On Engel and positive laws

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Abstract

Engel laws and positive laws have attracted attention of many authors. These laws have many common properties, e.g. each finitely generated group satisfying any of these laws has a finitely generated commutator subgroup. There are still many open problems concerning these laws.

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1 Notation

Let $F=\langle x, y\rangle$ be a free group of rank 2. We denote $x^{y^i}=y^{-i}x\,y^i$, $[x,y]=x^{-1}y^{-1}xy$, $[x,\,_0y]=x$ and $[x,\,_{i+1}y]=[[x,\,_iy],\,y]$. The law $[x,\,_ny]\equiv 1$ is the n-Engel law. We introduce the following subgroups.

$$E_n=\langle\,[x,\,_iy],\ 0\le i\le n\rangle,\quad E=\langle\,[x,\,_iy],\ 0\le i\rangle.$$

Let $\mathfrak{A}$ be the variety of all abelian groups, and $\mathfrak{A}_p$ – the variety of all abelian groups of exponent p. By $\mathfrak{N}_c$ we denote the variety of all nilpotent groups of nilpotency class c, and by $\mathfrak{S}_d$ – the variety of all soluble groups of solubility class d. By $\mathfrak{B}_e$ we denote so called restricted Burnside variety of exponent e, that is the variety generated by all finite groups of exponent e. It follows from Zelmanov positive solution of the Restricted Burnside Problem that all groups in $\mathfrak{B}_e$ are locally finite of exponent dividing e.

For the text below we have to recall the following

Definition A group G is called **locally graded** if every nontrivial finitely generated subgroup of G has a proper subgroup of finite index.

The class of locally graded groups was introduced in 1970 by S. N. Černikov [8] in order to avoid groups such as infinite Burnside groups or Ol'shanskii-Tarski monsters. This class contains all *soluble* groups, *locally finite* groups, *residually finite* groups. It is closed under taking *subgroups* and *extensions*. It is also closed under taking groups which are *locally-* or *residually-* in this class.

2 Positive laws

Positive laws are the laws of the form $u(x_1, x_2, \ldots. x_n)\equiv v(x_1, x_2, \ldots x_n)$, where u, v are distinct words in the free group $\langle x_1, x_2 \ldots.\rangle$, written without negative powers of the variables $x_1, x_2, \dots x_n$. The law is cancelled if u and v have different first (and last) letters. The degree of a cancelled law is the length of the longer word u or v.

Each positive law implies a binary positive law $u(x,y) = v(x,y)$ if substitute $x_i \to xy^i$.

It was shown by J.&T. Lewins [18], that if a group satisfies a positive law, then the variety it generates has a basis consisting of positive laws.

In 1953 A. I. Mal'tsev [21] and independently in 1963 B. H. Neumann and T. Taylor [27] proved that nilpotency can be defined by a positive law. Indeed, let $P \equiv Q$ be a positive law defining nilpotency of class $n-1$, and z be a variable which does not occure in P, Q. Let G satisfies the law $PzQ \equiv QzP$. This law implies $PQ \equiv QP$ and then $PQ^{-1}z \equiv zPQ^{-1}$. Hence the quotient of G by its center satisfies the law $P \equiv Q$, and is by assumption nilpotent of class $n-1$. So G is nilpotent of class n. Let $P_1 \equiv Q_1$ be the abelian law $xy \equiv yx$. Let $P_2 = P_1 z_1 Q_1$, $Q_2 = Q_1 z_1 P_1$ and inductively $P_n = P_{n-1} z_{n-1} Q_{n-1}$, and $Q_n = Q_{n-1} z_{n-1} P_{n-1}$. Then the positive law $P_n \equiv Q_n$ defines the variety $\mathfrak{N}_n$. Note that this law has $n+1$ variables $x, y, z_1, \dots, z_{n-1}$. If we put 1 for each z_i we get a binary positive law

$$P_n(x,y,1) \equiv Q_n(x,y,1).$$

It follows that groups which are nilpotent-by-(finite exponent), in particular virtually nilpotent groups (i.e. nilpotent-by-finite groups) satisfy positive laws.

