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SUMS OF THE RATIONAL POWERS OF ROOTS OF CUBIC POLYNOMIALS

Summary. In this paper a completely elementary method of generating the trigonometric equalities and identities is presented. Among them, the equalities connected with roots of Perrin's polynomial and the generalizations of known Ramanujan's equalities are proven.

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Streszczenie. W artykule przedstawiono elementarną metodę generowania równości i tożsamości trygonometrycznych. Między innymi wyprowadzono równości związane z pierwiastkami wielomianu Perrina oraz otrzymano uogólnienia znanych równości Ramanujana.

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1. Introduction

We will not try too hard to explain too much because it may cause only the confusion. So, instead of explaining, we will do our best to present, for many different ways, what it follows.

This paper, although published later than article [14], is an introduction to the discussed there topics, which include the elementary theory of equalities inspired by the following excellent Ramanujan formulae (1)-(3) (see also papers [2–4, 7, 10, 11, 13], where the history of these equalities is presented on the basis of sources contemporary to the Ramanujan Cubic Polynomials):

$$\left(\frac{1}{9}\right)^{1/3} - \left(\frac{2}{9}\right)^{1/3} + \left(\frac{4}{9}\right)^{1/3} = \left(\sqrt[3]{2} - 1\right)^{1/3}, \tag{1}$$

$$\left(\cos\frac{2\pi}{7}\right)^{1/3} + \left(\cos\frac{4\pi}{7}\right)^{1/3} + \left(\cos\frac{8\pi}{7}\right)^{1/3} = \left(\frac{5 - 3\sqrt[3]{7}}{2}\right)^{1/3}, \tag{2}$$

$$\left(\cos\frac{2\pi}{9}\right)^{1/3} + \left(\cos\frac{4\pi}{9}\right)^{1/3} + \left(\cos\frac{8\pi}{9}\right)^{1/3} = \left(\frac{3\sqrt[3]{9} - 6}{2}\right)^{1/3}, \tag{3}$$

as well as the identities inspired by these formulae and derived by the Authors in recent papers [9, 13, 15]:

$$\begin{aligned}
&\sqrt[3]{\frac{\cos(\boldsymbol{\alpha})}{\cos(4\boldsymbol{\alpha})}}\,\big(2\cos(\boldsymbol{\alpha})\big)^n + \sqrt[3]{\frac{\cos(2\boldsymbol{\alpha})}{\cos(\boldsymbol{\alpha})}}\,\big(2\cos(2\boldsymbol{\alpha})\big)^n + \sqrt[3]{\frac{\cos(4\boldsymbol{\alpha})}{\cos(2\boldsymbol{\alpha})}}\,\big(2\cos(4\boldsymbol{\alpha})\big)^n = \\
&= \sqrt[3]{\frac{\cos(2\boldsymbol{\alpha})}{\cos(\boldsymbol{\alpha})}}\big(2\cos(\boldsymbol{\alpha})\big)^{n+1} + \sqrt[3]{\frac{\cos(4\boldsymbol{\alpha})}{\cos(2\boldsymbol{\alpha})}}\big(2\cos(2\boldsymbol{\alpha})\big)^{n+1} + \sqrt[3]{\frac{\cos(\boldsymbol{\alpha})}{\cos(4\boldsymbol{\alpha})}}\big(2\cos(4\boldsymbol{\alpha})\big)^{n+1} = \\
&= \sqrt[3]{2\cos(\boldsymbol{\alpha})\,(2\cos(4\boldsymbol{\alpha}))^{3n+2}} + \sqrt[3]{2\cos(2\boldsymbol{\alpha})\,(2\cos(\boldsymbol{\alpha}))^{3n+2}} + \\
&\quad + \sqrt[3]{2\cos(4\boldsymbol{\alpha})\,(2\cos(2\boldsymbol{\alpha}))^{3n+2}} = \sqrt[3]{49}\,\Phi_n,
\end{aligned} \tag{4}$$

where $\boldsymbol{\alpha} = \frac{2\pi}{7}$, $\Phi_0 = 0$, $\Phi_1 = -1$, $\Phi_2 = 1$ and

$$\Phi_{n+3} + \Phi_{n+2} - 2\,\Phi_{n+1} - \Phi_n = 0, \qquad n \in \mathbb{Z}. \tag{5}$$

The adequate formula for $\boldsymbol{\beta} = \frac{2\pi}{9}$ has been presented by the Authors in paper [14].

It is obvious that formulae of type (4) are the Binet formulae for the respective recurrence relation (5). Problem of deriving formulae of type (4) reduces in practice to decomposing the respective characteristic polynomials and finding the initial values. The latter is especially a fundamental problem. In the current paper we present many of the completely determined results concerning this subject.

It is worth to emphasize that many of the sequences of integers, discussed in this paper, have been also presented in [8] (one can say that simultaneously with this paper, by giving the proper annotation).

Although the current paper is of the similar structure as paper [14], all the results presented here are in fact orginal (it concerns especially Section 3). In Section 2, where the technical grounds are presented, as a matter of fact the same theorems are given as in Section 2 of paper [14], but in contrast to [14], in this paper the proofs of theorems are additionally included.

2. Basic theorems

We will show now our crucial technical results. In fact, we present here all results from paper [14]. Additionally, the proofs of two first theorems are included here.

Let $f \in \mathbb{C}[\mathbb{X}]$ be such that

$$f(\mathbb{X}) = \mathbb{X}^3 + p\,\mathbb{X}^2 + q\,\mathbb{X} + r = (\mathbb{X} - \xi_1)(\mathbb{X} - \xi_2)(\mathbb{X} - \xi_3).$$

We start with the following result.

Theorem 1 (simple version). *If $p \in 3\sqrt[3]{r}$, then the values α, β, γ of complex roots $\sqrt[3]{\xi_1}$, $\sqrt[3]{\xi_2}$, $\sqrt[3]{\xi_3}$, respectively, can be chosen in such a way that*

$$\alpha + \beta + \gamma = 0.$$

In the sequel, if ξ_1, ξ_2, ξ_3 are the real numbers, then we can assume that also α, β, γ are all the real numbers.

