GROUP LAWS $[\mathbf{x}, \mathbf{y^{-1}}] \equiv \mathbf{u}(\mathbf{x}, \mathbf{y})$ AND VARIETAL PROPERTIES

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ABSTRACT. Let $F = \langle x, y\rangle$ be a free group. It is known that the commutator $[x, y^{-1}]$ cannot be expressed in terms of basic commutators, in particular in terms of Engel commutators. We show that the laws imposing such an expression define specific varietal properties. For a property $\mathcal{P}$ we consider a subset $U(\mathcal{P}) \subseteq F$ such that every law of the form $[x, y^{-1}] \equiv u,\ u \in U(\mathcal{P})$ provides the varietal property $\mathcal{P}$. For example, we show that each subnormal subgroup is normal in every group of a variety $\mathfrak{V}$ if and only if $\mathfrak{V}$ satisfies a law of the form $[x, y^{-1}] \equiv u$, where $u \in [F', \langle x\rangle]$.

1. INTRODUCTION

Let $F = \langle x, y\rangle$ be a noncyclic free group. We denote $x^y = y^{-1}xy$, $[x,\, _0y] = x$, $[x,\, _1y] = [x,\, y] = x^{-1}y^{-1}xy$, $[x,\, _{i+1}y] = [[x,\, _iy],\, y]$. If assume $x > y$ then the nontrivial commutators of the form $[x,\, _iy,\, _jx]$ are so called left-normed basic commutators.

Every group-law implies a 2-variable law, and each variety satisfies a law of the form $[x, y^{-1}] \equiv u$, for a word $u = u(x, y)$ in F. The commutator $[x, y^{-1}]$ cannot be expressed in terms of basic commutators (see e.g. [10], 36.24), so we consider the laws imposing such an expression. The laws with a similar expression may form the families of laws providing the same varietal property in the corresponding variety $\mathfrak{V}$. We look for a subset $U(\mathcal{P})$ in F such that the following two conditions imply each other:

(i) $\mathfrak{V}$ satisfies a law of the form $[x, y^{-1}] \equiv u, u \in U(\mathcal{P})$.

(ii) $\mathfrak{V}$ has the property $\mathcal{P}$.

In this paper we consider the varietal properties, first three of which are provided by so called restraining laws, Milnor laws and t-laws respectively.

$\mathcal{P}_1$: Each finitely generated group $G \in \mathfrak{V}$ has finitely generated G'.

$\mathcal{P}_2$: Each finitely generated metabelian group in $\mathfrak{V}$ has finitely generated G'.

$\mathcal{P}_3$: Subnormal subgroups are normal in every group $G \in \mathfrak{V}$.

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For each of these properties we describe a family $[x, y^{-1}] \equiv u,\ u \in U(\mathcal{P}_i)$ of laws providing the property $\mathcal{P}_i$.

Restraining laws:	$[x,\, y^{-1}] \equiv u, u \in \big\langle\, [x,\, _iy],\ i \geq 0 \,\big\rangle$,
Milnor laws:	$[x,\, y^{-1}] \equiv u, u \in \big\langle\, [x,\, _iy],\ i \geq 0 \,\big\rangle \cdot F''$,
t-laws:	$[x,\, y^{-1}] \equiv u, u \in \big\langle\, [\, F',\, \langle x \rangle\,]\, \big\rangle$,
Abelian and Pseudo-Abelian laws:	$[x,\, y^{-1}] \equiv u, u \in F''$.

2. Preliminaries

Following F. Point [13], we say that a group G has the **Milnor property** if *for all elements $g,\ h \in G$, the subgroup $\langle g^{h^i},\ i \in \mathbb{N} \rangle$ is finitely generated.* This property first was considered by J. Milnor ([9], Lemma 3). Later S. Rosset proved in [16] that a finitely generated group G satisfying this property has G' finitely generated. Groups satisfying the Milnor property were called by Y. Kim and A. Rhemtulla *restrained groups* [4].

