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SOLVING OF THE TWO-DIMENSIONAL UNSTEADY HEAT TRANSFER PROBLEM BY USING THE HOMOTOPY ANALYSIS METHOD

Summary. In this paper a solution of the two-dimensional unsteady heat transfer problem by using the homotopy analysis method is described. In presented method the functional series is generated. This paper contains the sufficient condition for convergence of this series. We also give the estimation of error of the approximate solution obtained by taking the partial sum of received series.

ROZWIĄZYWANIE DWUWYMIAROWEGO NIESTACJONARNEGO ZAGADNIENIA PRZEWODZENIA CIEPŁA PRZY WYKORZYSTANIU HOMOTOPIJNEJ METODY ANALIZY

Streszczenie. W artykule opisano rozwiązanie dwuwymiarowego niestacjonarnego zagadnienia przewodzenia ciepła przy wykorzystaniu homotopijnej metody analizy. W metodzie tej tworzony jest szereg funkcyjny. Podano warunek wystarczający zbieżności tego szeregu, a także oszacowanie błędu rozwiązania przybliżonego, które uzyskujemy, biorąc sumę częściową szeregu.

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1. Introduction

The homotopy analysis method was developed in the 90's of last century by Shijun Liao [13–16]. It is used for solving various types of the operator equations. In particular, this method was already applied for solving the integral equations [2, 9, 24], differential-integral equations [4, 7, 27] and differential equations [6, 18, 23], including the heat conduction problems [1, 5, 10–12, 26], as well as the fractional differential equations [3, 28].

In the current paper we describe the solution of two-dimensional unsteady heat transfer problem by application of the homotopy analysis method. In this method we generate the functional series, elements of which satisfy the differential equation resulting from the considered problem. The gained equation is easier to solve in comparison with the starting one. If the generated series is convergent then its sum is the solution of the starting equation. In this paper we present the sufficient condition for convergence of this series. This paper contains also the estimation of error of the approximate solution which we obtain by taking the partial sum of received series. Additionally we present the examples illustrating the usefulness of investigated method.

2. Problem and its solution

We will search for the solution of the two-dimensional unsteady heat transfer problem

$$\frac{\partial u(x,y,t)}{\partial t} = a\Big(\frac{\partial^2 u(x,y,t)}{\partial x^2} + \frac{\partial^2 u(x,y,t)}{\partial y^2}\Big), \qquad (x,y,t) \in D, \tag{1}$$

where $D = \{(x,y,t);\ x \in (b_1,b_2),\ y \in (d_1,d_2),\ t \in (0,t^*)\}$ and a is the heat diffusion coefficient. On boundary of the domain the Dirichlet boundary conditions are given

$$u(b_1,y,t) = \varphi_1(y,t), \qquad u(b_2,y,t) = \varphi_2(y,t), \qquad y \in [d_1,d_2],\ t \in (0,t^*), \tag{2}$$

$$u(x,d_1,t) = \theta_1(x,t), \qquad u(x,d_2,t) = \theta_2(x,t), \qquad x \in [b_1,b_2],\ t \in (0,t^*), \tag{3}$$

where φ_1, φ_2, θ_1 and θ_2 are continuous functions. The initial condition is of the form

$$u(x,y,0) = \psi(x,y), \qquad x \in [b_1,b_2],\ y \in [d_1,d_2]. \tag{4}$$

To solve the above problem we intend to use the homotopy analysis method. Using this method we are able to solve the operator equation

$$N(u(z)) = 0, \qquad z \in \Omega, \tag{5}$$

where N is a given operator and u is an unknown function. In the method we are looking for the solution in form of the series

$$u(z) = \sum_{m=0}^{\infty} u_m(z), \tag{6}$$

where u_0 is a given function and the rest of elements are determined by formula

$$L\big(u_m(z) - \chi_m\, u_{m-1}(z)\big) = h\, R_m\big(\overline{u}_{m-1}, z\big), \tag{7}$$

where L is the auxiliary linear operator with property $L(0) = 0$, $h \neq 0$ denotes the convergence control parameter, $\overline{u}_{m-1} = \{u_0(z), u_1(z), \ldots, u_{m-1}(z)\}$,

$$\chi_m = \begin{cases} 0 & m \leqslant 1, \\ 1 & m > 1 \end{cases} \tag{8}$$

and

$$R_m\big(\overline{u}_{m-1}, z\big) = \frac{1}{(m-1)!} \left(\frac{\partial^{m-1}}{\partial p^{m-1}} N\Big(\sum_{i=0}^{\infty} u_i(z)\, p^i \Big) \right)\bigg|_{p=0}. \tag{9}$$

More detailed description of the method may be found in literature [10, 14, 15].

