ZESZYTY NAUKOWE POLITECHNIKI ŚLĄSKIEJ 2014

Seria: MATEMATYKA STOSOWANA z. 4 Nr kol. 1920

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CERTAIN METHODS OF COMPLEMENTATION OF LINEAR EXTENSIONS OF DYNAMIC SYSTEMS TO REGULAR SYSTEMS

Summary. The paper presents a method of transformation of two weakly regular systems into one regular system. The method has been generalised to any number of weakly regular systems.

PEWNE METODY DOPEŁNIENIA LINIOWYCH ROZSZERZEŃ UKŁADÓW DYNAMICZNYCH DO UKŁADÓW REGULARNYCH

Streszczenie. W artykule przedstawiono metodę doprowadzenia dwóch układów słabo regularnych do jednego układu regularnego. Metoda została uogólniona na dowolną liczbę układów słabo regularnych.

2010 Mathematics Subject Classification: 34D10, 34D35, 37L25.
Keywords: dynamical system, regular system.
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Received: 14.07.2014.

1. Introduction

It is well known that examination of existence of invariant manifolds of dynamic systems is related to existence of the Green's function for the linearised system, i.e. for a linear extension of a dynamical system (see [1,5,6]). More precisely, if a linear extension has one Green's function, i.e. the system is regular, then the invariant manifold for a heterogeneous extension of the dynamical system can be expressed in an explicit integral form. This makes it possible to examine the smoothness of the invariant manifold. Deep research in this direction can be found in [2,7–9]. Book [3] shows that regularity of a linear extension of a dynamic system having the form of

$$\begin{cases} \dfrac{dx}{dt} = a\left(x\right), & x \in \mathbb{R}^m, \\ \dfrac{dy}{dt} = A\left(x\right) y, & y \in \mathbb{R}^n, \end{cases} \tag{1}$$

is equivalent to the existence of a certain non-degenerated quadratic form whose derivative, with respect to the tested system, is positive definite. Namely, we have the following theorem:

Theorem 1. *Let there be a quadratic form*

$$W = \langle S\left(x\right) y, y \rangle, \qquad y \in \mathbb{R}^n, \tag{2}$$

associated with symmetric matrix $S\left(x\right) \in C^1(\mathbb{R}^m)$, whose derivative with respect to the system of equations

$$\begin{cases} \dfrac{dx}{dt} = a\left(x\right), \\ \dfrac{dy}{dt} = -A^T\left(x\right) y, \end{cases} \tag{3}$$

is positive definite; thus

$$\dot{W} = \left\langle \left[\dot{S}\left(x\right) - S\left(x\right) A^T\left(x\right) - A\left(x\right) S\left(x\right)\right] y, y \right\rangle \geqslant \|y\|^2, \tag{4}$$

then system (1) will be weakly regular. If in addition we assume that

$$det S\left(x\right) \neq 0 \qquad \forall x \in \mathbb{R}^m, \tag{5}$$

then system (1) will be regular.

If a homogeneous linear extension has many different Green's functions, which is a strictly weakly regular system, then examination of smoothness of the invariant manifolds is rather difficult. Therefore, monograph [3] proposes complementing the linear extension to the form of a triangular regular system allowing it to obtain the Green's function for the initial linear extension as an n–dimensional block in a $2n$–dimensional Green's function. This result is formulated in the following theorem:

Theorem 2. *Let system (1) be weakly regular; then the extended system*

$$\begin{cases} \dfrac{dx}{dt} = a\,(x)\,, \\ \dfrac{dy}{dt} = A\,(x)\,y, \\ \dfrac{dz}{dt} = y - A^T\,(x)\,z, \end{cases} \tag{6}$$

is regular. Whereby the derivative of non-degenerated quadratic form

$$V_p = p\langle y,\, z\rangle + \langle S(x)z, z\rangle,$$

with respect to the system (6) is positive definite for sufficiently large values of parameter p.

