On S-length of groups

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Abstract

Let G be a group and S be a subsemigroup in G, generating G as a group. Every element in G is a product of elements from $S \cup S^{-1}$. An equality $G = S^{-1}S \cdots S^{-1}S$ allows to define an S-length $l(G)$ of the group G. The note concerns the problem posed by J. Krempa on possible values of $l(G)$. We show that for collapsing groups, supramenable groups and groups of a subexponential growth $l(G) \leq 2$. The S-length of a relatively free group can be equal to 1 or 2 or infinity, but it can not be equal to 3. The problem concerning other values is open.

Keywords: relatively free group, cancellative semigroup, S-length.

In the process of algebraic classifications in group theory there are used different parameters of groups defined by so called length or width functions. The notion of S-length of a group was suggested by J. Krempa in 1998 and some conjections were made on its value.

Let S be a subsemigroup in a group G, generating G as a group. If G is a finite group, it is clear that $G = S = S^{-1}$. If G is an abelian group then $G = S^{-1}S = S\,S^{-1}$, that is G is a group of fractions of S (see e.g. [1]). A natural question arises whether for each $k > 2$ there is a group such that $G = S^{-1}S \cdots S^{-1}S$ with no less than k factors. The aim of this note is to describe the groups of S-length equal to 1 and 2. We also give an example of a group with S-length greater or equal to 3 and show that the S-length of a relatively free group can not be equal to 3.

If G is a group and A is a subset in G then $sgp(A)$ will denote the subsemigroup of G generated by A. By $|A|$ we denote the cardinality of A.

1. S-length of a group

Let G be a finitely generated group and n be the smallest number of generators in G. A subsemigroup $S \subseteq G$ generated by any n-element set of generators in G will be called *a base semigroup of* G.

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The length of G with respect to a base semigroup S, denoted by $l(S,G)$ is the smallest natural k (if exists) such that

$$G = \underbrace{S^{-1}S\,S^{-1}S\cdots S}_{k}{}^{(-1)^k} = \underbrace{S\,S^{-1}S\,S^{-1}\cdots S}_{k}{}^{(-1)^{k-1}}. \qquad (1)$$

If such a k does not exist we assume $l(S,G) = \infty$.

Definition Let $X(G)$ denote the set of all base semigroups in G. Then the S-length of G is defined as

$$l(G) = sup\{l(S,G) : S \in X(G)\}.$$

Example 1 *Let $G = \langle a \rangle_2 * \langle b \rangle_3$ be the free product of finite cyclic groups of orders 2 and 3. Let $S_1 = sgp(a,b)$ and $S_2 = sgp(ab, ab^2)$. It is not difficult to see that $l(S_1,G) = 1$ (see also Example 4), while $l(S_2,G) \neq 1$, because each element in S_2 begins with a, so $S_2 \neq G$. We can see also that $a \notin S_2S_2^{-1}$ and $a = (ab)(ab^2)^{-1}(ab) = (ab)^{-1}(ab^2)(ab)^{-1} \in S_2\,S_2^{-1}S_2 \cap S_2^{-1}S_2\,S_2^{-1}$, hence we can conclude that $l(G) \geq 3$.* □

Example 2 *If G is a periodic group (e.g. a finite group) then $l(G) = 1$.* □

Example 3 *If F is a free noncyclic group and S is generated by a set $\{a, b, ...\}$ of free generators of F then for every k, $l(S,F) > k$, because the word $(ab^{-1})^k$ is not equal to any other word and needs $2k$ factors in (1). So we conclude that $l(F) = \infty$.* □

A word $w(x_1, ..., x_n)$ is called *positive* if it is written without inverses of x_i's. We give now necessary and sufficient conditions for a group G and a base semigroup $S \subseteq G$ to have $l(S,G) = 1$.

Proposition 1 *Let $S = sgp(a_1, a_2, ..., a_n)$ be a base semigroup in a group G. $l(S,G) = 1$ if and only if G has a defining relation $r(a_1, a_2, ..., a_n) = 1$, where r is a positive word, containing each generator.*

Proof Let F be a free group and $\mathcal{F}$ be a free subsemigroup in F, both generated by the set $X = \{x_1, x_2, ..., x_n\}$. By assumption G has a presentation $F/N = \langle x_1, x_2, ..., x_n \mid N \rangle$ for some $N \lhd F$, and the natural homomorphism $F \to G \cong F/N$ maps $\mathcal{F} \to S$.