A law is called *restraining* (or an $\mathfrak{R}$-law) if every group satisfying this law is restrained, or equivalently, if each finitely generated group G satisfying this law has finitely generated G' [6]. So we have the following.

Lemma 1 (cf. [6], [7])**.** *Let $\mathfrak{V}$ be a variety of groups. The following conditions are equivalent:*
1. $\mathfrak{V}$ satisfies a restraining law.
2. Each group $G \in \mathfrak{V}$ has the Milnor property.
3. Each finitely generated group $G \in \mathfrak{V}$ has G' finitely generated.

Note that the last of the above conditions is the property $\mathcal{P}_1$.

3. $[\mathbf{x},\, \mathbf{y^{-1}}]$ as a product of left-normed basic commutators and restraining laws

We denote by E a subgroup in F generated by all Engel commutators $[x,\, _iy]$, $i \geq 0$, that is

$$E := \langle x,\, [x,\, y],\, [x,\, _2y],\, [x,\, _3y], \ldots \rangle.$$

The subgroup E contains the left-normed basic commutators $[x,\, _iy,\, _jx]$, $i > 0$. Since the word $[x, y^{-1}]$ cannot be expressed modulo F'' in terms of basic commutators ([10], 36.24), it is interesting to consider the laws imposing this expression. We start with laws of the form

$$(1) \qquad [x, y^{-1}] \equiv u, \quad u \in E,$$

and show that it is the family of the restraining laws, defining varieties with properties listed in Lemma 1. It suffices to prove that $U(\mathcal{P}_1) = E$.

Theorem 1. *The implications $(i) \Leftrightarrow (ii)$ hold for a variety $\mathfrak{V}$, where*

- (i) *$\mathfrak{V}$ satisfies a law of the form $[x,\, y^{-1}] \equiv u,\ u \in E$.*
- (ii) *$\mathcal{P}_1$: Each finitely generated group $G \in \mathfrak{V}$ has G' finitely generated.*

Proof. It is shown in ([7], Corollary 5.4) that the following subgroups coincide for every n ($n \geq 0$): $\langle\, [x,\, _iy],\, 0 \leq i \leq n\,\rangle = \langle\, x^{y^i},\, 0 \leq i \leq n\,\rangle$. So we have

$$E := \langle\, [x,\, _iy],\ i \geq 0\ \rangle = \langle\, x^{y^i},\ i \geq 0\,\rangle. \tag{2}$$

$(i) \Rightarrow (ii)$. Assume that $\mathfrak{V}$ satisfies a law $[x, y^{-1}] \equiv u$, $u \in E$. Let $\mathfrak{V}$ and V denote the corresponding variety and the verbal subgroup in F, respectively. In view of Lemma 1, it suffices to show that each group $G \in \mathfrak{V}$ has the Milnor property, that is the subgroup $\langle\, x^{y^i},\ i \in \mathbb{N}\rangle$ is finitely generated modulo V. The following inclusions are written modulo V.

It follows by (2) that for $u \in E$ there is n, such that: u belongs to $\langle\, x^{y^i},\ 0 \leq i \leq n\rangle$, which implies by (i) that $[x, y^{-1}] \in \langle\, x^{y^i},\ 0 \leq i \leq n\,\rangle$, and hence

$$x^{y^{-1}} \in \langle x,\, x^y,\, x^{y^2},\, ...,\, x^{y^n}\,\rangle. \tag{3}$$

Conjugate (3) by y^{-1} then

$$x^{y^{-2}} \in \langle x^{y^{-1}},\, x,\, x^y, \ldots x^{y^{n-1}}\rangle \overset{(3)}{\subseteq} \langle x,\, x^y,\, x^{y^2}, \ldots\, x^{y^n}\,\rangle.$$

By repeating the conjugation we obtain for all $k > 0$

$$x^{y^{-k}} \in \langle x,\, x^y,\, x^{y^2},\, ...,\, x^{y^n}\,\rangle. \tag{4}$$

