Publ. Math. Debrecen
Manuscript (April 2008)

On locally graded n-Engel and positively n-Engel groups

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Abstract. We discuss four problems concerning n-Engel and so called *positively* n-Engel groups. As the answer to one of them we prove that in the class of locally graded groups every *positively* n-Engel group is locally nilpotent, which extends a similar result of D.M.Riley for residually finite groups.

1. Introduction

We discuss a number of problems concerning n-Engel and *positively* n-Engel groups (all definitions are given in the next section) studied since 1936 when M.Zorn proved that every finite n-Engel group is nilpotent. It is not true in general that an n-Engel group is nilpotent. Examples of non-nilpotent n-Engel groups can be found among 3- and 4-Engel groups ([1], [9], [22]). However, no examples of finitely generated non-nilpotent n-Engel groups are known.

Recall that an n-**variable law** $u(x_1, x_2, \dots, x_n) = v(x_1, x_2, \dots, x_n)$ is called **positive** if the words u, v do not involve inverses of any x_i's. The following questions are open.

Q1 Is every n-Engel group locally nilpotent? (In other words, is every finitely generated n-Engel group nilpotent?)
Q2 Is it true that there does not exist a finitely generated infinite simple n-Engel group?
Q3 Are n-Engel varieties defined by positive laws?
Q4 Is every *positively* n-Engel group locally nilpotent?

2000 *Mathematics Subject Classification:* Primary 20E10, Secondary 20E25.
Key words and phrases: n-Engel group, *positively* n-Engel group, positive law, local nilpotence.

Question **Q1** is approached in two main ways: one is to examine n-Engel groups for different n, the second is to investigate the problem in certain classes of groups. In the case of different n, an affirmative answer has been found only for $n = 2$ [14], for $n = 3$ [12] and for $n = 4$ [11]. As to the second approach, it has been shown that n-Engel groups are locally nilpotent if additionally they are soluble [8], residually finite [27], profinite [28] or compact [18]. Since all the above mentioned groups are locally graded, the most general answer so far was given by Y.K.Kim and A.H.Rhemtulla [13] in 1994 – namely, locally graded n-Engel groups are locally nilpotent.
We show (Proposition 3.1) that questions **Q1** and **Q2** are equivalent.

Question **Q3** was posed by A.I.Shirshov in 1963 (Problem 2.82, The Kourovka Notebook [25]). A positive answer has been given for 2- and 3-Engel groups [23] and for 4-Engel groups [24]. We show (Proposition 3.2) that question **Q3** has an affirmative answer more generally, that is for the class of locally graded groups.

The positive laws found by Shirshov for 2- and 3-Engel groups were generalized by D.M.Riley who defined *positively* n-Engel groups and gave an affirmative answer to question **Q4** for the class of residually finite groups ([20], Theorem A). D.M.Riley also pointed out that the result can be extended to the larger class $\mathcal{C}$ defined in [3]. In Theorem 4.2 we give an affirmative answer to question **Q4** for the class of locally graded groups which, as we show in Theorem 5.1, strictly contains the class $\mathcal{C}$.

2. Preliminaries

We recall some useful definitions, in particular of positively n-Engel groups [20], the class $\mathcal{C}$ [3] and locally graded groups [5].

Let $[a_1, a_2, \ldots, a_k] := [\ldots[[a_1, a_2], a_3], \ldots, a_k]$ denote a left-normed commutator.

A group is **nilpotent of class** n if it satisfies the law $[x_1, x_2, \ldots, x_{n+1}] = 1$.

A group is n**-Engel** if it satisfies the law $[x, y, y, \ldots, y] = 1$, where y occurs n times. In both definitions n is the smallest number with that property.

A.I.Mal'tsev [17] in 1953 and independently B.H.Neumann and T.Taylor [19] in 1963 proved that nilpotency of class n can be defined by a positive law.
If we take $\mu_1(x, y) := xy$, $\nu_1(x, y) := yx$ and define inductively for all $n \in \mathbb{N}$:

$$\mu_{n+1}(x, y, z_1, \ldots, z_n) := \mu_n(x, y, z_1, \ldots, z_{n-1}) z_n \nu_n(x, y, z_1, \ldots, z_{n-1}), \quad (1)$$

$$\nu_{n+1}(x, y, z_1, \ldots, z_n) := \nu_n(x, y, z_1, \ldots, z_{n-1}) z_n \mu_n(x, y, z_1, \ldots, z_{n-1}), \quad (2)$$

then by [19] a group is nilpotent of class n if and only if it satisfies the law

$$\mu_n(x, y, z_1, \dots, z_{n-1}) = \nu_n(x, y, z_1, \dots, z_{n-1}). \quad (3)$$

In 1963 A.I.Shirshov [23] proved that the law

$$xy^2x = yx^2y \quad (4)$$

defines 2-Engel groups and the following two laws define 3-Engel groups:

$$(xy^2x) \cdot (yx^2y) = (yx^2y) \cdot (xy^2x), \quad (5)$$

$$\big(xy^2x\big) \cdot xy \cdot \big(yx^2y\big) = \big(yx^2y\big) \cdot xy \cdot \big(xy^2x\big). \quad (6)$$