It turned out that the theorem remains true even for the system

$$\begin{cases} \dfrac{dx}{dt} = a\,(x)\,, \\ \dfrac{dy}{dt} = A\,(x)\,y, \\ \dfrac{dz}{dt} = B(x)y - A^T\,(x)\,z, \end{cases} \tag{7}$$

where matrix $B(x)$ is any positive definite matrix (or even negative definite). Theorem 2 can be further generalised to a wider class of systems having the form

$$\begin{cases} \dfrac{dx}{dt} = a\,(x)\,, \\ \dfrac{dy}{dt} = A\,(x)\,y + B_2(x)z, \\ \dfrac{dz}{dt} = B_1(x)y - A^T(x)z, \end{cases} \tag{8}$$

where $B_1(x)$, $B_2(x)$ are positive definite matrices. Then the derivative of the quadratic form $V = \langle y, z\rangle$ with respect to this system is positive definite, which means that the system is regular.

At this point a new question arises. If we fix the quadratic form $V = \langle y, z\rangle$, what conditions would have be met by the matrices of the system

$$\begin{cases} \dfrac{dx}{dt} = a\left(x\right), \\ \dfrac{dy}{dt} = A_{11}\left(x\right)y + A_{12}(x)z, \\ \dfrac{dz}{dt} = A_{21}(x)y + A_{22}(x)z, \end{cases} \tag{9}$$

in order for the derivative of this form with respect to this system to be positive definite, i.e. for the system to be regular. Of course, after the previous considerations, the solution to this problem seems to be trivial. It is sufficient that the following condition be satisfied: $A_{11} = -A_{22}^T$, and the matrices A_{12} and A_{21} be positive definite. However, the problem remains unsolved and becomes the starting point for a more thorough analysis of the issue of complementation of weakly regular linear extensions of dynamical systems to regular ones that have the only the Green's function (see [4]). This work aims to present the results obtained in this direction.

2. Main results

Previous studies have focused the complementation of one weakly regular system to a regular one. Currently, based on weak regularity of two systems, we will construct one regular system. Consider two systems of differential equations

$$\begin{cases} \dfrac{dx}{dt} = \omega(x), \\ \dfrac{dy}{dt} = A_1(x)y, \end{cases} \qquad \begin{cases} \dfrac{dx}{dt} = \omega(x), \\ \dfrac{dy}{dt} = A_2(x)y, \end{cases} \tag{10}$$

where $y \in \mathbb{R}^n$, $x \in \mathbb{R}^m$, $\omega(x) \in C_{Lip}(\mathbb{R}^m)$, $A_i(x) \in C^0(\mathbb{R}^m)$. The designations come from reference work [3].

The following statement is true.

Theorem 3. *If systems (10) are weakly regular, then the system*

$$\begin{cases} \dfrac{dx}{dt} = \omega(x), \\ \dfrac{dz_1}{dt} = [A_2(x) + \frac{1}{2}(A_1(x) + A_1^T(x)) - I_n]z_1 + [A_2^T(x) + A_1(x)]z_2, \\ \dfrac{dz_2}{dt} = [-A_2(x) + \frac{1}{2}(A_1(x) - A_1^T(x)) + I_n]z_1 - A_2^T(x)z_2, \\ \dfrac{dz_3}{dt} = [A_2(x) + \frac{1}{2}(A_1^T(x) - A_1(x)) + I_n]z_1 - [A_1(x) + A_2^T(x)]z_2 - A_1^T(x)z_3, \end{cases} \tag{11}$$

where $z_i \in \mathbb{R}^n$, $x \in \mathbb{R}^m$, $\omega(x) \in C_{Lip}(\mathbb{R}^m)$, $A_i(x) \in C^0(\mathbb{R}^m)$, is regular, i.e. has exactly one $3n \times 3n$ dimensional Green function.

Also, the derivative of the quadratic form

$$V_p = p^2\{\langle z_1, z_2\rangle + \langle z_1, z_3\rangle + \langle z_2, z_3\rangle\} + p\langle S_2(x)z_2, z_2\rangle + \langle S_1(x)z_3, z_3\rangle, \tag{12}$$

with respect to system (11) for sufficiently large values of $p >> 1$ is positive definite.

