)$, where A is obtained from B by the replacing the first its column by the column from 1's.

Note also that A. Heinz [2] proved that, in particular, $a(\Pi_{i=1}^n p_i)$, where p_i are distinct primes, equals to the number of Hamiltonian paths (or Gray codes) on n-cube with a marked starting node (see A003043 in [10]), while $b(\Pi_{i=1}^n p_i)$ equals to the number of directed Hamiltonian cycles on n-cube (see A003042 in [10]).

7. Conclusive remarks

It is interesting that together with the algorithm (27)-(30) for the evaluation of cycle index of permutations with restricted positions $C(A; t_1, t_2, ..., t_n)$ there is

a way to find an explicit representation for it. It appears that a unique way for that gives a so-called "method of index of arrangements" which was discovered by the author in [6]. This method was realized for finding explicit formulas for $C(A; t_1, t_2, ..., t_n)$ (with concrete examples) in [9]. Actually, in [9] we gave a wide development of Riordan-Kaplansky theory of rook polynomials [5].

Acknowledgment

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