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Solution of the solidification problem by using the variational iteration method

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Abstract

The paper presents the approximated solution of the solidification problem, modelled with the aid of the one-phase Stefan problem with the boundary condition of the second kind, by using the variational iteration method. For solving this problem one needs to determine the distribution of temperature in the given domain and the position of the moving interface. The proposed method of solution consists of describing the considered problem with a system of differential equations in a domain with known boundary, and solving the received system with the aid of VIM method. The accuracy of the obtained approximated solution is verified by comparing it with the analytical solution.

Keywords: Solidification process, Application of information technology to the foundry industry, Stefan problem, Moving boundary problem, Variational iteration method

1. Introduction

The Stefan problem is a mathematical model, describing the process of solidification of pure metals. It represents a group of thermal processes, during which the phase changes and the heat is absorbed or emitted, such as, for example, solidification of metals, melting of ice, freezing of water and soil, deep freezing of foodstuffs and so on [1-3]. Solving the Stefan problem consists of finding the temperature distribution in the given domain and the position of the moving interface (the freezing front), under the assumption of knowing the initial conditions, boundary conditions and thermo-physical properties of the considered materials.

In this paper, we consider the special case of Stefan problem, which is a one-phase problem, where the temperature at one side of the moving interface is constant and equal to the temperature of the phase change. In nature, such situation happens during the melting of ice, if it is under the melting temperature. Solving of such kind of problems was discussed for example in paper [4], where the analytical solution were found for some simple cases, or in paper [5], where the approximated methods must be applied.

Grzymkowski and Slota in [6] have applied the Adomian decomposition method, combined with some minimization method, for finding the approximate solution of the one-phase Stefan problem, while in [7] the Stefan problem was first approximated with the system of nonlinear ordinary differential equations, and next, the received system was solved with the aid of the Adomian decomposition method. The Adomian method and the Runge-Kutty method, applied for solving the nonlinear system of the ordinary differential equations, obtained by transforming the Stefan problem, are compared in [8].

In papers [9,10], an approximate solution of one-phase direct and inverse Stefan problem with the boundary condition of the first kind, by using the variational iteration method, is presented. In this procedure, function describing the moving interface is assumed in the form of a linear combination of given functions. Coefficients of this combination must be designated by minimising the proper functional. Different approach for solving the Stefan problem by using the variational iteration method is

proposed in [11], where the considered problem is first transformed into the unit square, and next, the transformed task was solved with the aid of VIM method. In this procedure, like in the one presented in the present study, using any additional optimization method is not needed.

The variational iteration method was introduced by Ji-Huan He, and, till now, this method was applied for solving a wide range of problems [12-28]. For example, in [15] the VIM method is used for finding the solution of the Laplace equation, in [16-19] for solving the heat transfer equation and wave equation, in [20] for solving the Couchy reaction-diffusion problem, in [21] for determining the solution of the hyperbolic differential equations and systems of partial differential equations in [22]. The discussed method is also applied for solving the inverse problems for the parabolic equation [23-25] and for finding the approximate solution of the inverse Stefan problem [9-10]. Besides, there are available publications describing the applications of the considered method for solving mathematical models, appearing in biology [26] and astrophysics [27]. Convergence of the VIM method is discussed by Tatari and Dehghan in [28].

In the present paper, we propose a new approach for using the variational iteration method for finding the approximate solution of the one-phase Stefan problem, with the boundary condition of the second kind.

2. Formulation of the problem

As it was said in the previous section, we deal in this paper with solving the one-phase Stefan problem, described in the domain $D = \{(x,t) : t \in [0,t^*),\ x \in [0,\xi(t)]\}$. Let us denote the particular parts of the domain *D*:

$$\Gamma_0 = \{(x,0) : x \in [0,s],\ s = \xi(0)\}, \tag{1}$$

$$\Gamma_1 = \{(0,t) : t \in [0,t^*)\}, \tag{2}$$

$$\Gamma_g = \{(x,t) : t \in [0,t^*),\ x = \xi(t)\}, \tag{3}$$

where the initial and boundary conditions are given. Considered problem is described by the equations:

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{a}\frac{\partial u(x,t)}{\partial t}, \qquad \text{in } D, \tag{4}$$

$$u(x,0) = \varphi(x), \qquad \text{on } \Gamma_0, \tag{5}$$

$$\xi(0) = s, \tag{6}$$

$$\frac{\partial u(0,t)}{\partial x} = \vartheta(t), \qquad \text{on } \Gamma_1, \tag{7}$$

$$u(\xi(t),t) = u^*, \qquad \text{on } \Gamma_g, \tag{8}$$

$$\lambda\frac{\partial u(x,t)}{\partial x} = \kappa\frac{\partial \xi(t)}{\partial t}, \qquad \text{on } \Gamma_g, \tag{9}$$

where $s > 0$, a is the thermal diffusivity, λ is the thermal conductivity, κ is the latent heat and fusion per an unit volume, u^* is the temperature of the phase change, *u* is the temperature, and *t* and *x* denote the time and spatial location variables, respectively.