Proof. Suppose that $p \in 3\sqrt[3]{r}$ and let $\sqrt[3]{r}$ be denoted by $\frac{p}{3}$. Then we have

$$f(\mathbb{X}) = \left(\mathbb{X} + \sqrt[3]{r}\right)^3 + \left(q - 3\left(\sqrt[3]{r}\right)^2\right)\mathbb{X},$$

hence

$$f(\mathbb{X}) = 0 \qquad \Leftrightarrow \qquad \mathbb{X} + \sqrt[3]{r} = \sqrt[3]{3\left(\sqrt[3]{r}\right)^2 - q}\;\sqrt[3]{\mathbb{X}}. \tag{6}$$

If we fix the value of $\sqrt[3]{3\left(\sqrt[3]{r}\right)^2 - q}$, then for the respective values α, β, γ of $\sqrt[3]{\xi_1}$, $\sqrt[3]{\xi_2}$, $\sqrt[3]{\xi_3}$ from (6) we get

$$\xi_1 + \xi_2 + \xi_3 + 3\sqrt[3]{r} = \sqrt[3]{3\left(\sqrt[3]{r}\right)^2 - q}\,(\alpha + \beta + \gamma),$$

i.e.

$$-p + 3\sqrt[3]{r} = 0 = \sqrt[3]{3\big(\sqrt[3]{r}\big)^2 - q}\,(\alpha+\beta+\gamma).$$

In the case of $q = 3\big(\sqrt[3]{r}\big)^2$ we have $f(\mathbb{X}) = \big(\mathbb{X} + \sqrt[3]{r}\big)^3$ and we can choose α, β, γ as any values of $\sqrt[3]{-\sqrt[3]{r}}$ and then also $\alpha+\beta+\gamma = 0$. □

Let us present the following version of Theorem 1 which holds for more general family of cubic polynomials.

Theorem 2 (general version). *If $3q \neq p^2$ or if $3q = p^2$ and $27r = p^3$, then there exists $c \in \mathbb{C}$ such that, for the respective complex values $\alpha \in \sqrt[3]{\xi_1 + c}$, $\beta \in \sqrt[3]{\xi_2 + c}$ and $\gamma \in \sqrt[3]{\xi_3 + c}$, we have*

$$\alpha + \beta + \gamma = 0.$$

If $\xi_1, \xi_2, \xi_3 \in \mathbb{R}$ and $3\,q \neq p^2$, then the above equality holds for

$$c = \frac{p^3 - 27\,r}{9(p^2 - 3q)} \tag{7}$$

(all roots should be real).

Whereas, if $3q = p^2$ and $27r \neq p^3$, then such constant $c \in \mathbb{C}$ does not exist.

Let $d \in \mathbb{R}$. If $\xi_1, \xi_2, \xi_3 \in \mathbb{R}$, then the following formula holds true

$$\sqrt[3]{\xi_1 + d} + \sqrt[3]{\xi_2 + d} + \sqrt[3]{\xi_3 + d} = \\ = \sqrt[3]{-P - 6\sqrt[3]{R} - \frac{3}{\sqrt[3]{2}}\left(\sqrt[3]{\mathcal{S} + \sqrt{\mathcal{T}}} + \sqrt[3]{\mathcal{S} - \sqrt{\mathcal{T}}}\right)},$$

where

$$\mathcal{S} := P\,Q + 6\,Q\sqrt[3]{R} + 6\,P\sqrt[3]{R^2} + 9\,R,$$
$$\mathcal{T} := P^2\,Q^2 - 4\,Q^3 - 4\,P^3\,R + 18\,P\,Q\,R - 27\,R^2,$$

$$\prod_{k=1}^{3}(\mathbb{X} - \xi_k - d) = \mathbb{X}^3 + P\,\mathbb{X}^2 + Q\,\mathbb{X} + R,$$

and

$$P = p - 3\,d,$$
$$Q = q + 3\,d^2 - 2\,d\,p,$$
$$R = r - d^3 + p\,d^2 - q\,d.$$

Additionally, we note that

$$\sqrt[3]{\xi_1^2}+\sqrt[3]{\xi_2^2}+\sqrt[3]{\xi_3^2}=\sqrt[3]{-p_2-6\sqrt[3]{r_2}-\frac{3}{\sqrt[3]{2}}\left(\sqrt[3]{\mathcal{S}_2+\sqrt{\mathcal{T}_2}}+\sqrt[3]{\mathcal{S}_2-\sqrt{\mathcal{T}_2}}\right)}, \quad (8)$$

where

$$\begin{aligned}
&\mathcal{S}_2 := p_2\,q_2+6\,q_2\sqrt[3]{r_2}+6\,p_2\sqrt[3]{r_2^2}+9\,r_2,\\
&\mathcal{T}_2 := p_2^2\,q_2^2-4\,q_2^3-4\,p_2^3\,r_2+18\,p_2\,q_2\,r_2-27\,r_2^2,\\
&p_2 = 2\,q-p^2,\\
&q_2 = q^2-2\,p\,r,\\
&r_2 = -r^2.
\end{aligned}$$

Proof. We propose the following sketch of the proof.

If $3q \neq p^2$, then there exist $u,\ v,\ w \in \mathbb{C}$, $v \neq 0$, such that

$$f(\mathbb{X}) = (\mathbb{X}+u)^3+v\,(\mathbb{X}+w).$$

Hence we obtain

$$f(\mathbb{X})=0 \qquad \Leftrightarrow \qquad \mathbb{X}+u=\sqrt[3]{v}\,\sqrt[3]{\mathbb{X}+w}.$$

Rest of the proof in this case runs in the same way like the proof of Theorem 1.

If $3q = p^2$, then we have

$$f(\mathbb{X})=\left(\mathbb{X}+\frac{p}{3}\right)^3+r-\left(\frac{p}{3}\right)^3$$

and equality (7) holds true only for $27r = p^3$.

