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A defining property of virtually nilpotent groups

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Abstract. We answer the question: *which property distinguishes the virtually nilpotent groups among the locally graded groups?* The common property of each finitely generated group to have a finitely generated commutator subgroup is not sufficient. However, the finitely generated commutator subgroup of $F_2(\operatorname{var} G)$, a free group of rank 2 in the variety defined by G, is the necessary and sufficient condition.

1. Introduction

A group is called virtually nilpotent (or nilpotent-by-finite) if it has a nilpotent subgroup of finite index. A group G is called locally graded if every nontrivial finitely generated subgroup of G has a proper subgroup of finite index. This class contains, for example, all locally soluble and all residually finite groups. It is closed under taking subgroups, extensions and cartesian products. The class of locally graded groups was introduced in 1970 by S. N. Černikov [2] in order to avoid groups such as infinite Burnside groups or Ol'shanskii–Tarski monsters.

It is well known that finitely generated virtually nilpotent groups are residually finite, whence locally graded. We look for a property which would distinguish them among the locally graded groups. For instance, every finitely generated virtually nilpotent group has a finitely generated commutator subgroup. However, the infinite torsion groups providing a negative solution to the General Burnside Problem (see e.g. [4]) or the groups of intermediate growth (see e.g. [3]) are

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residually finite (hence locally graded), have finitely generated commutator subgroups, but are not virtually nilpotent. So this condition is not sufficient. But it is enough to strengthen it to obtain the necessary and sufficient conditions for virtual nilpotency.

Let $F_2(\mathfrak{V})$ denote a free group of rank 2 in a variety $\mathfrak{V}$. In particular, $F_2'(\operatorname{var} G)$ denotes the commutator subgroup of the free group of rank 2 in the variety defined by G, which is the smallest class of groups containing G, closed under taking subgroups, homomorphic images and cartesian products [7]. Our main result is the following

Theorem. *A finitely generated locally graded group G is virtually nilpotent if and only if $F_2'(\operatorname{var} G)$ is finitely generated.*

2. Preliminaries

Let $g^h := h^{-1} g\, h$. We say[1] that a group G has the **Milnor Property** if for every $g, h \in G$ the subgroup $\langle g^{h^i},\ i \in \mathbb{N}\rangle$ is finitely generated.

In the following Lemma we show the connection between the commutator subgroup $F_2'(\mathfrak{V})$ and the Milnor Property of groups in the variety $\mathfrak{V}$.

Lemma 1. *Let $\mathfrak{V}$ be any variety. Then $F_2'(\mathfrak{V})$ is finitely generated if and only if every group of $\mathfrak{V}$ satisfies the Milnor Property.*

Proof. Let a, b be free generators of $F_2(\mathfrak{V})$ and let g, h be elements in an arbitrary group of $\mathfrak{V}$. Then the mapping $a \to g$, $b \to h$ defines the homomorphism $F_2(\mathfrak{V}) \to \langle g, h\rangle$. Therefore to prove the Lemma it is enough to show that $F_2'(\mathfrak{V})$ is finitely generated if and only if $\langle a^{b^i},\ i \in \mathbb{N}\rangle$ is finitely generated.

We do it in two steps. First, the subgroup $\langle a^{b^i},\ i \in \mathbb{N}\rangle$ is finitely generated if and only if $\langle a^{b^i}, i \in \mathbb{Z}\rangle$ is finitely generated. Indeed, if $\langle a^{b^i},\ i \in \mathbb{N}\rangle$ is generated by a^{b^j}, $1 \leq j \leq n$, then by means of the substitution $b \to b^{-1}$ we have that the subgroup $\langle a^{b^i},\ i \in \mathbb{Z}\rangle$ is generated by a^{b^j}, $-n \leq j \leq n$, whence finitely generated. Conversely, let $\langle a^{b^i},\ i \in \mathbb{Z}\rangle$ be finitely generated. Since it is invariant with respect to conjugation by b, we can choose the generators of the form a^{b^j} where $j > 0$. Thus the subgroup $\langle a^{b^i},\ i \in \mathbb{N}\rangle$ coincides with $\langle a^{b^i},\ i \in \mathbb{Z}\rangle$ and also is finitely generated.