Since V is fully invariant we can substitute $y \to y^{-1}$ to get for all $k > 0$

$$x^{y^k} \in \langle x,\, x^{y^{-1}},\, x^{y^{-2}}, \ldots,\, x^{y^{-n}}\,\rangle \overset{(4)}{\subseteq} \langle x,\, x^y,\, x^{y^2},\, ...,\, x^{y^n}\,\rangle. \tag{5}$$

In view of (4) and (5) it follows that the subgroup $\langle x^{y^i}, i \in \mathbb{N}\rangle$ is finitely generated modulo V. Hence each group $G \in \mathfrak{V}$ has the Milnor property and by Lemma 1, we have (ii).
$(ii) \Rightarrow (i)$. If each finitely generated group $G \in \mathfrak{V}$ has G' finitely generated then by Lemma 1, the subgroup $\langle\, x^{y^i},\, i \in \mathbb{N}\rangle$ is finitely generated (modulo V) by, say, a set $\{x,\, x^y,\, x^{y^2},\, ...,\, x^{y^n}\}$. Then $x^{y^{n+1}} \in \langle x,\, x^y,\, x^{y^2},\, ...,\, x^{y^n}\rangle$. Conjugation by $y^{-(n+1)}$ gives

$$x \in \langle\, x^{y^{-(n+1)}},\, x^{y^{-n}},\, x^{y^{-n+1}}, \ldots,\, x^{y^{-2}},\, x^{y^{-1}}\rangle.$$

Substitution $y \to y^{-1}$ implies that $x \in \langle\, x^y,\, x^{y^2},\, ...,\, x^{y^{n+1}}\rangle$. Now conjugation by y^{-1} gives $x^{y^{-1}} \in \langle\, x,\, x^y,\, x^{y^2},\, ...,\, x^{y^n}\rangle$, which leads to $[x,\, y^{-1}] \in E \cdot V$, and allows to conclude that F/V (and hence $\mathfrak{V}$) satisfies a law of the required form $[x,\, y^{-1}] \equiv u$, where $u \in E$. □

4. $[\mathbf{x}, \mathbf{y^{-1}}]$ as a product of basic commutators modulo $\mathbf{F''}$ and Milnor laws

Definition 1. *We call a law* the Milnor law *if it is not satisfied in any variety of the form $\mathfrak{A}_p\mathfrak{A}$ for a prime p.*

The choice of the name comes from the paper of F. Point [13] who introduced the laws (called the Milnor identities) by means of characteristic polynomials. By result of G. Endimioni [1], (see [14], Proposition 1.1), these laws are not satisfied in any variety of the form $\mathfrak{A}_p\mathfrak{A}$ for a prime p.

Lemma 2 (cf. [6], [7]). *Let F/V be a free group of rank 2 in a variety $\mathfrak{V}$. The following conditions are equivalent:*
1. $\mathfrak{V}$ does not contain a subvariety $\mathfrak{A}_p\mathfrak{A}$ for a prime p.
2. Each finitely generated metabelian group $G \in \mathfrak{V}$ has finitely generated G'.

Proof. $1 \Rightarrow 2$. If $\mathfrak{V}$ does not contain a subvariety $\mathfrak{A}_p\mathfrak{A}$ then $V \nsubseteq F''F'^p$ for any prime p. It follows that $F''V \nsubseteq F''F'^p$. By result of J. R. J. Groves ([2], Theorem C (ii)), the group $F/F''V$ is nilpotent-by-(finite exponent). Hence by [8], it satisfies a positive law, which is a restraining law. So by Lemma 1, all groups in *var* $F/F''V$ have finitely generated commutator subgroups and the condition 2 follows.
$2 \Rightarrow 1$. Let each finitely generated group $G \in \mathfrak{V}$ have G' finitely generated. If $\mathfrak{V}$ contains a subvariety $\mathfrak{A}_p\mathfrak{A}$, then $\mathfrak{V}$ contains the group $W = \langle a\rangle_p \wr \langle b\rangle$, the restricted wreath product of a cyclic group of order p, and an infinite cyclic group. The commutator subgroup W' contains elements $[a, b^i] = a^{-1}a^{b^i}$ for all $i \in \mathbb{Z}$, so W' has an infinite support and cannot be finitely generated. A contradiction. □