Discussion of two cases, when $27r = p^3$ and when $27r \neq p^3$, will be omitted here.

Relation (8) can be derived from the following calculations

$$\begin{aligned}
\mathbb{X}^3+p_2\,\mathbb{X}^2+q_2\,\mathbb{X}+r_2 &\overset{\mathbb{X}=x^2}{=} x^6+p_2\,x^4+q_2\,x^2+r_2=(x^2-\xi_1^2)(x^2-\xi_2^2)(x^2-\xi_3^2)=\\
&=(x-\xi_1)(x-\xi_2)(x-\xi_3)(x+\xi_1)(x+\xi_2)(x+\xi_3)=\\
&=(x^3+p\,x^2+q\,x+r)(x^3-p\,x^2+q\,x-r)=(x^3+q\,x)^2-(p\,x^2+r)^2=\\
&=x^6+(2\,q-p^2)\,x^4+(q^2-2\,p\,r)\,x^2-r^2.
\end{aligned}$$

□

We will formulate one more theorem describing some interesting recurrence sequence.

Theorem 3 (special case of the Newton-Girard formula). *If for the certain values* $\alpha \in \xi_1^{1/3}$, $\beta \in \xi_2^{1/3}$, $\gamma \in \xi_3^{1/3}$ *we have*

$$\alpha + \beta + \gamma = 0$$

and we set

$$S_n := \alpha^n + \beta^n + \gamma^n, \qquad n = 0, 1, 2, ...,$$

then from the Newton-Girard formula (see Remark 1 in [9]) we get

$$S_{n+3} = \alpha\,\beta\,\gamma\,S_n + \frac{1}{2}\,S_2\,S_{n+1}, \qquad n = 0, 1, 2, \ldots.$$

Here we have $\alpha\,\beta\,\gamma \in -\sqrt[3]{r}$, *whereas* S_2 *belongs to the right side of (8) for the respective values of all five complex roots appearing in this root of the third order (both* $\sqrt{\mathcal{T}}$ *are chosen with the same value).*

In the sequel, if $\xi_1, \xi_2, \xi_3 \in \mathbb{R}$ *and* $\mathcal{T}_2 \geqslant 0$, *then we can assume that all the above roots are real.*

Remark 4. More information about the generalizations of sequence S_n, its meaning and its properties can be found in paper [6].

3. Application of fundamental theorems

Our main aim is to give many specific examples of applications of the indicated theorems. We intend now to present several examples of polynomials with the known prime factors decompositions, to which these fundamental theorems will be applied. It will lead us to generate the new original equalities and identities, of the trigonometric nature especially.

1° We have the following decomposition

$$\mathbb{X}^3 + \mathbb{X}^2 - 2\,\mathbb{X} - 1 = \prod_{k=0}^{2} \big(\mathbb{X} - 2\cos(2^k \boldsymbol{\alpha})\big). \tag{9}$$

Hence and from Theorem 2 (formula (7)) we obtain

$$\sum_{k=0}^{2} \sqrt[3]{\cos(2^k \boldsymbol{\alpha}) + \frac{2}{9}} = 0. \tag{10}$$

This formula could be also obtained from the following elementary transformations

$$\mathbb{X}^3+\mathbb{X}^2-2\,\mathbb{X}-1=0 \iff \left(\mathbb{X}+\frac{1}{3}\right)^3=\frac{7}{3}\left(\mathbb{X}+\frac{4}{9}\right)$$
$$\iff 1+\sum_{k=0}^{2}2\cos\left(2^k\boldsymbol{\alpha}\right)=\sum_{k=0}^{2}\sqrt[3]{\frac{14}{3}\left(\cos\left(2^k\boldsymbol{\alpha}\right)+\frac{2}{9}\right)} \iff (10). \quad (11)$$

Moreover, from

$$\mathbb{X}^3=1+2\,\mathbb{X}-\mathbb{X}^2 \quad (12)$$

we obtain

$$1=\sum_{k=0}^{2}\sqrt[3]{1+2\cos\left(2^{k+1}\boldsymbol{\alpha}\right)-4\cos\left(2^k\boldsymbol{\alpha}\right)} \quad (13)$$

and

$$\left(\mathbb{X}+\sqrt{\frac{2}{3}}i\right)^3=(\sqrt{6}i-1)\mathbb{X}^2+1-\sqrt{\frac{8}{27}}i, \quad (14)$$

from which we receive the equality

$$1-\sqrt{6}i=\alpha_1+\alpha_2+\alpha_3 \quad (15)$$

for the respective complex roots

$$\alpha_k\in\sqrt[3]{(1-\sqrt{6}i)4\cos\left(2^k\boldsymbol{\alpha}\right)-1+\frac{2}{3}\sqrt{\frac{2}{3}}i}, \quad k=1,2,3. \quad (16)$$

2° From (9) the following decomposition can be deduced

$$\mathbb{X}^3-3\,\mathbb{X}^2-4\,\mathbb{X}-1=\prod_{k=0}^{2}\left(\mathbb{X}-\frac{\cos(2^k\boldsymbol{\alpha})}{\cos(2^{k+2}\boldsymbol{\alpha})}\right). \quad (17)$$

Thus, by Theorem 1 we obtain

$$\sum_{k=0}^{2}\sqrt[3]{\frac{\cos(2^k\boldsymbol{\alpha})}{\cos(2^{k+2}\boldsymbol{\alpha})}}=0, \quad (18)$$

i.e. $\Phi_0=0$ in formula (5).