Proof. Because of the weak regularity of systems (10) there exist symmetric matrices $S_i(x) \in C'(\mathbb{R}^m, \omega)$, $i = 1, 2$, satisfying the inequality

$$\left\langle [\dot{S}_i(x) - S_i(x)A_i^T(x) - A_i(x)S_i(x)]z, z \right\rangle \geqslant \|z\|^2, \tag{13}$$

whereby $S_i(x)$ may be a degenerated matrix.

Let

$$V_p = p^2\{\langle z_1, z_2\rangle + \langle z_1, z_3\rangle + \langle z_2, z_3\rangle\} + p\langle S_2(x)z_2, z_2\rangle + \langle S_1(x)z_3, z_3\rangle,$$

be a quadratic form with a parameter $p > 0$.

We will show that the derivative of this form with respect to system (11) for sufficiently large values of the parameter $p > 0$, is positive definite.

Let us denote

$$v = \langle z_1, z_2\rangle + \langle z_1, z_3\rangle + \langle z_2, z_3\rangle. \tag{14}$$

By calculating the derivative of the form v with respect to system (11) we obtain

$$\dot{v} = 2\langle I z_1, z_1\rangle.$$

Assuming that

$$w = p\langle S_2(x)z_2, z_2\rangle + \langle S_1(x)z_3, z_3\rangle, \tag{15}$$

the derivative of this form with respect to system (11) is equal to

$$\dot{w} = p\left\{\langle \dot{S}_2 z_2, z_2\rangle - \langle S_2 A_2^T z_2, z_2\rangle - \langle A_2 S_2 z_2, z_2\rangle\right\} + \\ + 2p\langle S_2[-A_2 + \frac{1}{2}(A_1 - A_1^T) + I]z_1, z_2\rangle + \langle \dot{S}_1 z_3, z_3\rangle - \langle S_1 A_1^T z_3, z_3\rangle - \\ - \langle A_1 S_1 z_3, z_2\rangle + 2\langle S_1[A_2 + \frac{1}{2}(A_1^T - A_1) + I]z_1, z_3\rangle - 2\langle S_1[A_1 + A_2^T]z_2, z_3\rangle.$$

Let

$$K_1 = \|S_2[-A_2 + \frac{1}{2}(A_1 - A_1^T) + I]\|_0,$$
$$K_2 = \|S_1[A_2 + \frac{1}{2}(A_1^T - A_1) + I]\|_0,$$
$$K_3 = \|S_1[A_1 + A_2^T]\|_0.$$

Using inequality (13), we obtain

$$\dot{w} \geqslant p\|z_2\|^2 + \|z_3\|^2 - 2pK_1\|z_1\|\|z_2\| - 2K_2\|z_1\|\|z_3\| - 2K_3\|z_2\|\|z_3\|.$$

Since $\dot{V}_p = p^2\dot{v} + \dot{w}$, the estimate of the formula is true

$$\dot{V}_p \geqslant 2p^2\|z_1\|^2 + p\|z_2\|^2 + \|z_3\|^2 - 2pK_1\|z_1\|\|z_2\| - \\ - 2K_2\|z_1\|\|z_3\| - 2K_3\|z_2\|\|z_3\|. \quad (16)$$

Consider the right hand-side of inequality (16) as a quadratic form Φ of three variables t_1, t_2, t_3:

$$\Phi(t_1, t_2, t_3) = 2p^2 t_1^2 + pt_2^2 + t_3^2 - 2pK_1 t_1 t_2 - 2K_2 t_1 t_3 - 2K_3 t_2 t_3,$$

which corresponds to the following matrix

$$T = \begin{pmatrix} 2p^2 & -pK_1 & -K_2 \\ -pK_1 & p & -K_3 \\ -K_2 & -K_3 & 1 \end{pmatrix}.$$

It is obvious that for sufficiently large values of the parameter $p > 0$ matrix T is positive definite, and thus the derivative of the quadratic form V_p with respect to system (11) is positive definite for sufficiently large values of parameter $p > 0$.