The question whether a finitely generated group satisfying a positive law must be nilpotent-by-finite has a negative answer. For example a free Burnside group $B(r,n), r > 1$ satisfying the law $x^n \equiv 1$, or a free finitely generated group satisfying the law $xy^n = y^n x$, are not nilpotent-by-finite for n sufficiently large by results of Novikov and Adian (see [1]).

The question whether a finitely generated group satisfying a positive law must be nilpotent-by-(finite exponent) was open for more then forty years. A counterexample was found in 1996 by Ol'shankii and Storozhev [25]. Their groups are not even (locally soluble)-by-(finite exponent).

However the affirmative answer was found for many types of groups. In 1997 it was done for so called class $\mathcal{C}$ ([5], Theorem B) defined inductively. The class $\mathcal{C}$, as it was shown later [4], consists of groups which locally are residually SB-groups (groups lying in finite products of varieties $\mathfrak{S}_d$ and $\mathfrak{B}_e$ for varying d, e). This result, combined with results of Kim and Rhemtulla ([15] Theorem A), saying that a finitely generated locally graded group G satisfying a positive law must be polycyclic-by-finite (hence in the class $\mathcal{C}$) implies the affirmative answer for locally graded groups. So since 1997 it is known that

a locally graded group G satisfying a positive law of degree n, must be in the product variety $\mathfrak{N}_c\mathfrak{B}_e$, where c and e depend on n only.

3 Three Questions on Engel laws

In 1936 M. Zorn proved that every finite n-Engel group is nilpotent. This is not true in general. An n-Engel law for $n > 2$ does not imply nilpotency (see [24] 34.62). In

1971 S. Bachmuth and H. Y. Mochizuki [2] constructed a non-soluble locally finite 3-Engel group of exponent 5, every n-generator subgroup of which is nilpotent of class at most $2n-1$. The local nilpotence of 4-Engel groups of exponent 5 was shown in 1997 by M. Vaughan-Lee [35].

The following question is open in general.

Q1: Is every n-Engel group locally nilpotent? (In other words, is every finitely generated n-Engel group nilpotent?)

Question Q1 is approached in two main ways: one is to examine n-Engel groups for different n. In that case an affirmative answer has been found

- 1942: for $n = 2$ – F. W. Levi [17],
- 1961: for $n = 3$ – H. Heineken [14],
- 2005: for $n = 4$ – G. Havas and M. R. Vaughan-Lee [13] (see also [32] for a simplification of a part of the proof of Havas and Vaughan-Lee).

The second approach is to investigate the problem in certain classes of groups. It has been shown that *n-Engel groups are locally nilpotent if additionally they* are

- 1953: soluble groups – K. W. Gruenberg [12],
- 1957: groups with the maximal condition – R. Baer [3],
- 1991: residually finite groups – J. S. Wilson [36],
- 1992: profinite groups – J. S. Wilson and E. I. Zelmanov [37],
- 1994: locally graded groups – Y. Kim and A. H. Rhemtulla [15],
- 1998: groups in the class $\mathcal{C}$ – R. Burns and Yu. Medvedev [6]. They proved that all n-Engel groups in the class $\mathcal{C}$ are contained in the product varieties $\mathfrak{N}_c\mathfrak{B}_e \cap \mathfrak{B}_e\mathfrak{N}_c$, where c, e depend on n only.
- 2003: compact groups – Yu. Medvedev [22].

Q2: Does there exist a finitely generated infinite simple n-Engel group?

The Questions **Q1, Q2** are equivalent in a sense that one has an affirmative answer if and only if the other has a negative answer.

Proposition 3.1 *There exists a non-(locally nilpotent) n-Engel group if and only if there exists a finitely generated infinite simple n-Engel group.*

Proof If there exists an n-Engel group G which is not locally nilpotent then by [15], G is not locally graded. Thus G must contain a finitely generated subgroup H, which has no proper subgroup of finite index. Since H is finitely generated, by Zorn's Lemma it has a maximal proper normal subgroup N. Then N is of infinite index and the factor H/N is a finitely generated infinite simple n-Engel group.