Moreover, from Theorem 3 one can derive the formula of the form (see Remark 1 in [9]):

$$S^1_{n+3}=\sqrt[3]{7}\,S^1_{n+1}+S^1_n,$$

where

$$S^1_n:=\sum_{k=0}^{2}\left(\sqrt[3]{\frac{\cos(2^k\boldsymbol{\alpha})}{\cos(2^{k+2}\boldsymbol{\alpha})}}\right)^n, \qquad n=0,1,2,...$$

(we have $S_0^1 = 3$, $S_1^1 = 0$, $S_2^1 \overset{(8)}{=} 2\sqrt[3]{7}$). We have also the following decomposition (verification of which can be obtained by induction):

$$S_n^1 = \widehat{a}_n + \widehat{b}_n \sqrt[3]{7} + \widehat{c}_n \sqrt[3]{49}, \quad n = 0, 1, 2, ...,$$

where

$$\begin{array}{ccc} \widehat{a}_{n+3} = \widehat{a}_n + 7\,\widehat{c}_n, & \widehat{b}_{n+3} = \widehat{b}_n + \widehat{a}_{n+1}, & \widehat{c}_{n+3} = \widehat{c}_n + \widehat{b}_{n+1}, \\ \widehat{a}_0 = 3, & \widehat{a}_1 = 0, & \widehat{a}_2 = 0, \\ \widehat{b}_0 = 0, & \widehat{b}_1 = 0, & \widehat{b}_2 = 2, \\ \widehat{c}_0 = 0, & \widehat{c}_1 = 0, & \widehat{c}_2 = 0. \end{array}$$

It could be easily verified that each of three sequences $\{\widehat{a}_n\}$, $\{\widehat{b}_n\}$ and $\{\widehat{c}_n\}$ satisfies the following recurrence relations

$$A_{n+9} - 3\,A_{n+6} + 3\,A_{n+3} - 7\,A_{n+2} - A_n = 0,$$

for every $n = 0, 1, 2, ...$, where, additionally, the given below initial values are needed

$$\begin{array}{cccccc} \widehat{a}_3 = 3, & \widehat{a}_4 = 0, & \widehat{a}_5 = 0, & \widehat{a}_6 = 3, & \widehat{a}_7 = 14, & \widehat{a}_8 = 0, \\ \widehat{b}_3 = 0, & \widehat{b}_4 = 0, & \widehat{b}_5 = 5, & \widehat{b}_6 = 0, & \widehat{b}_7 = 0, & \widehat{b}_8 = 8, \\ \widehat{c}_3 = 0, & \widehat{c}_4 = 4, & \widehat{c}_5 = 0, & \widehat{c}_6 = 0, & \widehat{c}_7 = 7, & \widehat{c}_8 = 0. \end{array}$$

3° While preparing article [14] we have discussed the following Johannes Kepler polynomial of the form

$$\mathbb{X}^3 - 7\,\mathbb{X}^2 + 14\,\mathbb{X} - 7 = \prod_{k=0}^{2} \left(\mathbb{X} - 4\sin^2(2^k\boldsymbol{\alpha})\right) \tag{19}$$

(numbers $4\sin^2(2^k\boldsymbol{\alpha})$, $k = 0, 1, 2$, are equal to the squares of chords A_1A_2, A_1A_3 and A_1A_4 of the regular heptagon $A_1A_2...A_7$ inscribed in the unit circle - see for example [1, 5]).

If we put

$$w(n) := \frac{\sqrt{7}}{7} \sum_{k=0}^{2} \operatorname{ctg}\left(2^{k+1}\boldsymbol{\alpha}\right) \cdot \left(2\sin\left(2^k\boldsymbol{\alpha}\right)\right)^{2n} \tag{20}$$

and

$$u(n) := \frac{\sqrt{7}}{7} \sum_{k=0}^{2} \operatorname{ctg}\left(2^{k+2}\boldsymbol{\alpha}\right) \cdot \left(2\sin\left(2^k\boldsymbol{\alpha}\right)\right)^{2n}, \tag{21}$$

then we also manage to find the following recursive formula

$$\begin{aligned} &w(0) = 1,\ w(1) = 3,\ w(2) = \mathbf{11}, \\ &w(n+3) = 7w(n+2) - 14w(n+1) + 7w(n) \end{aligned} \tag{22}$$

and

$$\begin{aligned} &u(0) = 1,\ u(1) = 3,\ u(2) = \mathbf{9}, \\ &u(n+3) = 7u(n+2) - 14u(n+1) + 7u(n), \end{aligned} \tag{23}$$

for every $n = 0, 1, 2, \ldots$, where $w(0)$ and $v(0)$ can be deduced from the following relation (see also A215575 in [8]):

$$\mathbb{X}^3 - 7\,\mathbb{X}^2 + 7\,\mathbb{X} + 7 = \prod_{k=0}^{2} \big(\mathbb{X} - \sqrt{7}\,\mathrm{ctg}(2^k \boldsymbol{\alpha})\big). \tag{24}$$

We note that $w(n) = A106460(n)$ for $n = 0, 1, 2, 3, 4$, and $u(n) = A215007(n)$ for $n \in \mathbb{N}$ (see [8]). Moreover, we obtain

$$w(n) = 2v(n) - v(n+1), \tag{25}$$

where

$$v(n) := \frac{2^{2n-1}}{\sqrt{7}} \sum_{k=0}^{2} \frac{\Big(\sin\big(2^k \boldsymbol{\alpha}\big)\Big)^{2n}}{\sin\big(2^{k+1} \boldsymbol{\alpha}\big)}, \tag{26}$$

for every $n = 0, 1, 2, \ldots$

We have

$$\begin{aligned} 2\sqrt{7}v(0) = \sum_{k=0}^{2} \frac{\cos^2(2^k\boldsymbol{\alpha}) + \sin^2(2^k\boldsymbol{\alpha})}{2\sin(2^k\boldsymbol{\alpha})\cos(2^k\boldsymbol{\alpha})} &= \frac{1}{2}\sum_{k=0}^{2} \mathrm{ctg}(2^k\boldsymbol{\alpha}) + \frac{1}{2}\sum_{k=0}^{2} \mathrm{tg}(2^k\boldsymbol{\alpha}) = \\ &= \frac{\sqrt{7}}{2} w(0) + \frac{1}{2}\sum_{k=0}^{2} \mathrm{tg}(2^k\boldsymbol{\alpha}), \end{aligned} \tag{27}$$

$$\sqrt{7}v(1) = \sum_{k=0}^{2} \frac{\sin^2(2^k\boldsymbol{\alpha})}{\sin(2^k\boldsymbol{\alpha})\cos(2^k\boldsymbol{\alpha})} = \sum_{k=0}^{2} \mathrm{tg}(2^k\boldsymbol{\alpha}) \tag{28}$$

and from (24) we get

$$\mathbb{X}^3 + \mathbb{X}^2 - \mathbb{X} + \frac{1}{7} = \prod_{k=0}^{2} \left(\mathbb{X} - \frac{\sqrt{7}}{7}\,\mathrm{tg}(2^k\boldsymbol{\alpha})\right), \tag{29}$$

which implies

$$\sum_{k=0}^{2} \operatorname{tg}(2^k \boldsymbol{\alpha}) = -\sqrt{7}, \quad v(1) = -1,\ v(0) = 0. \tag{30}$$