If $l(S,G) = 1$, then $G = S = S^{-1}$, and $F = \mathcal{F}N = \mathcal{F}^{-1}N$. Hence $(x_1x_2\cdots x_n)^{-1} \in F = \mathcal{F}N$. So the word $(x_1x_2\cdots x_n)^{-1}$ is in a coset sN for some positive word $s = s(x_1, ...x_n) \in \mathcal{F}$. Thus the required defining relator is $r := x_1x_2\cdots x_n \cdot s$ and the relation in G is $a_1a_2\cdots a_n \cdot s(a_1, ..., a_n) = 1$.

Conversely, let $G \cong F/N$. If $r \in N$ is a positive word containing each generator then for each fixed x_i we can write r as ux_iv for some $u, v \in \mathcal{F}$. Conjugation by u gives the defining relator $x_ivu \in N$, which implies that $x_i^{-1} \in vuN \subseteq \mathcal{F}N$. Hence $\mathcal{F}^{-1} \subseteq \mathcal{F}N$, $F = \mathcal{F}N$ and $G = S$. Since $G = G^{-1}$, we have $G = S = S^{-1}$ and $l(S, G) = 1$. □

Example 4 *In Example 1, the group $G = \langle a \rangle_2 * \langle b \rangle_3$, with a base semigroup $S_1 = sgp(a, b)$ has the defining relation $a^2b^3 = 1$. So $l(S_1, G) = 1$, however $l(G) \neq 1$.* □

We recall that a semigroup S satisfies Ore conditions if for every g, $h \in S$ there exist g', $h' \in S$ such that $gg' = hh'$ (right Ore condition) and there exist g'', $h'' \in S$ such that $g''g = h''h$ (left Ore condition) (for definition in different terminology see [1] §1.10). We give now necessary and sufficient conditions for a group G and a base semigroup $S \subseteq G$ to have $l(S, G) \leq 2$.

Proposition 2 *Let G be a group with a base semigroup S. $l(S, G) \leq 2$ if and only if S satisfies Ore conditions.*

Proof If $l(S, G) = 2$, then $G = S\,S^{-1} = S^{-1}S$ and for every g, $h \in S$, $h^{-1}g \in G = SS^{-1}$, which implies that there exist g', $h' \in S$ such that $h^{-1}g = h'g'^{-1}$ and the right Ore condition $gg' = hh'$ follows. The left Ore condition follows similarly from the equality $G = S^{-1}S$.

Conversely, the right Ore condition implies that for every g, $h \in S$ there exist g', $h' \in S$ such that $h^{-1}g = h'g'^{-1}$ which gives $S^{-1}S \subseteq SS^{-1}$. The opposite inclusion follows from the left Ore condition, so $SS^{-1} = S^{-1}S$. Each element of G is in a finite product of the form $S\,S^{-1}S\,S^{-1} \cdots S^{(-1)^{k-1}}$ with k, say, factors. If $k = 3$ then by the equality $S^{-1}S = SS^{-1}$ we have $S\,S^{-1}S = S(S\,S^{-1}) = S\,S^{-1}$. Similarly, for any k we can show by induction that $G = S^{-1}S = SS^{-1}$. Hence $l(S, G) \leq 2$. □

Theorem 1 *A group G without free noncyclic subsemigroup has $l(G) \leq 2$.*

Proof Let g, h be elements in G. Since by assumption the semigroup $sgp(g, h)$ is not free, there are two equal nontrivial words $u(g, h) = v(g, h)$. By using cancellation in G we can assume that $u(g, h)$ has g as the first letter and the first letter in $v(g, h)$ is h, so we obtain $g\,u_1(g, h) = h\,v_1(g, h)$, which implies the right Ore condition in every subsemigroup in G. For the left Ore condition we use the last letters in $u(g, h)$ and $v(g, h)$. Now the statement follows by Proposition 2. □

Corollary 1 *All collapsing groups [7], supramenable groups [8] and groups of a subexponential growth [4], [3], [2] have* $l(G) \leq 2$. $\square$