Theorem 2. *The implications $(i) \Leftrightarrow (ii)$ hold for a variety $\mathfrak{V}$, where*
- (i) *$\mathfrak{V}$ satisfies a law of the form $[x, y^{-1}] \equiv u, u \in EF''$.*
- (ii) *$\mathcal{P}_2$: Each finitely generated metabelian group $G \in \mathfrak{V}$ has finitely generated G'.*

Proof. $(i) \Rightarrow (ii)$. Assume that $\mathfrak{V}$ satisfies a law of the form $[x, y^{-1}] \equiv u,\ u \in EF''$. Then metabelian groups in $\mathfrak{V}$ satisfy the law of the form $[x, y^{-1}] \equiv u,\ u \in E$ and by Theorem 1, each finitely generated metabelian group $G \in \mathfrak{V}$ has a finitely generated G'.
$(ii) \Rightarrow (i)$. By Theorem 1, the group $F/F''V$ satisfies a law of the form $[x, y^{-1}] \equiv u,\ u \in E$, which implies that F/V (and hence $\mathfrak{V}$) satisfies a law of the required form $[x, y^{-1}] \equiv u, u \in EF''$. □

5. Varieties in which normality is a transitive relation

The groups in which normality is a transitive relation (t-groups) have been considered by many authors (see [15], 13.4), however no non-abelian infinite relatively free t-group was known till 1997 [5].

We consider the transitivity of normality (the t-property) as the varietal property and show that the laws providing the t-property are of the form $[x, y^{-1}] \equiv u$, where $U = \big[F', \langle x\rangle\big]$.

Theorem 3. *The implications $(i) \Leftrightarrow (ii)$ hold for a variety $\mathfrak{V}$, where*

(i) $\mathfrak{V}$ *satisfies a law of the form* $[x,\, y^{-1}] \equiv u, u \in \big[F', \langle x\rangle\big]$.
(ii) $\mathcal{P}_3$: *Subnormal subgroups are normal in every group* $G \in \mathfrak{V}$.

Proof. Since $[x,\, y^{-1}] = [x,\, y]^{-y^{-1}}$ and the subgroup $\big[F', \langle x\rangle\big]$ is invariant under the map $y \to y^{-1}$, it suffices to prove the theorem for the laws

$$(6) \qquad [x,\, y] \equiv u,\; u \in \big[F', \langle x\rangle\big].$$

$(i) \Rightarrow (ii)$. Assume that G is a group satisfying a law of the form (6). Let G have a normal subgroup H which has a normal subgroup K, that is

$$G \rhd H \rhd K.$$

If $g \in G$ and $k \in K$ then, we have by (6):

$$[k,\, g] \in \big[[\langle k\rangle, \langle g\rangle], \langle k\rangle\big] \subseteq \big[[K, G], K\big] \subseteq [H, K] \subseteq K,$$

which means that K is a normal subgroup of G. Hence each subnormal subgroup is normal in every group satisfying a law of the form (6).
$(ii) \Rightarrow (i)$. Let V be a verbal subgroup in F such that each subnormal subgroup is normal in F/V. Consider $F \rhd \langle x\rangle^F \rhd \langle x\rangle^{\langle x\rangle^F}$. Then by assumption, the subnormal subgroup $\langle x\rangle^{\langle x\rangle^F}$ is normal in F modulo V. Since it contains x, it must contain $\langle x\rangle^F$. So modulo V we have $\langle x\rangle^F \equiv \langle x\rangle^{\langle x\rangle^F}$. By commutator calculus $\langle x\rangle^F = \langle x\rangle F'$ and $\langle x\rangle^{\langle x\rangle^F} = \langle x\rangle\big[\langle x\rangle F', \langle x\rangle\big] = \langle x\rangle\big[F', \langle x\rangle\big]$. Thus