D.M.Riley in [20] generalized these positive laws and defined so called *positively* n-Engel group. Namely, a group is called ***positively*** n**-Engel** if it satisfies both of the following laws:

$$\mu_n(x, y, \underbrace{1, 1, \dots, 1}_{n-1}) = \nu_n(x, y, \underbrace{1, 1, \dots, 1}_{n-1}), \quad (7)$$

$$\mu_n(x, y, 1, xy, (xy)^2, \dots, (xy)^{n-2}) = \nu_n(x, y, 1, xy, (xy)^2, \dots, (xy)^{n-2}). \quad (8)$$

Note that the law (4) is of the form $\mu_2(x, y, 1) = \nu_2(x, y, 1)$ and laws (5), (6) are of the form $\mu_3(x, y, 1, 1) = \nu_3(x, y, 1, 1)$, $\mu_3(x, y, 1, xy) = \nu_3(x, y, 1, xy)$, which means that 2- and 3-Engel groups are *positively* 2- and 3-Engel groups, respectively.

The definition of the class $\mathcal{C}$ involves, among other things, the notion of the **restricted Burnside variety** of exponent e which is the variety generated by all finite groups of exponent e. It follows from Zelmanov's affirmative solution to the Restricted Burnside Problem (for more details see e.g. [26]) that all groups in the restricted Burnside variety of exponent e are locally finite of exponent dividing e.

An SB**-group** is one lying in some product of finitely many varieties, each of which is either a soluble or a restricted Burnside variety.

For any group-theoretic class $\mathcal{X}$ of groups, let $L\mathcal{X}$ denote the class of all groups locally in $\mathcal{X}$ and $R\mathcal{X}$ all groups residually in $\mathcal{X}$. Let Δ_1 denote the class of all SB-groups. Then define inductively for every natural n: $\Delta_{n+1} := L\Delta_n \cup R\Delta_n$. **The class** $\mathcal{C}$ is the union: $\mathcal{C} := \bigcup_{n\in\mathbb{N}} \Delta_n$.

A group G is called **locally graded** if every nontrivial finitely generated subgroup of G has a proper normal subgroup of finite index. The class of locally graded groups was introduced in 1970 by S.N.Černikov [5] in order to avoid groups such as infinite Burnside groups or Ol'shanskii-Tarski monsters. We recall the following

Properties of locally graded groups: *The class of locally graded groups contains all soluble, locally finite and residually finite groups. It is closed under taking subgroups and extensions. It is also closed under the operations L and R defined above i.e. a group which is locally-(locally graded) or residually-(locally graded) is locally graded.*

3. Questions Q1 –Q3

The following statement shows the equivalence of questions **Q1** and **Q2**.

Proposition 3.1. *There exists a non-(locally nilpotent) n-Engel group if and only if there exists a finitely generated infinite simple n-Engel group.*

PROOF. If there exists an n-Engel group G which is not locally nilpotent then by the result mentioned in the Introduction ([13], Corollary 4), G is not locally graded. Thus by definition, G must contain a finitely generated subgroup H, which has no proper subgroup of finite index. Since H is finitely generated, by Zorn's Lemma it has a maximal proper normal subgroup N. Then N is of infinite index and the factor H/N is a finitely generated infinite simple n-Engel group.

The "only if" part is obvious, since the only finitely generated nilpotent simple groups are cyclic of prime orders. □

The following statement gives an affirmative answer to question **Q3** for the class of locally graded groups.

Proposition 3.2. *A variety defined by a locally graded n-Engel group has a basis consisting of positive laws.*

PROOF. If G is a locally graded n-Engel group then G (by [13], Corollary 4) is locally nilpotent. If H is any 2-generator subgroup of G, then H is nilpotent and (by [10], Theorem 1) it is residually finite. So by the main result in [4] there exist integers c, e depending on n only such that all n-Engel groups in the class $\mathcal{C}$ (hence in particular all n-Engel residually finite groups) are contained in the variety satisfying the positive law $\mu_{c+1}(x^e, y^e, z_1^e, \ldots, z_c^e) = \nu_{c+1}(x^e, y^e, z_1^e, \ldots, z_c^e)$. This law implies the 2-variable law $\mu_{c+1}(x^e, y^e, \underbrace{1, \ldots, 1}_{c}) = \nu_{c+1}(x^e, y^e, \underbrace{1, \ldots, 1}_{c})$, satisfied in any 2-generator subgroup of G, whence we conclude that G satisfies this law. It was shown (in [15], Corollary, p. 7) that if a group satisfies a positive law, then the variety it generates has a basis consisting of positive laws, which completes the proof. $\square$

4. Positively n-Engel groups

Our main result concerns *positively* n-Engel groups. We show that if G is a finitely generated locally graded *positively* n-Engel group then G is nilpotent. Moreover, the nilpotency class of G is bounded by a function depending only on n and the minimal number of generators of G. We start with the following