Conversely, if there exists a finitely generated infinite simple n-Engel group then it is not nilpotent, since the only finitely generated nilpotent simple groups are cyclic of prime orders. □

The following question was posed by A. I. Shirshov in 1963 (Problem 2.82, The Kourovka Notebook [34]) and still is open.

Q3: Are n-Engel varieties defined by positive laws?

An affirmative answer has been given for 2- and 3-Engel groups – by A. I. Shirshov [30] and for 4-Engel groups – by G. Traustason [31].

4 Observation

It is not known whether n-Engel groups (for $n > 4$) satisfy positive laws, however the Engel laws and positive laws have some common properties:

First it was shown for finitely generated **residually finite groups G:**

- 1991: If G satisfies **an Engel law**, then G is nilpotent (Wilson [36]),
- 1993: If G satisfies **a positive law**, then G is virtually nilpotent (Semple and Shalev [29]).

Later – for finitely generated **locally graded groups G:**

- 1994: If G satisfies **an Engel law**, then G is virtually nilpotent (Kim and Rhemtulla [15]),
- 1997: If G satisfies **a positive law**, then G is virtually nilpotent (see page 2 and [7] Corollary 1).

The following question was posed by R.Burns: *What do the Engel laws and positive laws have in common that forces finitely generated locally graded groups satisfying them to be nilpotent-by-finite?*
The answer is that these laws have the same Engel construction.

5 Engel construction of the laws

Let $u(x, y)$ be a word, and S be a subset in the free group $F = \langle x, y\rangle$.

Definition 5.1 We say that a law $w(x, y) \equiv 1$ has construction

$$u(x, y) \,\widetilde{\in}\, S$$

if it is equivalent to a law $u(x, y) \equiv s(x, y)$ for some word $s(x, y) \in S$.

The laws with the same construction have similar properties. For example

- Every law of the form $[x, y] \equiv x^p$ for some prime p has construction $[x, y] \,\widetilde{\in}\, \{x^p,\, p \in \mathbb{P}\}$. These laws define abelian varieties $\mathfrak{A}_p$.
- The laws with construction $[x, y] \,\widetilde{\in}\, F''$ define varieties of groups with perfect commutator subgroups (i.e. $G' = G''$).

Definition 5.2 **The general Engel construction** is a construction of the form

$$x^{k_0}[x,y]^{k_1}\,[x,\,_2y]^{k_2}\ldots\,[x,\,_ny]^{k_n}\ \widetilde{\in}\ E',$$

where $E=\langle\,[x,\,_iy],\ 0\le i\rangle$, $n\in\mathbb{N}$, $k_i\in\mathbb{Z}$.

To show that every law is equivalent to a law which has the general Engel construction, we need the following technical results.

Lemma 5.3 *For $k\ge 1$ the following inclusions hold*

$$[x,y^{k+1}]\in\langle\,[x,\,_iy],\ 1\le i\le k\,\rangle\cdot[x,\,_{k+1}y], \tag{1}$$

$$[x,\,_{k+1}y]\in\langle\,[x,y^i],\ 1\le i\le k\,\rangle\cdot[x,y^{k+1}]. \tag{2}$$

Proof From the commutator identity $[a,bc]=[a,c][a,b][a,b,c]$ it follows

$$[x,y^{k+1}]=[x,y^k][x,y][[x,y],y^k] \tag{3}$$

$$[x,y,y^k]=[x,y]^{-1}[x,y^k]^{-1}[x,y^{k+1}] \tag{4}$$

For $k=1$ in (1) we have by (3): $[x,y^2]=[x,y]^2[x,y,y]\in\langle\,[x,y]\,\rangle\cdot[x,\,_2y]$. Now we assume that

$$[x,y^k]\in\langle\,[x,\,_iy],\ 1\le i\le(k-1)\,\rangle\cdot[x,\,_ky].$$

If replace $x\to[x,y]$, we obtain the following consequence of the assumption

$$[[x,y],y^k]\in\langle\,[[x,y],\,_iy],\ 1\le i\le(k-1)\,\rangle\cdot[[x,y],\,_ky]=\langle\,[x,\,_iy],\ 1<i\le k\,\rangle\cdot[x,\,_{k+1}y].$$