We have $v(n) = -A215008(n)$ for every $n = 0, 1, \ldots$ (see [8]). At last, we deduced that

$$\begin{aligned} A122068(n) &= \frac{1}{2}\big(u(n) + w(n)\big) = \\ &= \frac{1}{2}\big(A215007(n) + A215008(n+1) - 2 \cdot A215008(n)\big) = \text{(by (20) and (21))} \\ &= \frac{1}{2}\sum_{k=0}^{2}\left(1 - \frac{\sqrt{7}}{7}\operatorname{ctg}(2^k\boldsymbol{\alpha})\right)\big(2\sin(2^k\boldsymbol{\alpha})\big)^{2n}. \end{aligned}$$

The generating function of this sequence has the form

$$\frac{x(1-x)(1-3x)}{1-7x+14x^2-x^3}.$$

Remark 5. Let us put

$$A(n) = A(n-1) + A(n-2) + \frac{1}{7}A(n-3) \tag{31}$$

with $A(0) = 3,\ A(1) = 1,\ A(2) = 3$. The characteristic polynomial of the sequence $\{A(n)\}$ has the decomposition (compare this formula with (29))

$$\mathbb{X}^3 - \mathbb{X}^2 - \mathbb{X} - \frac{1}{7} = \prod_{k=0}^{2}\left(\mathbb{X} - 1 + \frac{4}{\sqrt{7}}\sin(2^k\boldsymbol{\alpha})\right). \tag{32}$$

Furthermore, we have $7^{\lfloor n/3 \rfloor} A(n) = A215828(n)$ and

$$(-\sqrt{7})^n A(n) = \sum_{k=0}^{2}\Big(\operatorname{tg}(2^k\boldsymbol{\alpha})\Big)^n. \tag{33}$$

4° We have (see [12]):

$$f(\mathbb{X}) := \mathbb{X}^3 - 3\,\mathbb{X} + 1 = \prod_{k=0}^{2}\big(\mathbb{X} - 2\cos(2^k\boldsymbol{\beta})\big). \tag{34}$$

Thus (by (7)):

$$\sum_{k=0}^{2}\sqrt[3]{6\cos(2^k\boldsymbol{\beta}) - 1} = 0 \tag{35}$$

and from $f(\mathbb{X}) = 0$ we obtain

$$\left(\mathbb{X}^2 - \frac{3}{2}\right)^2 = \frac{9}{4} - \mathbb{X}$$

which leads to equality

$$\sqrt{1 - \frac{8}{9}\cos(\boldsymbol{\beta})} + \sqrt{1 - \frac{8}{9}\cos(4\boldsymbol{\beta})} = 1 + \sqrt{1 - \frac{8}{9}\cos(2\boldsymbol{\beta})}, \tag{36}$$

since

$$\sum_{k=0}^{2} 4\cos^2(2^k\boldsymbol{\beta}) = \sum_{k=0}^{2}\Big(2 + 2\cos(2^{k+1}\boldsymbol{\beta})\Big) \overset{(34)}{=} 6.$$

Moreover, from (34) one can generate equality of the form

$$\sqrt[3]{\frac{\sin^2\frac{5\pi}{18}}{\sin\frac{7\pi}{18}\sin\frac{\pi}{18}}} + \sqrt[3]{\frac{\sin^2\frac{\pi}{18}}{\sin\frac{5\pi}{18}\sin\frac{7\pi}{18}}} = \sqrt[3]{\frac{\sin^2\frac{7\pi}{18}}{\sin\frac{\pi}{18}\sin\frac{5\pi}{18}}}. \tag{37}$$

Indeed, if $\mathbb{X}$ is a zero of $f(\mathbb{X})$, then we get

$$\mathbb{X}^3 = \frac{\mathbb{X}^2}{3 - \mathbb{X}^2} \quad \Leftrightarrow \quad \mathbb{X} = \sqrt[3]{\frac{\mathbb{X}^2}{3 - \mathbb{X}^2}}. \tag{38}$$

Since $\mathbb{X} = 2\cos\varphi$, for some $\varphi \in \{2\pi/9,\, 4\pi/9,\, 8\pi/9\}$ we get

$$\frac{\mathbb{X}^2}{3 - \mathbb{X}^2} = \frac{\cos^2\varphi}{\frac{3}{4} - \cos^2\varphi} = \frac{\sin^2\left(\frac{\pi}{2} - \varphi\right)}{\cos^2\frac{\pi}{6} - \cos^2\varphi} = \frac{\sin^2\left(\frac{\pi}{2} - \varphi\right)}{\sin\left(\varphi - \frac{\pi}{6}\right)\sin\left(\varphi + \frac{\pi}{6}\right)}. \tag{39}$$

Now, equality (37) follows easily from (38) and (39), since

$$\sum_{k=0}^{2}\cos\left(2^k\boldsymbol{\beta}\right) = 0.$$

Furthermore, from Theorem 3 we deduce

$$S^2_{n+3} = \sqrt[3]{3}\, S^2_{n+1} - \frac{1}{3}\, S^2_n, \quad n = 0, 1, 2, ..., \tag{40}$$

where

$$S^2_n := \sum_{k=0}^{2}\Big(2\cos(2^k\boldsymbol{\beta}) - \frac{1}{3}\Big)^{n/3}.$$