Now we will prove that quadratic form (12) is positive definite for $p \gg 0$. Let us write the matrix of quadratic form (12) as follows

$$S_p = \begin{pmatrix} 0 & \frac{1}{2}p^2 I_n & \frac{1}{2}p^2 I_n \\ \frac{1}{2}p^2 I_n & pS_2(x) & \frac{1}{2}p^2 I_n \\ \frac{1}{2}p^2 I_n & \frac{1}{2}p^2 I_n & S_1(x) \end{pmatrix}. \quad (17)$$

The matrix S_p can be expressed in the following form

$$S_p = p^2 J + p\bar{S}_2(x) + \bar{S}_1(x),$$

where

$$J = \frac{1}{2}\begin{pmatrix} 0 & I_n & I_n \\ I_n & 0 & I_n \\ I_n & I_n & 0 \end{pmatrix},$$

$$\bar{S}_1(x) = \mathrm{diag}(0, 0, S_1(x)),$$
$$\bar{S}_2(x) = \mathrm{diag}(0, S_2(x), 0).$$

We will show that matrix S_p^2 for sufficiently large values of parameter p is positive definite.

Because $S_p^2 = p^4 J^2 + p^3(J\bar{S}_2(x) + \bar{S}_2(x)J) + p^2(J\bar{S}_1(x) + \bar{S}_2^2(x) + \bar{S}_1(x)J) + \bar{S}_1^2(x)$, then assuming $u = [u_1, u_2, u_3]$, $u_i \in \mathbb{R}^n$, we obtain

$$\langle S_p^2 u, u\rangle = p^4\langle J^2 u, u\rangle + p^3\langle [J\bar{S}_2(x) + \bar{S}_2(x)J]u, u\rangle + \\ + p^2\langle [J\bar{S}_1(x) + \bar{S}_2^2(x) + \bar{S}_1(x)J]u, u\rangle + \langle \bar{S}_1^2(x)u, u\rangle.$$

Let us estimate each component of $\langle S_p^2 u, u\rangle$; thus

$$\langle J^2 u, u\rangle \geqslant \frac{1}{4}\left(\|z_1 + z_2 + z_3\|^2 + \|z_1\|^2 + \|z_2\|^2 + \|z_3\|^2\right) \geqslant \frac{1}{4}\|u\|^2,$$
$$\langle [J\bar{S}_2(x) + \bar{S}_2(x)J]u, u\rangle \geqslant -M_2\|u\|^2,$$
$$\langle [J\bar{S}_1(x) + \bar{S}_2^2(x) + \bar{S}_1(x)J]u, u\rangle \geqslant -M_1\|u\|^2,$$
$$\langle \bar{S}_1^2(x)u, u\rangle \geqslant -M_0\|u\|^2,$$

where $M_i = \mathrm{const} > 0$. Therefore, we obtain the estimate:

$$\langle S_p^2 u, u\rangle \geqslant (\frac{1}{4}p^4 - p^3 M_2 - p^2 M_1 - M_0)\|u\|^2.$$

It follows that for sufficiently large values of parameter $p > 0$ matrix S_p^2 is positive definite, and hence $\det S_p^2 \neq 0$ and, consequently, $\det S_p \neq 0$ for all $x \in \mathbb{R}^m$.

We have proven that quadratic form (12) has a positive definite derivative with respect to system (11) and matrix S_p associated with this form is non-degenerated for sufficiently large values of parameter $p > 0$, so system (11) is regular, i.e. has exactly one Green's function. □

In the case of two weakly regular systems

$$\begin{cases} \dfrac{dx}{dt} = \omega(x), \\ \dfrac{dy}{dt} = A_i(x)y, \qquad i = 1, 2, \end{cases}$$

where $y \in \mathbb{R}^n$, $x \in \mathbb{R}^m$, $\omega(x) \in C_{Lip}(\mathbb{R}^m)$, $A_i(x) \in C^0(\mathbb{R}^m)$ the structure of such a matrix is obtained

$$P(x) = \begin{pmatrix} A_2 + \frac{1}{2}(A_1 + A_1^T) - I_n & A_2^T + A_1 & 0 \\ -A_2 + \frac{1}{2}(A_1 - A_1^T) + I_n & -A_2^T & 0 \\ A_2 + \frac{1}{2}(A_1^T - A_1) + I_n & -[A_1 + A_2^T] & -A_1^T \end{pmatrix},$$

that the system

$$\begin{cases} \dfrac{dx}{dt} = \omega(x), \\ \dfrac{dy}{dt} = P(x)z, \qquad z \in \mathbb{R}^{3n}, \end{cases} \tag{18}$$

is regular. Let us illustrate this in the following example.