Now we apply the assumption and its consequence to (3) to get required (1),

$$[x,y^{k+1}]\in\langle\,[x,\,_iy],\ 1\le i\le k\,\rangle\cdot[x,\,_{k+1}y].$$

For $k=1$ in (2) we have by (4): $[x,y,y]\in\langle\,[x,y]\,\rangle\cdot[x,y^2]$. Now assume

$$[x,\,_ky]\in\langle\,[x,y^i],\ 1\le i\le(k-1)\,\rangle\cdot[x,y^k],$$

then if replace here $x\to[x,y]$ and write $[[x,y],\,_ky]$ as $[x,\,_{k+1}y]$, we have

$$[x,\,_{k+1}y]\in\langle\,[[x,y],y^i],\ 1\le i\le(k-1)\,\rangle\cdot[[x,y],y^k]\overset{(4)}{\subseteq}$$

$$\langle\,[x,y]^{-1}[x,y^i]^{-1}[x,y^{i+1}],\ 1\le i\le(k-1)\,\rangle\cdot[x,y]^{-1}[x,y^k]^{-1}[x,y^{k+1}]\subseteq$$

$$\langle\,[x,y^i],\ 1\le i\le k\,\rangle\cdot[x,y^{k+1}].$$

□

Corollary 5.4 *The following subgroups coincide for every n, $(n\ge 0)$.*

$$E_n:=\langle\,[x,\,_iy],\ 0\le i\le n\,\rangle=\langle x^{y^i},\ 0\le i\le n\,\rangle.$$

Proof By (1) and (2) we have the following equality

$$\langle\, [x,\, _iy],\, 0\!\le\! i\!\le\! n\,\rangle = \langle\, x,\, [x,y^i],\, 0\!\le\! i\!\le\! n\,\rangle.$$

Now it suffices to show that $\langle\, x,\, [x,y^i],\, 1\le i\le n\,\rangle = \langle x^{y^i},\ 0\le i\le n\rangle$. Indeed, the normal closure of x, denoted by $\langle x\rangle^F$, is freely generated by all conjugates $x^{y^i}, i\in\mathbb{Z}$ (see [20], p.138). Hence in the subgroup $\langle x^{y^i}, 0\le i\le n\,\rangle$ we can replace the free generators $x^{y^i}, i\!\ne\!0$ by $x^{-1}x^{y^i} = [x,y^i]$, which gives the required equality. □

We can now prove that every binary law has the general Engel construction.

Theorem 5.5 *Every binary law is equivalent to a law which has construction*

$$x^{k_0}[x,y]^{k_1}\,[x,\,_2y]^{k_2}...\,[x,\,_ny]^{k_n}\,\widetilde{\in}\,E',$$

where $E\!=\!\langle\,[x,\,_iy],\ 0\!\le\! i\rangle,\ n\in\mathbb{N},\ k_i\in\mathbb{Z}$.

Proof By ([24], 12.12) every word is equivalent to a pair of words, one of the form x^k; $k\ge 0$, the other a commutator word. Thus every binary law is equivalent to a law $x^k w(x,y)\equiv 1$, where $w\in F'$. Since $F'\subseteq\langle x\rangle^F$, w is a product of some x^{y^i} with say, $i\!\ge\!-m$. Conjugation by y^m gives us the equivalent law $w\equiv 1$ where $w\in\langle x^{y^i},\ 0\!\le\! i\,\rangle$. Since by Corollary 5.4, $E := \langle\,[x,\,_iy],\, 0\!\le\! i\rangle = \langle x^{y^i},\ 0\!\le\! i\rangle$, we have

$$w\in\langle\,[x,\,_iy],\,0\!\le\! i\,\rangle = E.$$

Hence every commutator law is equivalent to a law $w\equiv 1$, where the word w is a product of commutators $[x,\,_iy],\ 0\!\le\! i$. By ordering these factors *modulo* E', we get the law which has construction

$$[x,y]^{k_1}\,[x,\,_2y]^{k_2}...\,[x,\,_ny]^{k_n}\,\widetilde{\in}\,E', \qquad k_i\in\mathbb{Z}.$$

Now we add x^{k_0} if the initial law defines a variety of exponent k_0. □

The parameters n, k_i and a subset $S\subseteq E'$ define specific Engel constructions, responsible for some properties of groups. However the same property can be defined by different equivalent constructions.