To give an example of a group with S-length greater or equal to 3 we recall that the restricted wreath product (see [6] p.45) $G = A\, wr\langle b\rangle$ of a group A and an infinite cyclic group $\langle b\rangle$ is a semidirect product of W and $\langle b\rangle$ where $W = \prod^{\times} A^{b^i}$ is the direct product of copies of A, numbered by elements of $\langle b\rangle$. Instead of A^{b^0} we write A, and $A^{b^i} = b^{-i} A\, b^i$. Every element $g \in G$ can be uniquely written as

$$g = b^j w$$

where w is a product of commuting factors a^{b^i}, $\ a^{b^i} = b^{-i} a\, b^i$, $\ a^{b^i} b^j = b^j\, a^{b^{i+j}}$.

Theorem 2 *If* $G = \langle a\rangle_2\, wr\langle b\rangle$ *and* $S = sgp(a, b)$, *then* $l(S, G) = 3$.

Proof Note that $a^b\, a^{b^2} = b^{-1}\, a\, b^{-1} a\, b^2 = b^{-2} abab \notin S\, S^{-1}$, hence $l(S, G) > 2$. To prove that $l(S, G) = 3$ it suffices to show that $G = \langle b\rangle\, S\, \langle b\rangle = \langle b\rangle\, S^{-1} \langle b\rangle$. Since $a = a^{-1}$, we have $a \in S \cap S^{-1}$. Let $w \in W$. Then w can be written as a product of, say k, commuting factors

$$w = a^{b^{i_1}} a^{b^{i_2}} \cdots a^{b^{i_k}} = b^{-i_1}\, a\, b^{i_1 - i_2} a\, b^{i_2 - i_3} \cdots b^{i_{k-1} - i_k} a\, b^{i_k}. \tag{2}$$

For example $a^{b^5} a^{b^3} a^{b^{-1}} = b^{-5} a\, b^{5-3} a\, b^{3+1} a\, b^{-1} = b^{-5} a\, b^2 a\, b^4 a\, b^{-1} \in \langle b\rangle\, S\, \langle b\rangle$. Since a^{b^i} commute, we can assume in (2) $i_1 > i_2 > ... > i_k$, then $w \in b^{-i_1} S b^{i_k}$.

If write the factors a^{b^i} in reverse order then $i_1 < i_2 < ... < i_k$, and $w \in b^{-i_1} S^{-1} b^{i_k} \subseteq \langle b\rangle\, S^{-1} \langle b\rangle$. Thus $W \subseteq \langle b\rangle\, S\, \langle b\rangle \cap \langle b\rangle\, S^{-1} \langle b\rangle$ and multiplication by $\langle b\rangle$ implies $G = \langle b\rangle\, S\, \langle b\rangle = \langle b\rangle\, S^{-1} \langle b\rangle$. Hence $G = S^{-1} S\, S^{-1} = S\, S^{-1} S$ and $l(S, G) = 3$. $\square$

2. S-length of relatively free groups

A group G satisfies a law $u(x_1, ..., x_m) \equiv v(x_1, ..., x_m)$ if for every elements $g_1, ..., g_m$ in G the equality $u(g_1, ..., g_m) = v(g_1, ..., g_m)$ holds. A law $u(x_1, ..., x_m) \equiv v(x_1, ..., x_m)$ is called an *m-variable positive law* if the words u, v are positive words. The positive law is called *balanced* if for each variable x_i its exponent sum is the same in u and in v. The simplest example of the balanced positive law is the abelian law. Note that a non-balanced law implies a law $x^m \equiv 1$ for some $m \in \mathbb{N}$ and implies the balanced law $x^m y^m \equiv y^m x^m$.

A group G is called *relatively free of rank* n if it is free in some variety of groups, and has a set of free generators of cardinality n. We formulate the statements 13.11, 13.52, 13.53, 13.25 from [6] in the following

Lemma 1 *Let* $\{a_i\}$ *be a set of free generators in a relatively free group* G.
(i) If $\{b_j\}$ *is another set of free generators in* G*, then* $|\{a_i\}| = |\{b_j\}|$.
(ii) $|\{a_i\}|$ *is the smallest number of generators in* G.
(iii) Every relator on the generators in $\{a_i\}$ *is a law in the group* G.
(iv) Every mapping $\{a_i\} \to G$ *can be extended to an endomorphism of* G.