Second, $\langle a^{b^i},\ i \in \mathbb{Z}\rangle$ is finitely generated if and only if $F_2'(\mathfrak{V})$ is finitely generated. Indeed, since the normal closure of a in $F_2(\mathfrak{V})$ is equal to $\langle a^{b^i},\ i \in \mathbb{Z}\rangle = \langle a\rangle F_2'(\mathfrak{V})$, the "if" part is clear. For the converse note that if $\langle a^{b^i},\ i \in \mathbb{Z}\rangle$

[1] after F. Point [8]

is finitely generated then in particular $F_2(\mathfrak{V})$ satisfies the Milnor Property (since this is a relatively free group). Now the "only if" part follows from the results of S. Rosset ([9], Lemmas 2, 3, Corollary 1) who proved that if G is a finitely generated group satisfying the Milnor property then G' is finitely generated. □

3. Proof of necessity

As we mentioned, every finitely generated virtually nilpotent group is locally graded. So, in order to prove the "only if" part of the Theorem it suffices to prove the following.

Proposition 1. *If G is a finitely generated virtually nilpotent group then the commutator subgroup $F_2'(\operatorname{var} G)$ is finitely generated.*

Proof. By assumption, the variety $\operatorname{var} G$ is contained in the product of a nilpotent variety and a locally finite variety (see e.g. [7], 52.11). Therefore $F_2(\operatorname{var} G)$ lies in this product of the varieties, whence it is virtually nilpotent. Then it satisfies the maximal condition (ibid. 31.13). Hence, every subgroup in $F_2(\operatorname{var} G)$ – in particular $F_2'(\operatorname{var} G)$ – is finitely generated. □

4. Proof of sufficiency

We have to show that if G is locally graded and $F_2'(\operatorname{var} G)$ is finitely generated, then G is virtually nilpotent.

The idea of the proof is the following. We first show that the assumption on $F_2'(\operatorname{var} G)$ implies that every finitely generated residually finite group in $\operatorname{var} G$ is virtually nilpotent (Lemma 4), and then, that the finite residual R of the group G (which is the intersection of all subgroups of finite index in G) is finitely generated (Lemma 5). Finally, we show that if R is not trivial, then it is not locally graded (Lemma 6), which contradicts the assumption that G is locally graded. Then the statement follows.

We start with the result similar to that in [5].

Lemma 2. *Let $\mathfrak{V}$ be any variety. Then $F_2'(\mathfrak{V})$ is finitely generated if and only if $\mathfrak{V}$ satisfies a law of the form*

$$x^{y^n} \equiv w, \quad \text{where} \quad w \in \langle x, x^y, x^{y^2}, \dots, x^{y^{n-1}} \rangle. \tag{1}$$

PROOF. Let $F_2(\mathfrak{V})$ be freely generated by a and b. If $F_2'(\mathfrak{V})$ is finitely generated then by Lemma 1 the subgroup $\langle a^{b^i},\ i \in \mathbb{N}\rangle$ is finitely generated. Hence there exists n such that a^{b^n} belongs to $\langle a, a^b, a^{b^2}, \dots, a^{b^{n-1}}\rangle$. Since every relation on free generators in a relatively free group is a law, the group $F_2(\mathfrak{V})$ (and hence $\mathfrak{V}$) satisfies the law of the form (1).