Additionally, we can receive by induction the following decomposition

$$S^2_n = a_n + b_n\sqrt[3]{3} + c_n\sqrt[3]{9}, \quad n = 0, 1, 2, ...,$$

where, by (40), we have

$$\begin{array}{lll} a_{n+3} = 3\,c_{n+1} - \frac{1}{3}\,a_n, & b_{n+3} = a_{n+1} - \frac{1}{3}\,b_n, & c_{n+3} = b_{n+1} - \frac{1}{3}\,c_n, \\ a_0 = 3, & a_1 = 0, & a_2 = 0, \\ b_0 = 0, & b_1 = 0, & b_2 = 2, \\ c_0 = 0, & c_1 = 0, & c_2 = 0. \end{array} \tag{41}$$

Similarly like in case of sequence $\{S_n^1\}_{n=0}^{\infty}$, also in this case it can be easily verified that each of sequences $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ and $\{c_n\}_{n=0}^{\infty}$ satisfies the following recurrence relations

$$A_{n+9} + A_{n+6} - \frac{8}{3}A_{n+3} + \frac{1}{27}A_n = 0,$$

for every $n = 0, 1, 2, ...$, where, additionally, the given below initial values are needed

$$\begin{array}{llllll} a_3 = -1, & a_4 = 0, & a_5 = 0, & a_6 = \frac{19}{3}, & a_7 = 0, & a_8 = 0, \\ b_3 = 0, & b_4 = 0, & b_5 = -\frac{5}{3}, & b_6 = 0, & b_7 = 0, & b_8 = \frac{62}{9}, \\ c_3 = 0, & c_4 = 2, & c_5 = 0, & c_6 = 0, & c_7 = -\frac{7}{3}, & c_8 = 0. \end{array}$$

From relations (41) we obtain

$$a_{3n+1} = a_{3n+2} = b_{3n} = b_{3n+1} = c_{3n} = c_{3n+2} = 0, \tag{42}$$

for every $n = 0, 1, 2, ...$, and

$$a_{3n} \neq 0, \quad b_{3n+2} \neq 0, \quad c_{3n+1} \neq 0, \tag{43}$$

for every $n = 1, 2, ...$ Last three relations arise easily from the following more general facts

$$a_{3n}c_{3n+1} < 0, \quad b_{3n+2}c_{3n+1} < 0, \quad a_{3n}b_{3n-1} < 0,$$

for every $n = 1, 2, ...$

At last, from (42) and (43) we get

$$S_{3n}^2 = a_{3n}, \qquad S_{3n+1}^2 = c_{3n+1}\sqrt[3]{9},$$

$$S_{3n+2}^2 = b_{3n+2}\sqrt[3]{3},$$

for every $n = 1, 2, ...$

Remark 6. Using Remark 1 from [9] we deduce the following relation

$$S^3_{n+3} = \sqrt[3]{9}S^3_{n+1} + S^3_n \tag{44}$$

and $S^3_0 = 3,\ S^3_1 = 1,\ S^3_2 = 2\sqrt[3]{9}$, where

$$S^3_n := \left(\frac{\cos\boldsymbol{\beta}}{\cos(2\boldsymbol{\beta})}\right)^{n/3} + \left(\frac{\cos(2\boldsymbol{\beta})}{\cos(4\boldsymbol{\beta})}\right)^{n/3} + \left(\frac{\cos(4\boldsymbol{\beta})}{\cos\boldsymbol{\beta}}\right)^{n/3}. \tag{45}$$

On the other hand, from (44) we obtain

$$S^3_n = x_n + \sqrt[3]{9}y_n + \sqrt[3]{81}z_n \tag{46}$$

where

$$\begin{array}{ccc} x_{n+3} = x_n + 9z_{n+1}, & y_{n+3} = y_n + x_{n+1}, & z_{n+3} = z_n + y_{n+1}, \\ x_0 = 3, & x_1 = 0, & x_2 = 0, \\ y_0 = 0, & y_1 = 0, & y_2 = 2, \\ z_0 = 0, & z_1 = 0, & z_2 = 0. \end{array} \tag{47}$$

From (47) we receive

$$\begin{array}{ll} x_{3n} > 0 \text{ and } 3|x_{3n}, & x_{3n+1} = x_{3n+2} = 0,\ n = 0,1,\ldots, \\ y_{3n-1} > 0 \text{ and } 3|(y_{3n-1}+1), & y_{3n} = y_{3n-2} = 0,\ n = 1,2,\ldots, \\ z_{3n+1} > 0 \text{ and } 3|(z_{3n+1} - z_{3n-2} + 1), & z_{3n} = z_{3n-1} = 0,\ n = 1,2,\ldots, \end{array} \tag{48}$$

moreover, all three sequences $\{x_{3n}\}_{n=1}^{\infty},\ \{y_{3n-1}\}_{n=1}^{\infty},\ \{z_{3n+1}\}_{n=1}^{\infty}$ are increasing. Thus we get

$$\begin{gathered} S^3_{3n} = x_{3n} \text{ for } n = 0,1,2,\ldots, \text{ and} \\ S^3_{3n-1} = \sqrt[3]{9}y_{3n-1},\ S^3_{3n+1} = \sqrt[3]{81}z_{3n+1} \text{ for } n = 1,2,\ldots \end{gathered} \tag{49}$$

At last it can be obtained the recurrence relation:

$$\mathbb{X}_{n+3} - 3\mathbb{X}_{n+2} - 6\mathbb{X}_{n+1} - \mathbb{X}_n = 0$$

for every of the following sequences $\{x_{3n}\},\ \{y_{3n-1}\},\ \{z_{3n+1}\},\ \{S^3_{3n}\},\ \{S^3_{3n-1}\},$ $\{S^3_{3n+1}\}$.