We can show that for a relatively free group G with a base semigroup S the number $l(S, G)$ is an invariant of G, not depending on the choice of S.

Lemma 2 *The* S*-length* $l(G)$ *of a relatively free group* G *is equal to* $l(S, G)$ *for any base semigroup* S *on a set of free generators.*

Proof Let G be a relatively free group. Let S_1 and S_2 be two base semigroups in G generated by the sets of free generators $\{a_i\}$ and $\{b_i\}$ respectively. We show that the equality $l(S_1, G) = l(S_2, G)$ holds. By (i) of Lemma 1, the sets $\{a_i\}$ and $\{b_i\}$ have the same cardinality and the map $a_i \to b_i$ defines by (iv) of Lemma 1, an automorphism α in G, such that $S_1^\alpha = S_2$. Thus α, applied to the expression (1) written with respect to S_1, changes S_1 to S_2 and hence $l(S_1, G) \geq l(S_2, G)$. Similarly, by applying the automorphism α^{-1} to (1) for S_2, we get $l(S_2, G) \geq l(S_1, G)$, which gives the equality $l(S_1, G) = l(S_2, G)$.

If a relatively free group G of rank n is a non-Hopf group [5] then there exists a set $\{b_i\}$ of n generators, which generate G not freely. Let S_2 be the base semigroups on the set $\{b_i\}$ and S_1 be the base semigroups on the set of free generators $\{a_i\}$. The map $a_i \to b_i$ defines by (iv) of Lemma 1, an epi-endomorphism in G which implies that $l(S_1, G) \geq l(S_2, G)$. Hence we conclude that $l(G) = (S_1, G)$. □

Theorem 3 *The* S*-length* $l(G)$ *of a relatively free group* G *is less or equal to 2 if and only if* G *satisfies a positive law.*

Proof Let $l(G) \leq 2$ and let S be a base semigroup generated by a set of free generators $\{a_i\}$ in G. In view of Lemma 2, $l(S, G) \leq 2$ and by Proposition 2, S satisfies Ore conditions. So for generators a_1, a_2 there exist $s_1, s_2 \in S$ such that $a_1 s_1 = a_2 s_2$, then by (iii) of Lemma 1, G satisfies a positive law.

Conversely, let G satisfies a positive law $u'(x_1, ..., x_n) = v'(x_1, ..., x_n)$. Then G satisfies a binary positive law $u(x, y) = v(x, y)$, obtained by substitution $x_i \to xy^i$. It follows that G does not contain a free subsemigroup and the statement follows by Theorem 1. □

Corollary 2 *Let G be a relatively free group.*
1. *$l(G) = 1$ if and only if G satisfies a law $x^n = 1$ for some $n \in \mathbb{N}$.*
2. *$l(G) = 2$ if and only if G satisfies only balanced non-trivial positive laws.*
$\square$

There is a conjecture that for a relatively free group G the S-length $l(G)$ can be only 1, 2 or infinity. We prove that it can not be equal to 3.

Theorem 4 *If G is a relatively free group, then $l(G) \neq 3$.*

Proof If G is a relatively free group and $l(G) = 3$ then the equality $G = S\,S^{-1}S = S^{-1}S\,S^{-1}$ implies by (iii) of Lemma 1, that G must satisfy a law

$$x\,y^{-1}z \equiv a^{-1}b\,c^{-1}, \tag{3}$$

where a, b, c are positive words in a free group F generated by $x, y, z, ...$
The law (3) can be written as

$$b \equiv a(x\,y^{-1}z)c. \tag{4}$$

We introduce three maps in the free group, defined by

$$\alpha : y \to 1, \quad \beta : y \to x, \quad \gamma : y \to z,$$

and each leaves other generators fixed. Each of these maps changes (4) into a positive law. If $l(G) = 3$ then by Theorem 3, each of these positive laws must be trivial. We show that it is not possible.

If apply α to (4), we obtain a positive law $b^\alpha \equiv a^\alpha xzc^\alpha$ which must be trivial that is we have the equality

$$b^\alpha \doteq a^\alpha xz\,c^\alpha,$$

where the symbol "$\doteq$" denotes the equality of words in a free group.