6 Engel construction and properties of $\mathfrak{R}$-laws

In 1968 Milnor ([23] Lemma 3) proved that if for all elements g, h in a finitely generated group G the subgroup $\langle g^{h^i},\ i\in\mathbb{N}\rangle$ is finitely generated, and A is an abelian normal subgroup in G so that G/A is cyclic then A is finitely generated.

The property that *for all elements* g, h *in a finitely generated group* G*, the subgroup* $\langle g^{h^i},\ i\!\in\!\mathbb{N}\rangle$ *is finitely generated* was considered by many authors. It is called *the Milnor property* in (F. Point [26]). The groups with this property are called *restrained* (Kim and Rhemtulla [15]). We shall call this property *the restraining property* and varieties consisting of groups with this property – *the restrained varieties.*

Definition 6.1 A law is an $\mathfrak{R}$-law (*restraining law*) if for all elements g, h in a group G satisfying this law, the subgroup $\langle g^{h^i},\ i\in\mathbb{N}\rangle$ is finitely generated.

By another words, a law is an $\mathfrak{R}$-law if it provides the restraining property.

In 1976 Rosset [28] noticed that in the mentioned above Milnor's proof the assumption that A is abelian can be dropped. He proved that *if for all elements g, h in a group G the subgroup $\langle g^{h^i},\ i\in\mathbb{N}\rangle$ is finitely generated* then

- (i) the commutator subgroup G' is finitely generated and
- (ii) if G/N is cyclic then N is finitely generated.

In view of (i) we obtain another equivalent definition of the $\mathfrak{R}$-laws [19].

Definition 6.2 A law $w \equiv 1$ is an $\mathfrak{R}$-law if every finitely generated group G satisfying this law has its commutator subgroup G' finitely generated.

The following observation shows more clearly how restrained varieties in general are related to Engel varieties.

Proposition 6.3 *A law is an $\mathfrak{R}$-law if and only if it implies a law with the following Engel construction*

$$[x,\ _ny]\ \widetilde{\in}\ \langle x,\ [x,y],\ [x,\ _2y],\ ...,\ [x,\ _{n-1}y]\,\rangle. \tag{5}$$

Proof A variety is restrained if and only if for some positive integer n it satisfies a law with construction

$$x^{y^n}\,\widetilde{\in}\,\langle\, x,\ x^y, x^{y^2}, ...,\ x^{y^{n-1}}\rangle, \tag{6}$$

which in view of (1) is equivalent to a law with construction (5). □

We summarize that a law $w \equiv 1$ is an $\mathfrak{R}$-law if one of the following holds

- Each group satisfying $w \equiv 1$ is restrained,
- Each finitely generated group satisfying $w \equiv 1$ has G' finitely generated,
- The law $w \equiv 1$ implies a law with the Engel construction $[x,\ _ny]\ \widetilde{\in}\ E_{n-1}$.

Corollary 6.4 *All Engel laws and all positive laws are $\mathfrak{R}$-laws.*

Proof Each Engel law has the required Engel construction. In [28] Rosset proved that a finitely generated group of subexponential growth is restrained. He used only fact that such a group does not contain a free subsemigroup (by growth reason). Since groups satisfying positive laws also do not contain free subsemigroups the same proof implies that groups satisfying positive laws are restrained (see also (e.g. [12], [16], [26], [5] p.520). So we conclude that positive laws are the $\mathfrak{R}$-laws. □

We need the following

Proposition 6.5 *Every finitely generated residually finite group G satisfying an $\mathfrak{R}$-law is nilpotent-by-finite, $G \in \mathfrak{N}_c\mathfrak{B}_e$, where c, e depend on the law only.*