5° We have decomposition of the form (see [12]):

$$\mathbb{X}^3 - 3\,\mathbb{X} + \sqrt{3} = \big(\mathbb{X} - 2\sin(\boldsymbol{\beta})\big)\big(\mathbb{X} + 2\sin(2\boldsymbol{\beta})\big)\big(\mathbb{X} - 2\sin(4\boldsymbol{\beta})\big). \tag{50}$$

Hence, by (7), we obtain

$$\sqrt[3]{6\sin(\boldsymbol{\beta})-\sqrt{3}}+\sqrt[3]{6\sin(4\boldsymbol{\beta})-\sqrt{3}}=\sqrt[3]{6\sin(2\boldsymbol{\beta})+\sqrt{3}}. \tag{51}$$

Thus, by Theorem 3, one can generate the formula

$$S^4_{n+3}=\sqrt[3]{3}\,S^4_{n+1}-\frac{\sqrt{3}}{3}\,S^4_n,\quad n=0,1,2,...,$$

where

$$S^4_n:=\left(2\,\sin(\boldsymbol{\beta})-\frac{\sqrt{3}}{3}\right)^{n/3}+\left(-2\,\sin(2\boldsymbol{\beta})-\frac{\sqrt{3}}{3}\right)^{n/3}+\left(2\,\sin(4\boldsymbol{\beta})-\frac{\sqrt{3}}{3}\right)^{n/3}.$$

The following relations could be obtained:

$$S^4_0=3,\ S^4_1=0,\ S^4_2=-2\cdot 3^{-5/9}S^4_{-1}. \tag{52}$$

The sequence $\{S^4_n\}_{n=0}^{\infty}$ will be still discussed by us in the near future.

Remark 7. From (34) and (50) there can be generated the following decompositions

$$\mathbb{X}^3+\sqrt{3}\,\mathbb{X}^2-3\,\mathbb{X}-\frac{\sqrt{3}}{3}=\big(\mathbb{X}-\operatorname{ctg}\boldsymbol{\beta}\big)\big(\mathbb{X}+\operatorname{ctg}(2\boldsymbol{\beta})\big)\big(\mathbb{X}-\operatorname{ctg}(4\boldsymbol{\beta})\big) \tag{53}$$

and

$$\mathbb{X}^3+3\sqrt{3}\,\mathbb{X}^2-3\,\mathbb{X}-\sqrt{3}=\big(\mathbb{X}-\operatorname{tg}\boldsymbol{\beta}\big)\big(\mathbb{X}+\operatorname{tg}(2\boldsymbol{\beta})\big)\big(\mathbb{X}-\operatorname{tg}(4\boldsymbol{\beta})\big) \tag{54}$$

which by Theorem 2 implies

$$\sqrt[3]{\operatorname{ctg}\boldsymbol{\beta}+\frac{\sqrt{3}}{9}}+\sqrt[3]{\operatorname{ctg}(4\boldsymbol{\beta})+\frac{\sqrt{3}}{9}}=\sqrt[3]{\operatorname{ctg}(2\boldsymbol{\beta})-\frac{\sqrt{3}}{9}} \tag{55}$$

and

$$\sqrt[3]{\operatorname{tg}\boldsymbol{\beta}+\frac{\sqrt{3}}{3}}+\sqrt[3]{\operatorname{tg}(4\boldsymbol{\beta})+\frac{\sqrt{3}}{3}}=\sqrt[3]{\operatorname{tg}(2\boldsymbol{\beta})-\frac{\sqrt{3}}{3}}. \tag{56}$$

6° We have (see [16]):

$$f_1(\mathbb{X}):=\mathbb{X}^3+\frac{1-\sqrt{13}}{2}\,\mathbb{X}^2-\mathbb{X}+\frac{\sqrt{13}-3}{2}=$$
$$=\left(\mathbb{X}-2\cos\frac{2\pi}{13}\right)\left(\mathbb{X}-2\cos\frac{6\pi}{13}\right)\left(\mathbb{X}-2\cos\frac{8\pi}{13}\right) \tag{57}$$

and

$$\mathbb{X}^3+\frac{1+\sqrt{13}}{2}\,\mathbb{X}^2-\mathbb{X}-\frac{\sqrt{13}+3}{2}= \\ =\left(\mathbb{X}-2\cos\frac{4\pi}{13}\right)\left(\mathbb{X}-2\cos\frac{10\pi}{13}\right)\left(\mathbb{X}-2\cos\frac{12\pi}{13}\right) \quad (58)$$

which implies, by (7):

$$\sqrt[3]{\cos\frac{2\pi}{13}+c_1}+\sqrt[3]{\cos\frac{6\pi}{13}+c_1}+\sqrt[3]{\cos\frac{8\pi}{13}+c_1}=0,$$

where $c_1:=\frac{7\sqrt{13}-31}{18(\sqrt{13}-1)}$ and

$$\sqrt[3]{\cos\frac{4\pi}{13}+c_2}+\sqrt[3]{\cos\frac{10\pi}{13}+c_2}+\sqrt[3]{\cos\frac{12\pi}{13}+c_2}=0,$$

where $c_2:=\frac{7\sqrt{13}+31}{18(\sqrt{13}+1)}$, respectively.