Example

Let us consider two weakly regular systems of equations

$$\begin{cases} \dfrac{dx}{dt} = \sin x, \quad x \in \mathbb{R}, \\ \dfrac{dy}{dt} = 3(\cos x)y, \quad y \in \mathbb{R}, \end{cases} \qquad \begin{cases} \dfrac{dx}{dt} = 1, \quad x \in \mathbb{R}, \\ \dfrac{dy}{dt} = -(\operatorname{tgh} x)y, \quad y \in \mathbb{R}. \end{cases}$$

We will show that the system

$$\begin{cases} \dfrac{dx_1}{dt} = \sin x_1, \\ \dfrac{dx_2}{dt} = 1, \\ \dfrac{dy_1}{dt} = [-1 + 3\cos x_1 - \operatorname{tgh} x_2]y_1 + [3\cos x_1 - \operatorname{tgh} x_2]y_2, \\ \dfrac{dy_2}{dt} = [1 + \operatorname{tgh} x_2]y_1 + [\operatorname{tgh} x_2]y_2, \\ \dfrac{dy_3}{dt} = [1 - \operatorname{tgh} x_2]y_1 - [3\cos x_1 - \operatorname{tgh} x_2]y_2 - [3\cos x_1]y_3, \end{cases} \tag{19}$$

is regular.

Let's take the quadratic form

$$V_p = p^2(y_1y_2 + y_1y_3 + y_2y_3) + p(\operatorname{tgh} x_2)y_2^2 - (\cos x_1)y_3^2,$$

and assume that

$$v_1 = y_1y_2 + y_1y_3 + y_2y_3.$$

Then the derivative of v_1 with respect to system (19) is equal to

$$\dot{v}_1 = 2y_1^2.$$

Similarly, by calculating the derivative $v_2 = (\operatorname{tgh} x_2)y_2^2$ with respect to system (19), we obtain

$$\begin{aligned}\dot{v}_2 &= \frac{1}{\operatorname{ctgh}^2 x_2} y_2^2 + 2(\operatorname{tgh} x_2)y_2\{[1+\operatorname{tgh} x_2]y_1 + [\operatorname{tgh} x_2]y_2\} = \\ &= \left[\frac{1}{\operatorname{ctgh}^2 x_2} + 2(\operatorname{tgh} x_2)^2\right] y_2^2 + 2\operatorname{tgh} x_2(1+\operatorname{tgh} x_2)y_1y_2 \geqslant \\ &\geqslant y_2^2 - 4|y_1||y_2|.\end{aligned}$$

Finally, by calculating the derivative of the form $v_3 = (-\cos x_1)y_3^2$ with respect to system (19), we obtain

$$\begin{aligned}\dot{v}_3 &= y_3^2 \sin^2 x_1 - 2y_3\{[1-\operatorname{tgh} x_2]y_1 - [3\cos x_1 - \operatorname{tgh} x_2]y_2 - [3\cos x_1]y_3\} = \\ &= y_3^2(\sin^2 x_1 + 6\cos^2 x_1) - 2[1-\operatorname{tgh} x_2]y_1y_3 + 2[3\cos x_1 - \operatorname{tgh} x_2]y_2y_3 \geqslant \\ &\geqslant y_3^2 - 4|y_1||y_3| - 8|y_2||y_3|.\end{aligned}$$

Eventually, the derivative of quadratic form V_p with respect to system (19) satisfies the inequality

$$V_p \geqslant 2p^2y_1^2 + py_2^2 - 4p|y_1||y_2| + y_3^2 - 4|y_1||y_3| - 8|y_2||y_3|.$$

Let us consider the right hand-side of the above inequality as quadratic form Φ dependent on three variables t_1, t_2, t_3:

$$\Phi(t_1, t_2, t_3) = 2p^2t_1^2 + pt_2^2 - 4pt_1t_2 + t_3^2 - 4t_1t_3 - 8t_2t_3.$$

The matrix associated with this form is the following

$$T = \begin{pmatrix} 2p^2 & -2p & -2 \\ -2p & p & -4 \\ -2 & -4 & 1 \end{pmatrix}.$$

Matrix T is positive definite for $p > 20$; hence system (19) is regular for $p > 20$.