Proof By result of G. Endimioni [9] every finitely generated residually finite group G in a variety is nilpotent-by-finite if and only if the variety does not contain a group $C_p wr C$ for all primes p. Since the groups $C_p wr C$ have infinitely generated commutator subgroups they do not satisfy an $\mathfrak{R}$-law and hence the first statement follows. It is also shown in [9] that c and e depend only on the variety and hence on the $\mathfrak{R}$-law (see also [6] Theorem A). □

A result of a similar kind as the above by Endimioni has been obtained by G. Traustason [33]. Namely, the property that every finitely generated residually nilpotent group in the variety is nilpotent holds if and only if the variety contains neither $C_p wr C$ nor $C wr C_p$. It is interesting to note that the Engel varieties satisfy but the varieties with positive laws do not in general satisfy this property.

Lemma 6.6 *Every finitely generated group G satisfying an $\mathfrak{R}$-law has its finite residual R (intersection of all subgroups of finite index in G) finitely generated.*

Proof The group G/R is residually finite and by assumption satisfies an $\mathfrak{R}$-law, hence by Proposition 6.5, G/R contains a nilpotent subgroup H/R of finite index. Now, since $|G:H| = |(G/R):(H/R)| < \infty$ and G is finitely generated, both H and H/R are finitely generated. Being a finitely generated nilpotent group, H/R is polycyclic (see [24] 31.12). So there is a finite subnormal series with cyclic factors $H = N_0 \rhd N_1 \rhd \cdots \rhd N_m = R$. Then by means of m successive applications of Rosset's result mentioned as (ii) on the page 7, we conclude that R is finitely generated. □

Corollary 6.4 and the following Theorem answer the question *What do the Engel laws and positive laws have in common that forces finitely generated locally graded groups satisfying them to be nilpotent-by-finite?*

Theorem 6.7 (cf. [19]) *Let G be a finitely generated group satisfying an $\mathfrak{R}$-law. Then either G is nilpotent-by-finite or its finite residual has a finitely generated, infinite simple quotient.*

Proof By Lemma 6.6, finite residual R of G is finitely generated. Let R contain a proper subgroup T say, of finite index. By a result of M. Hall, R has only finitely many subgroups of index $k = |R : T|$ and their intersection K is characteristic of finite index in R. So K is normal in G and $K \subseteq T \subsetneq R$. Since R/K is finite and G/R is nilpotent-by-finite, the isomorphism $(G/K)/(R/K) \cong G/R$ implies that G/K is finite-by-(nilpotent-by-finite). Since finite-by-nilpotent group is nilpotent-by-finite, whence G/K is nilpotent-by-finite and then residually finite. Then $R \subseteq K$, which contradicts to $K \subseteq T \subsetneq R$ unless $R = 1$ in which case G is nilpotent-by-finite by Proposition 6.5.

If R contains no proper subgroup of finite index then by Zorn Lemma it has a maximal proper normal subgroup N of infinite index. The quotient R/N is the finitely generated infinite simple group satisfying the $\mathfrak{R}$-law. □

Moreover, in view of Proposition 6.5 we obtain the following.

Corollary 6.8 *For every $\mathfrak{R}$-law there exist positive integers c and e depending only on the law, such that every locally graded group satisfying this law lies in the product variety $\mathfrak{N}_c\mathfrak{B}_e$.*

Note There are non-(locally graded), non-(virtually nilpotent) finitely generated groups satisfying $\mathfrak{R}$-laws. For example free Burnside groups $B(r,n)$ for $r > 1$ and sufficiently large n, the groups satisfying the law $xy^n = y^n x$ also for n sufficiently large. Another example was given by Ol'shanskii and Storozhev in [25]. The $\mathfrak{R}$-laws in these examples are positive laws.

In [19] a problem was formulated whether there are $\mathfrak{R}$-laws that imply neither positive nor Engel law? Note that every law $[x, {}_n y] \equiv [x, {}_m y]$ for $n > m$ has Engel construction (5), so it is an $\mathfrak{R}$-law. Every finite group satisfies such a law, hence the law need not be an Engel law. We do not know whether it is then a positive law.