Since b is a positive word and is a pre-image of the word $a^\alpha xzc^\alpha$, it must have a form

$$b \doteq b_1(x\,y^k z)\,b_2,$$

where b_1, b_2 are positive words, $k \geq 0$, $b_1^\alpha \doteq a^\alpha$, $b_2^\alpha \doteq c^\alpha$. We use this form of b to rewrite the law (3) as

$$xy^{-1}z \equiv (a^{-1}b_1)xy^kz(b_2c^{-1}). \tag{5}$$

Since $b_1^\alpha \doteq a^\alpha$, the word $a^{-1}b_1$ becomes trivial under α replacing $y \to 1$, therefore it has the exponent sums of x and of z equal to zero, and similarly for b_2c^{-1}, which we denote as

$$\sigma_x(a^{-1}b_1) = 0, \quad \sigma_z(a^{-1}b_1) = 0, \quad \sigma_x(b_2c^{-1}) = 0, \quad \sigma_z(b_2c^{-1}) = 0. \quad (6)$$

If apply β to (5), we get the law

$$z \equiv (a^{-1}b_1)^\beta (xx^kz)(b_2c^{-1})^\beta$$

which must be trivial because otherwise it gives a nontrivial positive law. So we have the equality

$$z \doteq \underbrace{(a^{-1}b_1)^\beta xx^k}\cdot \underline{z} \cdot \underbrace{(b_2c^{-1})^\beta}.$$

Since $\sigma_z((a^{-1}b_1)^\beta) = \sigma_z(a^{-1}b_1) \overset{(6)}{=} 0$, the underlined z can not be cancelled from the left and similarly not from the right. Then the underbraced words must be trivial. In particular, $(a^{-1}b_1)^\beta xx^k \doteq 1$. It follows that $\sigma_x((a^{-1}b_1)^\beta) = -k-1$. Since for any word w, $\sigma_x(w^\beta) = \sigma_x(w) + \sigma_y(w)$, we have

$$-k-1 = \sigma_x((a^{-1}b_1)^\beta) = \sigma_x(a^{-1}b_1) + \sigma_y(a^{-1}b_1) \overset{(6)}{=} 0 + \sigma_y(a^{-1}b_1),$$

that is

$$\sigma_y(a^{-1}b_1) = -k-1 < 0.$$

Now we apply γ to (5), and the law we get must again be trivial

$$x \doteq (a^{-1}b_1)^\gamma \cdot \underline{x} \cdot z^kz\,(b_2c^{-1})^\gamma.$$

Since $\sigma_x(a^{-1}b_1)^\gamma = \sigma_x(a^{-1}b_1) \overset{(6)}{=} 0$, the underlined x can not be cancelled from the left and similarly not from the right. Hence $(a^{-1}b_1)^\gamma \doteq 1$. Then the exponent sum of z in $(a^{-1}b_1)^\gamma$ must be also zero and we have

$$0 = \sigma_z((a^{-1}b_1)^\gamma) = \sigma_z(a^{-1}b_1) + \sigma_y(a^{-1}b_1) = 0 - k - 1 < 0,$$

a contradiction. □

The following Example illustrates the above reasoning for the specific law of the form $x\,y^{-1}z \equiv a^{-1}b\,c^{-1}$. We show how it implies a positive law.

Example 5 *Let* $a = zy^3xy,\ b = zyxyxyzyx,\ c = xy$ *then the law* (3) *is*

$$x\,y^{-1}z \equiv (zy^3xy)^{-1}zyxyxyzyx\,(xy)^{-1}, \quad (7)$$

If apply α to (7) we get $xz \equiv (zx)^{-1}b^{\alpha}(x)^{-1}$, which is trivial because

$$b^{\alpha} \doteq (zyxyxyzyx)^{\alpha} \doteq (zx)xz(x).$$

By applying β to (7) we get the law which again is trivial

$$z \doteq \underbrace{(zx^5)^{-1}(zx^3)x^2}\cdot z \cdot \underbrace{(x^2)(x^2)^{-1}}.$$

By applying γ to (7) we get a nontrivial law

$$x \equiv (z^4xz)^{-1}z^2xzxz^2zx(xz)^{-1}.$$

which implies a positive law $(z^4xz)x(xz) \equiv z^2xzxz^3x$, or shorter

$$z^2xzx^2z \equiv xzxz^3x. \ \square$$

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