For the considered equation (1) we have

$$N\big(u(x,y,t)\big) = a \left(\frac{\partial^2 u(x,y,t)}{\partial x^2} + \frac{\partial^2 u(x,y,t)}{\partial y^2} \right) - \frac{\partial u(x,y,t)}{\partial t}. \tag{10}$$

As the linear operator L we may take operator $\frac{\partial^2}{\partial x^2}$ or operator $\frac{\partial^2}{\partial y^2}$:

$$L_1(u) = \frac{\partial^2 u}{\partial x^2}, \qquad L_2(u) = \frac{\partial^2 u}{\partial y^2}. \tag{11}$$

Then we may use the averaging method, similarly as it is done in the Adomian decomposition method [8]. By using this approach, in our case, we solve two problems with different choice of the linear operator, averaged solutions of which will give the solution of the initial problem.

After simple transformations we get for $m \geqslant 1$:

$$R_m\big(\overline{u}_{m-1}, x, y, t\big) = N(u_{m-1}(x,y,t)). \tag{12}$$

In this way, for $m = 1$ we obtain two partial differential equations ($k = 1, 2$):

$$L_k\big(u_{k,1}(x,y,t)\big) = h\left(a\,\frac{\partial^2 u_{k,0}(x,y,t)}{\partial x^2} + a\,\frac{\partial^2 u_{k,0}(x,y,t)}{\partial y^2} - \frac{\partial u_{k,0}(x,y,t)}{\partial t}\right), \quad (13)$$

while for $m \geqslant 2$ the equations take the form ($k = 1, 2$):

$$L_k\big(u_{k,m}(x,y,t)\big) = L_k\big(u_{k,m-1}(x,y,t)\big) + \\ + h\left(a\,\frac{\partial^2 u_{k,m-1}(x,y,t)}{\partial x^2} + a\,\frac{\partial^2 u_{k,m-1}(x,y,t)}{\partial y^2} - \frac{\partial u_{k,m-1}(x,y,t)}{\partial t}\right). \quad (14)$$

In order to ensure the uniqueness of solution we have to complete the above equations by some additional conditions. For that purpose we use the boundary conditions (2)–(3). For $m = 1$ and the first equation ($k = 1$) we put the conditions in the following form

$$u_{1,0}(b_1,y,t) + u_{1,1}(b_1,y,t) = \varphi_1(y,t), \quad (15)$$

$$u_{1,0}(b_2,y,t) + u_{1,1}(b_2,y,t) = \varphi_2(y,t), \quad (16)$$

whereas for the second equation ($k = 2$) the conditions take the form

$$u_{2,0}(x,d_1,t) + u_{2,1}(x,d_1,t) = \theta_1(x,t), \quad (17)$$

$$u_{2,0}(x,d_2,t) + u_{2,1}(x,d_2,t) = \theta_2(x,t). \quad (18)$$

Next, for $m > 1$ we impose, respectively, the following conditions

$$u_{1,m}(b_1,y,t) = 0, \quad (19)$$

$$u_{1,m}(b_2,y,t) = 0, \quad (20)$$

and

$$u_{2,m}(x,d_1,t) = 0, \quad (21)$$

$$u_{2,m}(x,d_2,t) = 0. \quad (22)$$

As the initial approximation we may take the function describing the initial condition ($k = 1, 2$):

$$u_{k,0}(x,y,t) = \psi(x,y). \quad (23)$$

Averaging the results we have

$$u_m(x,y,t) = \frac{1}{2}\big(u_{1,m}(x,y,t) + u_{2,m}(x,y,t)\big), \qquad m = 0, 1, 2, \ldots. \quad (24)$$

Choosing appropriately the value of convergence control parameter h we may affect the area of convergence of series (6) and the rate of this convergence [15,17, 19]. One way of choosing this value is the so-called "optimization method" [15,25]. In this method we define the squared residual of governing equation

$$E_n(h) = \iiint\limits_D \Big(N\left[\widehat{u}_n(x,y,t)\right]\Big)^2 \, dx\, dy\, dt, \tag{25}$$

where $\widehat{u}_n$ is the approximate solution specified as

$$\widehat{u}_n(x,y,t) = \sum_{m=0}^{n} u_m(x,y,t). \tag{26}$$

Optimal value of the convergence control parameter will be obtained by setting minimum of the squared residual. In this method we define also the so-called effective region of the convergence control parameter

$$\mathbf{R}_h = \big\{h : \lim_{n\to\infty} E_n(h) = 0\big\}. \tag{27}$$

Choosing the value of convergence control parameter different from the optimal value, but still belonging to the effective region, we still get the convergent series however with the lower convergence rate.