7 Special kind of $\mathfrak{R}$-laws, called L_n

We denote by L_n the laws of the form $[x,y] \equiv [x, {}_n y]$, where $n > 1$. These laws are the $\mathfrak{R}$-laws because they have construction of the form (5) if we write them as $[x, {}_n y] \equiv [x, y]$.

Proposition 7.1 (cf. [10]) *Every metabelian group and every finite group satisfying the law L_n is abelian.*

Proof If substitute $[y, {}_{n-1}x]$ for y in L_n, we get $[x, [y, {}_{n-1}x]] \equiv [x, {}_n[y, {}_{n-1}x]]$. By taking inverse and interchanging $x \rightleftarrows y$ we obtain a law with construction $[x, {}_n y] \,\widetilde{\in}\, F''$ which in view of L_n, implies a law with construction $[x, y] \,\widetilde{\in}\, F''$. Clearly, each metabelian group satisfying a law with such construction is abelian.

Now, if there exists a finite non-abelian group satisfying the law L_n, then there exists such a group G of the smallest order. By O. Schmidt, a finite group G, all whose proper subgroups are abelian, is metabelian. Hence by the above, G must be abelian, which is a contradiction. □

So every law of the form $[x, y] \equiv [x, {}_n y]$, $n > 1$ is either abelian or pseudo-abelian (see [24] Problem 5). It was conjectured in 1966 by N. Gupta [10] that each such a law is abelian. The proofs were given only for $n = 2$ and $n = 3$. We have looked for a shorter proof and its possible extension for $n > 3$. Our proofs are based on the following

Observation *Let a group G satisfy the law L_n. If an element $b \in G$ is conjugate to its inverse then $b^2 = 1$.*

Proof Let $b^{-1} = b^a$ say, then $b^2 = (b^a)^{-1}b = [a, b] \equiv [a, {}_n b] = [b^2, {}_{n-1}b] = 1$. □

In both cases for L_2 and L_3 we could deduce the law $[x, y]^2 \equiv 1$, which leads to the abelian law. To do this we used the following

Proposition 7.2 *In the free group F the word $[x,\, _ny^{-1}]$ is conjugate to $[x,\, _ny]^{(-1)^n}$ for $n = 1, 2, 3$.*

Proof It suffices to check the following commutator identities which are the equalities in the free group F. Namely (i) $[x, y^{-1}] = [x, y]^{-y^{-1}}$ which is clear,

$$(ii)\ [x,\, _2y^{-1}] = [x,\, _2y]^{y^{-1}[x,y]^{-1}y^{-1}}, \qquad (iii)\ [x,\, _3y^{-1}] = [x,\, _3y]^{-y^{-1}[x,y]^{-1}y^{-2}}.$$

We prove here only (iii) for which we use the commutator identities (i) and

$$(iv)\ [a^{-1}, b] = [a, b]^{-a^{-1}}, \qquad (v)\ a^b = a[a, b], \qquad (vi)\ [a, bc] = [a, c][a, b]^c.$$

Namely, $[x,\, _3y^{-1}] \overset{(i)}{=} [[[x,\, y]^{-y^{-1}},\, y]^{-y^{-1}},\, y]^{-y^{-1}} = [[[x,\, y]^{-1},\, y]^{-1},\, y]^{-y^{-3}} \overset{(iv)}{=}$

$$[[x,\, _2y]^{[x,\,y]^{-1}},\, y]^{-y^{-3}} = [[x,\, _2y],\, y^{[x,\,y]}]^{-[x,\,y]^{-1}y^{-3}} \overset{(v)}{=}$$

$$[[x,\, _2y],\ y\cdot[x,\, _2y]^{-1}]]^{-[x,\,y]^{-1}y^{-3}} \overset{(vi)}{=} [x,\, _3y]^{-[x,\,_2y]^{-1}[x,\,y]^{-1}y^{-3}} = [x,\, _3y]^{-y^{-1}[x,\,y]^{-1}y^{-2}}.$$

□

However Proposition 7.2 is not valid for $n = 4$. We conjecture that the laws L_n need not be abelian for $n > 3$.

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