Moreover, from (57) we receive for $\mathbb{X}=2\cos\varphi$, $\varphi\in\{\frac{2\pi}{13},\frac{6\pi}{13},\frac{8\pi}{13}\}$:

$$f_1(\mathbb{X})=0 \quad\Rightarrow\quad \mathbb{X}=\sqrt[3]{\frac{\sqrt{13}-1}{2}\,\mathbb{X}^2+\mathbb{X}+\frac{3-\sqrt{13}}{2}} \quad\Rightarrow \\ \Rightarrow\quad 2\cos\varphi=\sqrt[3]{(\sqrt{13}-1)\cos(2\varphi)+2\cos\varphi+\frac{\sqrt{13}+1}{2}},$$

which concludes

$$\sqrt[3]{\frac{7-\sqrt{13}}{2}}=\sqrt[3]{\left(\frac{\sqrt{13}-1}{2}\right)^2}=\sqrt[3]{2\cos\frac{4\pi}{13}+\frac{\sqrt{13}+1}{3}\cos\frac{2\pi}{13}+\frac{\sqrt{13}+7}{6}}+ \\ +\sqrt[3]{2\cos\frac{12\pi}{13}+\frac{\sqrt{13}+1}{3}\cos\frac{6\pi}{13}+\frac{\sqrt{13}+7}{6}}+ \\ +\sqrt[3]{2\cos\frac{10\pi}{13}+\frac{\sqrt{13}+1}{3}\cos\frac{8\pi}{13}+\frac{\sqrt{13}+7}{6}}.$$

Similarly, from (58) we obtain the equality

$$\sqrt[3]{\frac{7+\sqrt{13}}{2}}=\sqrt[3]{\left(\frac{\sqrt{13}+1}{2}\right)^2}=\sqrt[3]{2\cos\frac{8\pi}{13}+\frac{1-\sqrt{13}}{3}\cos\frac{4\pi}{13}+\frac{7-\sqrt{13}}{6}}+ \\ +\sqrt[3]{2\cos\frac{6\pi}{13}+\frac{1-\sqrt{13}}{3}\cos\frac{10\pi}{13}+\frac{7-\sqrt{13}}{6}}+ \\ +\sqrt[3]{2\cos\frac{2\pi}{13}+\frac{1-\sqrt{13}}{3}\cos\frac{12\pi}{13}+\frac{7-\sqrt{13}}{6}}.$$

7° Let us consider the Perrin polynomial (see Remark 8 below):

$$f_2(\mathbb{X}) := \mathbb{X}^3 - \mathbb{X} - 1 = \left(\mathbb{X} - \tau_0^{-1}\right)\left(\mathbb{X} - i\sqrt{\tau_0}\, e^{i\Psi}\right)\left(\mathbb{X} + i\sqrt{\tau_0}\, e^{-i\Psi}\right),$$

where $\tau_0 := 0.754877666...$ is the only positive root of polynomial $\tau^3 + \tau^2 - 1$ and

$$\Psi := \arcsin\left(\frac{1}{2\sqrt{\tau_0^3}}\right).$$

By (7) we get

$$\alpha + \beta + \gamma = 0,$$

for the respective complex values of $\alpha \in \sqrt[3]{1 + \tau_0^{-1}}$, $\beta \in \sqrt[3]{1 + i\sqrt{\tau_0}\, e^{i\Psi}}$, $\gamma \in \sqrt[3]{1 - i\sqrt{\tau_0}\, e^{-i\Psi}}$. Moreover, we have

$$f_2(\mathbb{X}) = 0 \qquad \Leftrightarrow \qquad \left(\mathbb{X} - \frac{\sqrt{3}}{3}\, i\right)^3 = 1 + \frac{\sqrt{3}}{9}\, i\left(1 - (3\,\mathbb{X})^2\right)$$

which gives the relation

$$-\sqrt{3}\, i = \alpha_1 + \beta_1 + \gamma_1,$$

for the respective complex values of $\alpha_1 \in \sqrt[3]{1 + \frac{\sqrt{3}}{9}\, i\left(1 - 9\,\tau_0^{-2}\right)}$, $\beta_1 \in \sqrt[3]{1 + \frac{\sqrt{3}}{9}\, i\left(1 + 9\,\tau_0\, e^{i2\Psi}\right)}$ and $\gamma_1 \in \sqrt[3]{1 + \frac{\sqrt{3}}{9}\, i\left(1 + 9\,\tau_0\, e^{-i2\Psi}\right)}$.

Remark 8. The Perrin polynomial is the characteristic polynomial of the known Perrin recurrence sequence (see [8] A001608):

$$a_n = a_{n-2} + a_{n-3}, \qquad n = 3, 4, ...,$$
$$a_0 = 3, \quad a_1 = 0, \quad a_2 = 2.$$

The Binet formula of this sequence has the form

$$a_n = \tau_0^{-n} + i^n \tau_0^{n/2}\left(e^{i(\pi+\Psi)n} + e^{-i\Psi n}\right) =$$
$$= \tau_0^{-n} + 2(-1)^n \tau_0^{n/2} \cos\left(\left(\frac{\pi}{2} + \Psi\right)n\right), \qquad n \in \mathbb{N}_0.$$

Final remark

Our colleague from Russia Sergey Markelov in private correspondence informed us about deriving several new formulae of the type discussed in the current paper.

Among others, he has found the following ones:

$$\sqrt[3]{\cos\frac{\pi}{13}+\cos\frac{5\pi}{13}}+\sqrt[3]{\cos\frac{3\pi}{13}+\cos\frac{11\pi}{13}}+\sqrt[3]{\cos\frac{7\pi}{13}+\cos\frac{9\pi}{13}}= \\ =\sqrt[3]{\frac{7-3\sqrt[3]{13}}{2}}, \quad (59)$$

$$\sqrt[3]{\cos\frac{\pi}{31}+\cos\frac{15\pi}{31}+\cos\frac{23\pi}{31}+\cos\frac{27\pi}{31}+\cos\frac{29\pi}{31}}+ \\ +\sqrt[3]{\cos\frac{3\pi}{31}+\cos\frac{7\pi}{31}+\cos\frac{17\pi}{31}+\cos\frac{19\pi}{31}+\cos\frac{25\pi}{31}}+ \\ +\sqrt[3]{\cos\frac{5\pi}{31}+\cos\frac{9\pi}{31}+\cos\frac{11\pi}{31}+\cos\frac{13\pi}{31}+\cos\frac{21\pi}{31}}= \\ =\sqrt[3]{\frac{3\sqrt[3]{62}-11}{2}}.$$

Moreover, he is convinced that identities like this one, but for other denominators also exist, for example for all prime p that can be presented as n^2+n+1. More information can be found in http://ru-math.livejournal.com/79774.html – a Russian internet forum where sums of this type are considered and where some efforts are made to present them in the general context.

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