In this way the problem is reduced to solution of the sequence of differential equations (13) and (14) with the proper boundary conditions and to determination of the value of convergence control parameter. Obtained equations are easier to solve in comparison with the initial partial differential equation (the unknown functions appear only once in every equation).

Similarly as for the one-dimensional case [10,11] and for the two-dimensional steady problem [5], for the discussed case as well we may prove the adequate theorems concerning the convergence of obtained series and estimation of the error of approximate solution. Proofs of these theorems run analogically as proofs of the corresponding theorems in [10,11,17,19–22].

Theorem 1. *Let functions u_m, $m \geqslant 1$, be determined in the way described above. If series $\sum_{m=0}^{\infty} u_m$ is convergent, then its sum satisfies the considered equation.*

Theorem 2. *Let functions u_m, $m \geqslant 1$, be determined as it was described above. If parameter h is selected in such a way that there exist the constants $\beta_h \in (0,1)$ and $m_0 \in \mathbb{N}$ such that for each $m \geqslant m_0$ the following inequality*

$$\|u_{m+1}\| \leqslant \beta_h \,\|u_m\|, \tag{28}$$

is satisfied, then the series $\sum_{m=0}^{\infty} u_m$ is uniformly convergent.

Theorem 3. *If assumptions of Theorem 2 are satisfied and additionally $n \in \mathbb{N}$ and $n \geqslant m_0$, then we get the following estimation of error of approximate solution*

$$\|u - \widehat{u}_n\| \leqslant \frac{\beta_h^{n+1-m_0}}{1-\beta_h} \|u_{m_0}\|. \tag{29}$$

3. Examples

Example 1

In the first example we put $b_1 = 0$, $b_2 = 1$, $d_1 = 0$, $d_2 = 1$, $t^* = 1$, $a = 1$ and

$$\begin{aligned} \varphi_1(y,t) &= y^2 + 4\,t, & \varphi_2(y,t) &= y^2 + y + 4\,t + 1, \\ \theta_1(x,t) &= x^2 + 4\,t, & \theta_2(x,t) &= x^2 + x + 4\,t + 1, \\ \psi(x,y) &= x^2 + x\,y + y^2. \end{aligned}$$

As the initial approximations $u_{1,0}$ and $u_{2,0}$ we take the function describing the initial condition

$$u_{1,0}(x,y,t) = u_{2,0}(x,y,t) = x^2 + x\,y + y^2.$$

In result of proper calculations we obtain successively

$$\begin{aligned} u_0(x,y,t) &= x^2 + x\,y + y^2, \\ u_1(x,y,t) &= 4\,t + h\left(x^2 - x - y + y^2\right), \\ u_2(x,y,t) &= h^2\left(x^2 - x - y + y^2\right), \\ u_3(x,y,t) &= h^2\,(h+1)\left(x^2 - x - y + y^2\right), \\ u_4(x,y,t) &= h^2\,(h+1)^2\left(x^2 - x - y + y^2\right), \\ u_5(x,y,t) &= h^2\,(h+1)^3\left(x^2 - x - y + y^2\right). \end{aligned}$$

Generally, we have

$$u_m(x,y,t) = h^2\,(h+1)^{m-2}\left(x^2 - x - y + y^2\right), \qquad m \geqslant 2.$$

Thus we get the exact solution of considered equation

$$\begin{aligned} u(x,y) = \sum_{m=0}^{\infty} u_m(x,y) &= x^2 + x\,y + y^2 + 4\,t + h\left(x^2 - x - y + y^2\right) + \\ &+ \left(x^2 - x - y + y^2\right) h^2 \sum_{m=2}^{\infty} (h+1)^{m-2} = x^2 + x\,y + y^2 + 4\,t, \end{aligned}$$

if only the geometric series, occurring in the above equation, is convergent. It will be for $h \in (-2, 0)$. In this way we get the effective region of the convergence control parameter $\mathbf{R}_h = (-2, 0)$.