Conversely, if $\mathfrak{V}$ satisfies a law of the form (1) then

$$a^{b^n} \in \langle a, a^b, a^{b^2}, \dots, a^{b^{n-1}}\rangle. \tag{2}$$

Conjugation by b gives

$$a^{b^{n+1}} \in \langle a^b, a^{b^2}, \dots, a^{b^{n-1}}, a^{b^n}\rangle \overset{(2)}{\subseteq} \langle a, a^b, a^{b^2}, \dots, a^{b^{n-1}}\rangle.$$

By repeating this step we obtain $a^{b^{n+i}} \in \langle a, a^b, a^{b^2}, \dots, a^{b^{n-1}}\rangle$ for $i \geq 0$. It follows that $\langle a^{b^i},\ i \in \mathbb{N}\rangle$ is finitely generated and then $\mathfrak{V}$ satisfies the Milnor property. Now by Lemma 1, $F_2'(\mathfrak{V})$ is finitely generated. □

Let C_n and C denote the cyclic group of order n and the infinite cyclic group, respectively. The next lemma is an immediate consequence of the result by R. G. BURNS and YU. MEDVEDEV [1]

Lemma 3 (cf. [1] Dichotomy Theorem)**.** *If for all n, the law $w \equiv 1$ is not satisfied in the restricted wreath product $C_n \wr C$, then every finitely generated residually finite group satisfying this law is virtually nilpotent.* □

We recall that the restricted wreath product $W := C_n \wr C$ is a two-generator metabelian group with the infinitely generated commutator subgroup. Indeed, if C_n and C are generated by elements a and b, respectively, then the commutator subgroup W' contains elements $[a, b^i] = a^{-1}a^{b^i}$ for all $i \in \mathbb{Z}$, hence W' has an infinite support and cannot be finitely generated.

Lemma 4. *If $F_2'(\mathfrak{V})$ is finitely generated then every finitely generated residually finite group in $\mathfrak{V}$ is virtually nilpotent.*

PROOF. If $F_2'(\mathfrak{V})$ is finitely generated, then for every two-generated group G in $\mathfrak{V}$, G' is also finitely generated. Thus none of wreath products $C_n \wr C$ can lie in the variety $\mathfrak{V}$ and the result follows from Lemma 3. □

Lemma 5. *Let $F_2'(\mathfrak{V})$ be finitely generated. If $G \in \mathfrak{V}$ is a finitely generated group, then the finite residual R in G is finitely generated.*

Proof. By Lemma 4, the finitely generated residually finite group G/R is virtually nilpotent. So G/R contains a nilpotent subgroup, H/R say, of finite index. Thus H/R is finitely generated and nilpotent. Hence H/R is polycyclic (see e.g. [7] 31.12). So there exists a finite subnormal series with cyclic factors $H = N_0 \rhd N_1 \rhd \cdots \rhd N_m = R$. Since by Lemma 1, G satisfies the Milnor Property we can successively apply Lemma 2 of Rosset [9], which says that if G satisfies the Milnor Property and G/N is cyclic then N is finitely generated. Then, after m steps, we conclude that R is finitely generated. □

Lemma 6. *Let G be a finitely generated group and R – its finite residual. If G/R is virtually nilpotent, $R \neq 1$ and R is finitely generated then R is not locally graded.*

Proof. Assume the contrary, that the nontrivial finitely generated group R is locally graded. By definition it means that R contains a proper subgroup T (that is $T \subsetneq R$) of finite index. By a known result of M. Hall, R has only a finite number of subgroups of index $|R : T|$ and their intersection, M say, is characteristic of finite index in R. Hence M is normal in G and $M \subseteq T$, whence $M \subsetneq R$ and R/M is finite. Since by assumption G/R is virtually nilpotent (that is nilpotent-by-finite), we conclude that G/M is finite-by-nilpotent-by-finite. Thus it is nilpotent-by-finite, whence residually finite, that contradicts the definition of R. □

Now we can prove the "if" part of the Theorem.

Proposition 2. *Let G be a finitely generated, locally graded group, such that $F_2'(\operatorname{var} G)$ is finitely generated. Then G is virtually nilpotent.*

Proof. If R is the finite residual of G, then by Lemma 4, G/R is virtually nilpotent, whence by Lemma 5, R is finitely generated. Now, if $R \neq 1$, then by Lemma 6 it is not locally graded, which is impossible, since a subgroup of a locally graded group has to be locally graded itself. So, R must be trivial, which means that G is virtually nilpotent, which finishes the proof. □

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