3. Generalisation of the results

Let us consider k systems of differential equations

$$\begin{cases} \dfrac{dx}{dt} = \omega(x), \\ \dfrac{dy}{dt} = A_i(x)y, \qquad i = 1, 2, \ldots, k, \end{cases} \tag{20}$$

where $y \in \mathbb{R}^n$, $x \in \mathbb{R}^m$, $t \in \mathbb{R}$, $\omega(x) \in C_{Lip}(\mathbb{R}^m)$, $A_i(x) \in C^0(\mathbb{R}^m)$. Assume that each of these systems is weakly regular, i.e. for each of these systems there exists at least one Green's function.

The problem is to find for $k \geqslant 3$ a matrix

$$P(x) = \tilde{P}(A_1, A_2, \ldots, A_k),$$

with dimensions $(k+1)n \times (k+1)n$ so that the system

$$\begin{cases} \dfrac{dx}{dt} = \omega(x), \\ \dfrac{dz}{dt} = P(x)z, \qquad z \in \mathbb{R}^{(k+1)n}, \end{cases} \tag{21}$$

is regular, namely that it has exactly one the Green's function.

At the beginning, consider a case where $k = 3$. Let us express system (21) in the following form

$$\begin{cases} \dfrac{dx}{dt} = \omega(x), \\ \dfrac{dz_1}{dt} = P_{11}(x)z_1 + \ldots + P_{14}(x)z_4, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \dfrac{dz_4}{dt} = P_{41}(x)z_1 + \ldots + P_{41}(x)z_4, \quad z_i \in \mathbb{R}^n. \end{cases} \tag{22}$$

By calculating the derivative of a quadratic form

$$V(z) = 2[\langle z_1,\, z_2\rangle + \langle z_1,\, z_3\rangle + \langle z_1,\, z_4\rangle + \langle z_2,\, z_3\rangle + \langle z_2,\, z_4\rangle + \langle z_3,\, z_4\rangle], \tag{23}$$

we obtain

$$\dot{V}(z) = \langle [SP(x) + P^T(x)S]z,\, z\rangle,$$

where

$$S = \begin{pmatrix} 0 & I & I & I \\ I & 0 & I & I \\ I & I & 0 & I \\ I & I & I & 0 \end{pmatrix}. \tag{24}$$

Suppose that the quadratic form satisfies the estimate

$$\dot{V}(z) \geqslant \|z_1\|^2. \tag{25}$$

In order for this condition to be fulfilled, it suffices that

$$SP(x) = \mathrm{diag}(B_0, 0, 0, 0) + M(x), \tag{26}$$

where matrices $M(x)$ and $B(x)$ meet the conditions

$$M^T(x) \equiv -M(x), \qquad \langle B_0(x)z_1,\, z_1\rangle \geqslant \beta_0\|z_1\|^2, \quad \beta_0 > 0. \tag{27}$$

Matrix $P(x)$ can be expressed in the following form

$$\begin{aligned} P(x) &= S^{-1}[\mathrm{diag}(B_0,0,0,0) + M(x)] = \\ &= \frac{1}{3}\begin{pmatrix} -2I & I & I & I \\ I & -2I & I & I \\ I & I & -2I & I \\ I & I & I & -2I \end{pmatrix}\begin{pmatrix} B_0(x) & M_{12}(x) & M_{13}(x) & M_{14}(x) \\ -M_{12}^T(x) & 0 & M_{23}(x) & M_{24}(x) \\ -M_{13}^T(x) & -M_{23}^T(x) & 0 & M_{34}(x) \\ -M_{14}^T(x) & -M_{24}^T(x) & -M_{34}^T(x) & 0 \end{pmatrix} = \\ &= [P_{ij}(x)]_{i,j=1}^4, \end{aligned} \tag{28}$$