Because in this case the squared residual for $n \geqslant 3$ is equal to

$$E_n(h) = 16\,(h+1)^{2n-2},$$

therefore the optimal value of convergence control parameter h is equal to -1. In Figure 1 the plot of logarithm of the squared residual E_7 is shown.

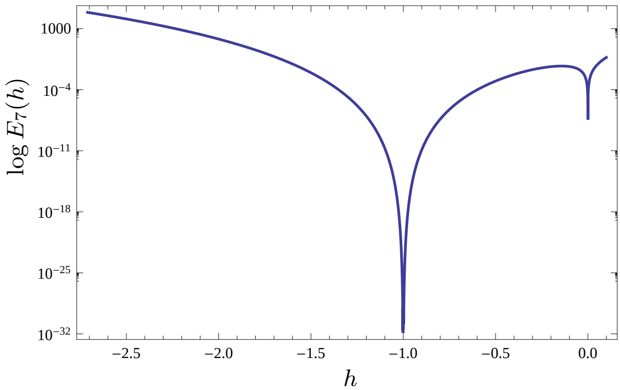

Fig. 1. Logarithm of the squared residual E_7
Rys. 1. Logarytm kwadratu reszt E_7

Example 2

In the second example we assume $b_1 = 0$, $b_2 = 1$, $d_1 = 0$, $d_2 = 1$, $t^* = 1$, $a = 1/2$ and

$$\varphi_1(y,t) = e^{y+t}, \qquad \varphi_2(y,t) = e^{y+t+1},$$
$$\theta_1(x,t) = e^{x+t}, \qquad \theta_2(x,t) = e^{x+t+1},$$
$$\psi(x,y) = e^{x+y}.$$

In this case the exact solution is given by function

$$u_e(x,y,t) = e^{x+y+t}.$$

As the initial approximations $u_{1,0}$ and $u_{2,0}$ we take the function describing the initial condition

$$u_{1,0}(x,y,t) = u_{2,0}(x,y,t) = e^{x+y}.$$

After the first step of calculations we obtain

$$u_1(x,y,t) = \frac{1}{2}\left(e^t - h - 1\right)\left(e^x + e^y + (e-1)\left(x\,e^y + y\,e^x\right)\right) + h\,e^{x+y}.$$

Plot of logarithm of the squared residual E_7 is displayed in Figure 2. In this case the optimal value of convergence control parameter h is equal to -2. Whereas we could not determine the exact form of the effective region of the convergence control parameter. However it is known that interval $(-\frac{7}{2}, 0)$ is included in this region.

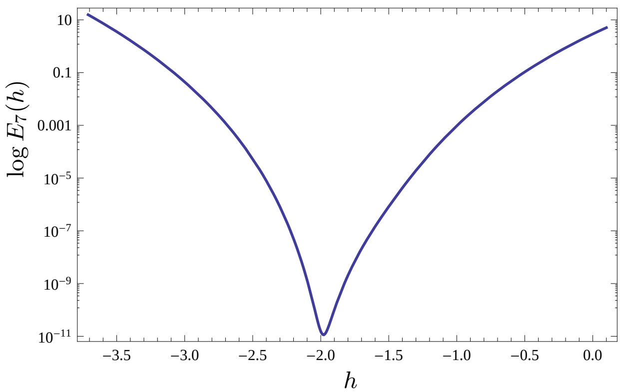

Fig. 2. Logarithm of the squared residual E_7
Rys. 2. Logarytm kwadratu reszt E_7

In this example we are not able to determine sum of the series, therefore we collected in Table 1 the absolute (Δ) and relative errors (δ) of the approximate solutions $\widehat{u}_n$ estimating the exact solution u_e. The errors decrease rapidly and computation of only four terms provides the error lower than 0.0075%. Figure 3 presents the distribution of absolute errors of the exact solution approximation obtained for $n = 10$ and $t = 1/2$ and $t = 1$. As indicated by the example, for properly chosen values of the convergence control parameter h, if it is impossible to predict the general form of function u_m or to calculate the sum of series in (6), it is sufficient to make use of the sum of several first functions u_m to provide a very good approximation of the sought solution.