$$\langle x\rangle F' \equiv \langle x\rangle\big[F', \langle x\rangle\big],$$

which implies (for some k) a law $[x, y] \equiv x^k u$, where $u \in \big[F', \langle x\rangle\big]$. The latter implies $x^k \equiv 1$ and the required law $[x, y] \equiv u,\; u \in \big[F', \langle x\rangle\big]$. $\square$

The following Proposition shows that each variety with transitivity of normality is either abelian or pseudo-abelian, that is *a non-abelian variety without non-abelian metabelian groups.* The problem of existence of such a variety was posed in ([10], Problem 5). The first examples of the pseudo-abelian varieties were given by A. Yu. Olshanskii [11], [12].

Proposition 1. *A variety with transitivity of normality has no non-abelian metabelian groups.*

Proof. In view of Theorem 3, it suffices to show that each law of the form $[x, y] \equiv u,\; u \in \big[F', \langle x\rangle\big]$ implies a law of the form $[x, y] \equiv v,\; v \in F''$.

If put $[x, y]$ instead of x in (6), we obtain $[x, y, y] \equiv v \in F''$, which implies that each 2-generator metabelian group satisfying a law (6), is 2-nilpotent. Since the values of the word $u \in \big[F', \langle x\rangle\big]$ in the 2-nilpotent group are trivial, each 2-generator metabelian group satisfying (6) is abelian. Hence a law of the form $[x, y] \equiv v,\; v \in F''$ follows. $\square$

Question The question whether the converse implication holds, that is whether each pseudo-abelian law implies transitivity of normality, is open.

This question was first formulated in [3]. A positive answer is known [5] only for the pseudo-abelian varieties constructed by A. Yu. Ol'shanskii.

6. Varieties of 2-Engel groups with G' of finite exponent.

We show that each law in the family

$$[x,\, y^{-1}] \equiv [x,\, y]^k,\ \ k \in \mathbb{Z} \tag{7}$$

defines a variety of 2-Engel groups G with G' of finite exponent unless $k = -1$.

Theorem 4. *The implications* $(i) \Leftrightarrow (ii)$ *hold for a variety* $\mathfrak{V}$*, where*

(i) $\mathfrak{V}$ *satisfies a law of the form,* $[x,\, y^{-1}] \equiv [x,\, y]^k,\ k \in \mathbb{Z},\, k \neq -1.$

(ii) $\mathfrak{V}$ *consists of* 2*-Engel groups* G *with* $(G')^{k+1} = \{e\},\ k \neq -1.$

Proof. $(ii) \Rightarrow (i)$. Condition (ii) implies that G satisfies the laws $[[x,\, y],\, y] \equiv 1$ and $[x,\, y]^{k+1} \equiv 1$. Since $[[x,\, y],\, y] = [x,\, y^{-1}]^y [x,\, y]^y$, the law $[[x,\, y],\, y] \equiv 1$ is equivalent to

$$[x,\, y^{-1}] \equiv [x,\, y]^{-1}. \tag{8}$$

The second law can be written as $[x,\, y]^k \equiv [x,\, y]^{-1}$. Then in view of (8), $[x,\, y]^k \equiv [x,\, y]^{-1} \equiv [x,\, y^{-1}]$, which gives the required law $[x,\, y^{-1}] \equiv [x,\, y]^k$.

$(i) \Rightarrow (ii)$. The inverse of the right-hand part of (7) can be obtained by interchanging $x \rightleftarrows y$, hence the same holds for the left-hand part, that is $[x,\, y^{-1}]^{-1} \equiv [y,\, x^{-1}]$. By the commutator identity $[x,\, y^{-1}] = [x,\, y]^{-y^{-1}}$, this implies $[x,\, y]^{y^{-1}} \equiv [y,\, x]^{-x^{-1}}$. Conjugation by y implies $[[y,\, x],\, x^{-1}y] \equiv 1$, which, by mapping $y \to xy$ gives $[[y,\, x],\, y] \equiv 1$, and hence $[[x,\, y],\, y] \equiv 1$.