Proposition 4.1. *Every finitely generated finite-by-nilpotent group is nilpotent-by-finite.*

Proof. Let G be a finitely generated group and let N be a finite normal subgroup such that G/N is nilpotent of class c, that is $\gamma_{c+1}(G) \subseteq N$. Since N is a normal subgroup of G, the centralizer C of N in G is obviously normal. Next, since N is finite, all conjugacy classes in G of elements in N are finite, so the centralizers of all elements in N have finite indices. Therefore the centralizer C, as their intersection, has finite index also. Furthermore, $\gamma_{c+2}(C) = [\gamma_{c+1}(C), C] \subseteq [N, C] = 1$, so C is nilpotent. Thus C is a nilpotent normal subgroup of finite index in G, which means that G is nilpotent-by-finite as required. $\square$

Theorem 4.2. *Every finitely generated locally graded positively n-Engel group G is nilpotent of class depending only on n and the minimal number of generators of G.*

Proof. If G satisfies the assumptions and R is the intersection of all normal subgroups of finite index in G, then G/R is a finitely generated residually finite *positively* n-Engel group, so (by [20], Theorem A) it is nilpotent. Note that G, being *positively* n-Engel, satisfies positive laws, whence (by [3], p. 520) for every $a, b \in G$ the subgroup $\langle a^{\langle b\rangle}\rangle$ is finitely generated (here $\langle x\rangle$ denotes the cyclic group generated by x and $a^b = b^{-1}ab$).

Next, as a finitely generated nilpotent group, G/R is polycyclic (by [21], 5.2.18). Hence there exists a finite subnormal series $R = H_0 \leq H_1 \leq \ldots \leq H_q = G$ with all factors H_i/H_{i-1} cyclic. Since H_q and $\langle a^{\langle b\rangle}\rangle$ for all $a, b \in H_q$ are finitely generated and H_q/H_{q-1} is cyclic, it follows that H_{q-1} is finitely generated (see e.g. [13], Lemma 1). Since H_{q-1} has exactly the same properties as H_q and

H_{q-1}/H_{q-2} is again cyclic, we conclude that H_{q-2} is finitely generated. We continue in this fashion obtaining that $H_0 = R$ is finitely generated.

If $R \neq 1$, then being a nontrivial finitely generated subgroup of the locally graded group, R contains a proper normal subgroup T of finite index which (by [16], Ch. IV, Theorem 4.7) contains a subgroup K characteristic in R and of finite index in R. Hence $K \leq T \lneqq R$ which implies $K \lneqq R$. Since K is characteristic in R and $R \triangleleft H$, then $K \triangleleft H$. Now, since R/K is finite and G/R is nilpotent, then from $(G/K)/(R/K) \cong G/R$ it follows that G/K is a finitely generated finite-by-nilpotent group. Thus by Proposition 4.1, G/K is finitely generated nilpotent-by-finite, and hence by [10], it is residually finite. This means that the intersection of all normal subgroups of finite index in G is a subgroup of K, that is $R \leq K$. Together with $K \lneqq R$ this gives a contradiction. Hence $R = 1$ so G is a residually finite *positively* n-Engel group and (by [20], Theorem A) it must be nilpotent of class depending only on n and the minimal number of generators, as required. □

It is also worth mentioning that as a corollary of Theorem A in [20], D.M.Riley deduced that for finitely generated residually finite groups the properties of being n-Engel and *positively* m-Engel are equivalent for certain related n, m. By imitating Riley's proof, we obtain the following

Corollary 4.3. *Let G be a locally graded group and let m, n be natural numbers.*
(i) If G is positively m-Engel, then G is n-Engel for some n depending on m only.
(ii) If G is n-Engel, then G is positively m-Engel for some m depending on n only. □

5. The class $\mathcal{C}$ and locally graded groups

D.M.Riley has observed that his result on *positively* n-Engel groups ([20], Theorem A) can be extended from the class of residually finite groups to the class $\mathcal{C}$ (see page 3 for the definition). We show now that Theorem 4.2 from the previous section actually extends Riley's result to a much larger class of groups than even $\mathcal{C}$.

Theorem 5.1. *The class $\mathcal{C}$ is strictly contained in the class of locally graded groups.*

Proof. As shown (in [2], Theorem 1(i)) every group in the class $\mathcal{C}$ is locally-(residually-SB), whence (by Properties listed on page 4) is a locally graded group.

The strictness of the inclusion follows from the existence of a non-residually finite group G of intermediate growth, constructed by Anna Erschler in [6] (in fact, A.Erschler obtained a continuum of such groups). As shown (in [6], Theorem 1) G is an extension of a finite group by the residually finite group of intermediate growth constructed by R.I.Grigorchuk in [7]. Thus the group G, as an extension in the class of locally graded groups, is locally graded. Since (by [2], Corollary 2) every group of intermediate growth in the class $\mathcal{C}$ is residually finite, G does not belong to $\mathcal{C}$, which completes the proof. $\square$

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