Table 1
Errors of the exact solution approximation ($\Delta\widehat{u}_n$ – absolute error, $\delta\widehat{u}_n$ – percentage relative error)

n	$\Delta\widehat{u}_n$	$\delta\widehat{u}_n$ [%]
1	1.434	14.323
2	$6.615 \cdot 10^{-2}$	0.720
3	$1.444 \cdot 10^{-2}$	0.160
4	$6.749 \cdot 10^{-4}$	$7.498 \cdot 10^{-3}$
5	$1.478 \cdot 10^{-4}$	$1.642 \cdot 10^{-3}$
6	$6.732 \cdot 10^{-6}$	$8.489 \cdot 10^{-5}$
7	$1.029 \cdot 10^{-6}$	$2.295 \cdot 10^{-5}$
8	$2.565 \cdot 10^{-8}$	$5.708 \cdot 10^{-7}$
9	$1.056 \cdot 10^{-8}$	$2.357 \cdot 10^{-7}$
10	$2.621 \cdot 10^{-10}$	$5.849 \cdot 10^{-9}$

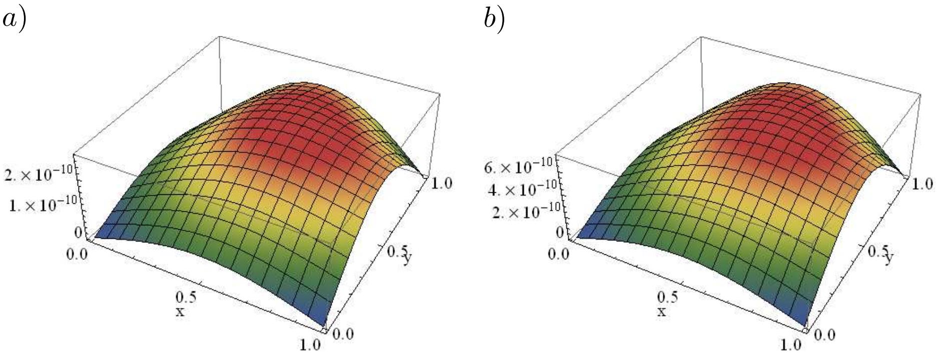

Fig. 3. Distribution of error of the exact solution approximation for $n = 10$ and $t = 1/2$ (a) or $t = 1$ (b)

Rys. 3. Rozkład błędów rozwiązania przybliżonego dla $n = 10$ oraz $t = 1/2$ (a) lub $t = 1$ (b)

Example 3

In the next example we assume $b_1 = 0$, $b_2 = 1$, $d_1 = 0$, $d_2 = 1$, $t^* = 1$, $a = 1/4$ and

$$\varphi_1(y,t) = 500 + \frac{y - y^2 - t}{5}, \qquad \varphi_2(y,t) = 500 + \frac{y - y^2 - t}{5},$$
$$\theta_1(x,t) = 500 + \frac{x - x^2 - t}{5}, \qquad \theta_2(x,t) = 500 + \frac{x - x^2 - t}{5},$$
$$\psi(x,y) = 500 + \frac{x - x^2 + y - y^2}{5}.$$

As the initial approximations $u_{1,0}$ and $u_{2,0}$, as usually, we take the function describing the initial condition

$$u_{1,0}(x,y,t) = u_{2,0}(x,y,t) = 500 + \frac{x - x^2 + y - y^2}{5}.$$

In result of proper calculations we obtain successively

$$\begin{aligned}
u_0(x,y,t) &= 500 + \frac{x - x^2 + y - y^2}{5},\\
u_1(x,y,t) &= \frac{1}{20}\big(h\,(x - x^2 + y - y^2) - 4\,t\big),\\
u_2(x,y,t) &= \frac{1}{80}\,h^2\,\big(x - x^2 + y - y^2\big),\\
u_3(x,y,t) &= \frac{1}{320}\,h^2\,(h+4)\,\big(x - x^2 + y - y^2\big),\\
u_4(x,y,t) &= \frac{1}{1280}\,h^2\,(h+4)^2\,\big(x - x^2 + y - y^2\big),\\
u_5(x,y,t) &= \frac{1}{5120}\,h^2\,(h+4)^3\,\big(x - x^2 + y - y^2\big).
\end{aligned}$$

Generally, we have

$$u_m(x,y,t) = \frac{1}{5\cdot 4^m}\,h^2\,(h+4)^{m-2}\,\big(x - x^2 + y - y^2\big), \qquad m \geqslant 2.$$