where

$$\begin{aligned} P_{11}(x) &= \frac{1}{3}(-2B_0(x) - \sum_{i=2}^{4} M_{1i}^T(x)), \\ P_{21}(x) &= \frac{1}{3}(B_0(x) + 2M_{12}^T(x) - M_{13}^T(x) - M_{14}^T(x)), \\ P_{22}(x) &= \frac{1}{3}(M_{12}(x) - M_{23}^T(x) - M_{24}^T(x)), \\ P_{33}(x) &= \frac{1}{3}(M_{13}(x) + M_{23}(x) - M_{34}^T(x)), \\ P_{44}(x) &= \frac{1}{3}(M_{14}(x) + M_{24}(x) + M_{34}(x)). \end{aligned} \tag{29}$$

Denoting

$$\bar{P}(x) = [P_{ij}(x)]_{i,j=2}^4, \qquad \bar{z} = [z_2, z_3, z_4], \tag{30}$$

let us consider the following system

$$\begin{cases} \dfrac{dx}{dt} = \omega(x), \\ \dfrac{d\bar{z}}{dt} = \bar{P}(x)\bar{z}, \end{cases} \tag{31}$$

If matrix $\bar{P}(x)$ of system (31) has a special block form

$$\bar{P}(x) = \begin{pmatrix} -A_1^T(x) & 0 & 0 \\ * & -A_2^T(x) & 0 \\ * & * & -A_3^T(x) \end{pmatrix}, \tag{32}$$

then we obtain the following equations

$$\begin{aligned} P_{22}(x) &= -A_1^T(x), \quad P_{23}(x) = 0, \\ P_{33}(x) &= -A_2^T(x), \quad P_{24}(x) = 0, \\ P_{44}(x) &= -A_3^T(x), \quad P_{34}(x) = 0. \end{aligned} \tag{33}$$

based on which one can uniquely determine matrices $M_{ij}(x)$, $i = 1, 2, 3$, $j = 2, 3, 4$. Using (29), equations (33) take the following form

$$M_{12}(x) - M_{23}^T(x) - M_{24}^T(x) = -3A_1^T(x), \tag{34}$$

$$M_{13}(x) + M_{23}(x) - M_{34}^T(x) = -3A_2^T(x), \tag{35}$$

$$M_{14}(x) + M_{24}(x) + M_{34}(x) = -3A_3^T(x), \tag{36}$$

and

$$M_{13}(x) - 2M_{23}(x) - M_{34}^T(x) = 0, \tag{37}$$

$$M_{14}(x) - 2M_{24}(x) + M_{34}(x) = 0, \tag{38}$$

$$M_{14}(x) + M_{24}(x) - 2M_{34}(x) = 0. \tag{39}$$

By subtracting equation (37) from equation (35), equation (38) from (36) and equation (39) from equation (36), we obtain, respectively

$$M_{23}(x) = -A_2^T(x), \qquad M_{24}(x) = -A_3^T(x), \qquad M_{34}(x) = -A_3^T(x), \tag{40}$$

and, hence, we can determine the remaining matrices $M_{ij}(x)$:

$$\begin{aligned} M_{12}(x) &= -3A_1^T(x) - A_2(x) - A_3(x), \\ M_{13}(x) &= -2A_2^T(x) - A_3(x), \\ M_{14}(x) &= -A_3^T(x). \end{aligned} \tag{41}$$

If systems of equations (20) are weakly regular, then, because the conditions (33) for matrix $\bar{P}(x)$ take place, the derivative of the quadratic form

$$\bar{V}(x, \bar{z}) = \langle S(x)\bar{z}, \bar{z} \rangle, \tag{42}$$

with respect to system (31) is positive definite, i.e.

$$\dot{\bar{V}}(x, \bar{z}) = \left\langle [\dot{S}(x) + S(x)\bar{P}(x) + \bar{P}^T(x)S(x)]\bar{z}, \bar{z} \right\rangle \geqslant \|\bar{z}\|^2. \tag{43}$$