By combining the law in (i) and (8) we obtain $[x,\, y]^{k+1} \equiv 1$. To get $(G')^{k+1} = \{e\}$, it suffices to show that the 2-engel group is metabelian. Indeed, by ([10], 34.31) it satisfies $[[[x,\, y],\, z],\, t] \equiv 1$. By (8), we have $[[x,\, y],\, z^{-1}] \equiv [[x,\, y],\, z]^{-1}$. And now by the commutator identity $[a,\, bc] = [a,\, c][a,\, b][[a,\, b],\, c]$ we have the metabelian law

$$[[x,\, y],\, [z,\, t]] = [[x,\, y],\, z^{-1}t^{-1}z\,t] \equiv 1.$$

□

References

[1] G. Endimioni, On the locally finite p-groups in certain varieties of groups. *Quart. J. Math. Oxford Ser.* **48**(2), (1997), 169—178.

[2] J.R.J.Groves, Varieties of soluble groups and a dichotomy of P.Hall, *Bull. Austral. Math. Soc.,* **5**, (1971), 391–410.

[3] L.G. Kovács and M.F. Newman, Hanna Neumann's problems on varieties of groups, in *Proc. Second Internat. Conf. Theory of Groups*, Lecture Notes in Math. **372**, *Springer-Verlag, Berlin-Heidelberg-New York,* 1974, 417–431.

[4] Y. Kim and A. H. Rhemtulla, On locally graded groups, *Proceedings of the Third International Conference on Group Theory,* Pusan, Korea 1994, Springer-Verlag, Berlin-Heidelberg-New York (1995), 189–197.

[5] O. Macedońska, A. Storozhev, Varieties of t-groups, *Communications in Algebra* **25**(5), (1997), 1589–1593.

[6] O. Macedońska, What do the Engel laws and positive laws have in common, *Fundamental and Applied Mathematics,* **14**(7), (2008), 175–183 (in Russian). English transl. *J. of Math. Sciences,* **164**(2), (2010), 272–280, http://www.springerlink.com/content/2u1h4hukl573532v/fulltext.pdf

[7] O. Macedońska, W. Tomaszewski, On Engel and positive laws, *London Math. Soc. Lecture Notes* (2011).

[8] A. I. Mal'tsev, Nilpotent semigroups, *Ivanov. Gos. Ped. Inst. Uc. Zap.* **4**, (1953), 107–111 (in Russian).

[9] J. Milnor, Growth of finitely generated solvable groups, *J. Diff. Geom.* **2**, (1968), 447–449.

[10] H. Neumann, VARIETIES OF GROUPS, *Springer-Verlag, Berlin-Heidelberg-New York,* 1967.

[11] A.Yu.Ol'shanskii, Varieties in which all finite groups are abelian, *Mat. Sbornik,* **126**, (168), (1), (1985), 59–82 (in Russian).

[12] Ol'shanskii, A.Yu. *Geometry of defining relations in groups*; Mathematics and its applications (Soviet Series), 70; Kluwer Academic Publishers: Dordrecht, 1991.

[13] F. Point, Milnor identities, *Comm. Algebra* **24**(12), (1996), 3725–3744.

[14] F. Point, *Milnor Property in Finitely Generated Soluble Groups*, Comm. Algebra **31**(3), (2003), 1475–1484.

[15] D.J.S. Robinson, A COURSE IN THE THEORY OF GROUPS, *Springer-Verlag Berlin, Heidelberg, New York, 1982.*

[16] S. Rosset, A property of groups of non-exponential growth, *Proc. Amer. Math. Soc.* **54**, (1976), 24–26.

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