Thus we get the exact solution of investigated equation

$$u(x,y) = \sum_{m=0}^{\infty} u_m(x,y) = 500 + \frac{x - x^2 + y - y^2 - t}{5},$$

if only the appropriate geometric series is convergent, which will happen for $h \in (-8,0)$. In this way we also get the effective region of the convergence control parameter $\mathbf{R}_h = (-8,0)$. The squared residual for $n \geqslant 1$ is equal to

$$E_n(h) = \frac{1}{4^{2n+1}}\,h^2\,(h+4)^{2n-2}.$$

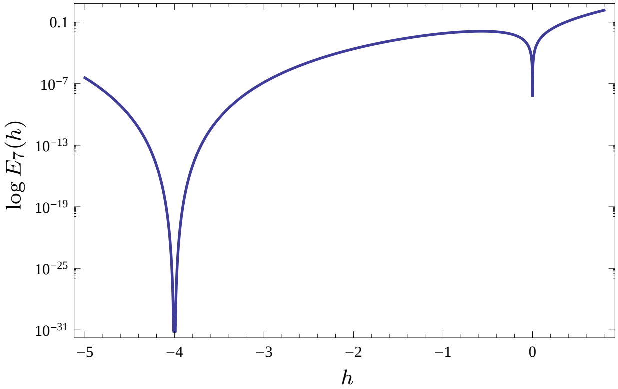

Fig. 4. Logarithm of the squared residual E_7
Rys. 4. Logarytm kwadratu reszt E_7

Hence, the optimal value of convergence control parameter h is equal to -4. In Figure 4 the plot of logarithm of the squared residual E_7 is shown.

Let us consider the more general case with the undetermined value of the thermal diffusivity coefficient a and $b_1 = 0$, $b_2 = 1$, $d_1 = 0$, $d_2 = 1$, $t^* = 1$, with the initial-boundary conditions defined by functions

$$\varphi_1(y,t) = 500 + \frac{y - y^2 - 4\,a\,t}{5}, \qquad \varphi_2(y,t) = 500 + \frac{y - y^2 - 4\,a\,t}{5},$$
$$\theta_1(x,t) = 500 + \frac{x - x^2 - 4\,a\,t}{5}, \qquad \theta_2(x,t) = 500 + \frac{x - x^2 - 4\,a\,t}{5},$$
$$\psi(x,y) = 500 + \frac{x - x^2 + y - y^2}{5}.$$

In this case we get

$$u_0(x,y,t) = 500 + \frac{x - x^2 + y - y^2}{5},$$
$$u_1(x,y,t) = \frac{1}{5}\,a\left(h\,(x - x^2 + y - y^2) - 4\,t\right),$$
$$u_2(x,y,t) = \frac{1}{5}\,a^2\,h^2\left(x - x^2 + y - y^2\right),$$
$$u_3(x,y,t) = \frac{1}{5}\,a^2\,h^2\,(a\,h + 4)\left(x - x^2 + y - y^2\right),$$
$$u_4(x,y,t) = \frac{1}{5}\,a^2\,h^2\,(a\,h + 4)^2\left(x - x^2 + y - y^2\right),$$
$$u_5(x,y,t) = \frac{1}{5}\,a^2\,h^2\,(a\,h + 4)^3\left(x - x^2 + y - y^2\right),$$

that is, in general

$$u_m(x,y,t) = \frac{1}{5}\,a^2\,h^2\,(a\,h+4)^{m-2}\left(x-x^2+y-y^2\right), \qquad m \geqslant 2.$$

Hence we get the exact solution of discussed equation

$$u(x,y) = \sum_{m=0}^{\infty} u_m(x,y) = 500 + \frac{x-x^2+y-y^2-4\,a\,t}{5},$$

if only $h \in (-2/a, 0)$. In this way we also obtain the effective region of the convergence control parameter $\mathbf{R}_h = (-2/a, 0)$. The squared residual for $n \geqslant 1$ is equal to

$$E_n(h) = 64\,a^4\,h^2\,(a\,h+1)^{2n-2}.$$

Hence, the optimal value of convergence control parameter h is equal to $-\frac{1}{a}$.

The obtained above equality seems to be a more general regularity. In all calculated examples of the two-dimensional unsteady heat conduction problem, in these ones presented in this work as well as in some other ones, the optimal value of convergence control parameter h was always equal to $-\frac{1}{a}$.

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