When conditions (25) and (43) are met, the derivative of the quadratic form

$$pV(z) + \bar{V}(x, \bar{z}), \tag{44}$$

with respect to the system (21) for sufficiently large values of parameter $p > 0$ is positive definite. Since quadratic form (44) for sufficiently large values of parameter p is non-degenerated, then system (21) is regular, wherein matrix $P(x)$ is of the following form

$$P(x) = \frac{1}{3}\begin{pmatrix} -2I & I & I & I \\ I & -2I & I & I \\ I & I & -2I & I \\ I & I & I & -2I \end{pmatrix} \times$$
$$\times \begin{pmatrix} B_0 & -3A_1^T - A_2 - A_3 & -2A_2^T - A_3 & -A_3^T \\ 3A_1 + A_2^T + A_3^T & 0 & -A_2^T & -A_3^T \\ 2A_2 + A_3^T & A_2 & 0 & -A_3^T \\ A_3 & A_3 & A_3 & 0 \end{pmatrix}, \tag{45}$$

where $B_0(x) \in C^0(\mathbb{R}^m)$ is any positive definite matrix.

In a case where $k > 3$ matrix $P(x)$ is of the following form

$$P(x) = \frac{1}{k}\begin{pmatrix} -(k-1)I & I & I & \dots & I \\ I & -(k-1)I & I & \dots & I \\ I & I & -(k-1)I & \dots & I \\ \dots & \dots & \dots & \dots & \dots \\ I & I & I & \dots & -(k-1)I \end{pmatrix} \times$$
$$\times \begin{pmatrix} B_0(x) & M_{12}(x) & M_{13}(x) & \dots & M_{1,k+1}(x) \\ -M_{12}^T(x) & 0 & M_{23}(x) & \dots & M_{2,k+1}(x) \\ -M_{13}^T(x) & -M_{23}^T(x) & 0 & \dots & M_{3,k+1}(x) \\ \dots & \dots & \dots & \dots & \dots \\ -M_{1,k+1}^T(x) & -M_{2,k+1}^T(x) & -M_{3,k+1}^T(x) & \dots & 0 \end{pmatrix}. \tag{46}$$

As before, assuming that matrix $\bar{P}(x) = [P_{ij}]_{i,j=2}^k$ has the following special block form

$$\bar{P}(x) = \begin{pmatrix} -A_1^T(x) & 0 & 0 & \dots & 0 \\ * & -A_2^T(x) & 0 & \dots & 0 \\ * & * & -A_3^T(x) & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ * & * & * & \dots & -A_k^T(x) \end{pmatrix}, \tag{47}$$

matrices M_{ij} can be uniquely determined

$$\begin{gathered} M_{23}(x) = -A_2^T(x), \\ M_{24}(x) = M_{34}(x) = -A_3^T(x), \\ M_{25}(x) = M_{35}(x) = M_{45}(x) = -A_4^T(x), \\ \dots\dots\dots\dots \\ M_{2,k+1}(x) = M_{3,k+1}(x) = M_{4,k+1}(x) = \dots = M_{k,k+1}(x) = -A_k^T(x), \end{gathered} \tag{48}$$

and

$$\begin{aligned} M_{12}(x) &= -kA_1^T(x) - \sum_{i=2}^{k} A_i(x), \\ M_{13}(x) &= -(k-1)A_2^T(x) - \sum_{i=3}^{k} A_i(x), \\ &\vdots \\ M_{1p}(x) &= -(k-p+2)A_p^T(x) - \sum_{i=p}^{k} A_i(x), \\ M_{1,k+1}(x) &= -A_k^T(x). \end{aligned} \tag{49}$$

Therefore, the following theorem is true.

Theorem 4. *Let systems (20) be weakly regular; then the system*

$$\begin{cases} \dfrac{dx}{dt} = \omega(x), \\ \dfrac{dz}{dt} = P(x)z, \quad z = (z_1, z_2, \dots, z_k), \end{cases} \tag{50}$$

where $z_i \in \mathbb{R}^n$, $x \in \mathbb{R}^m$, $\omega(x) \in C_{Lip}(\mathbb{R}^m)$, $P(x) \in C^0(\mathbb{R}^m)$, *is regular, i.e. has exactly one* $(k \cdot n) \times (k \cdot n)$ *dimensional Green's function, wherein matrix* $P(x)$ *is defined by formula (46).*

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