As solving the Stefan problem, formulated with equations (4)-(9), we understand to determine the temperature distribution *u(x,t)* and function $\xi(t)$, describing the moving interface position. The functions $\varphi(x)$, $\vartheta(t)$ and the coefficients a, λ, κ, u^* and s are given, in the considered problem.

3. Basic ideas of the variational iteration method

The variational iteration method is used for solving a wide range of nonlinear operator equations [12-14] of the form:

$$L(u(t)) + N(u(t)) = f(t), \tag{10}$$

where *L* is a linear operator, *N* is a nonlinear operator, *f(t)* is some given function and *u(t)* is an unknown function.

The VIM method consists of constructing a correctional functional, associated with the equation (10):

$$u_n(t) = u_{n-1}(t) + \int_0^t \lambda(s)(L(u_{n-1}(s)) + N(\tilde{u}_{n-1}(s)) - f(s))ds, \tag{11}$$

where $\tilde{u}_{n-1}$ is a restricted variation [12-14] (it means $\partial\tilde{u}_{n-1} = 0$), and $\lambda(s)$ is a general Lagrange multiplier [12,13,29], optimally determined by using the variational theory [12-14,30].

After determining the value of $\lambda(s)$, we obtain the following iterational formula:

$$u_n(t) = u_{n-1}(t) + \int_0^t \lambda(s)(L(u_{n-1}(s)) + N(u_{n-1}(s)) - f(s))ds, \tag{12}$$

from which the solution (approximate or exact) of equation (10) can be calculated. Of course, an initial approximation $u_0(t)$ has to be known.

4. Method of solution

In the first step, the considered problem, formulated for the curvilinear domain *D*, is transformed for the domain of rectangular geometry. We do that by using the following substitution:

$$\begin{cases} y = \dfrac{x}{\xi(t)}, \\ \tau = t. \end{cases} \tag{13}$$

The above substitution transforms the domain D into the domain $\overline{D}$ of the form: $\overline{D} = \{(y,\tau) : \tau \in [0,t^*),\ y \in [0,1]\}$, presented in the Figure 1, and the boundaries Γ_i for the boundaries $\overline{\Gamma}_i$, $i \in \{0,1,g\}$, where:

$$\overline{\Gamma}_0 = \{(y,0) : y \in [0,1]\},$$

$$\overline{\Gamma}_1 = \{(0,\tau) : \tau \in [0,t^*)\},$$

$$\overline{\Gamma}_g = \{(1,\tau) : \tau \in [0,t^*)\}.$$

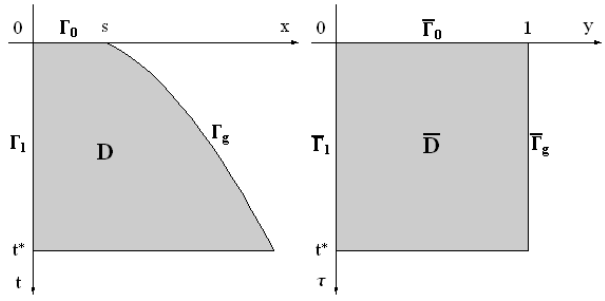

Fig. 1. Domains D and $\overline{D}$

The substitution of the variables causes, that the considered system of equations takes the form:

$$\frac{\partial u(y,\tau)}{\partial \tau} = \frac{y}{\xi(\tau)}\frac{\partial \xi(\tau)}{\partial \tau}\frac{\partial u(y,\tau)}{\partial y} + \frac{a}{(\xi(\tau))^2}\frac{\partial^2 u(y,\tau)}{\partial y^2}, \quad \text{in } \overline{D}, \tag{14}$$

$$\frac{\partial \xi(\tau)}{\partial \tau} = \frac{\lambda}{\kappa \xi(\tau)}\frac{\partial u(y,\tau)}{\partial y}, \quad \text{on } \overline{\Gamma}_g, \tag{15}$$

with the conditions:

$$\xi(0) = s, \tag{16}$$

$$u(y,0) = \varphi(s\ y), \quad \text{on } \overline{\Gamma}_0, \tag{17}$$

$$\frac{\partial u(0,\tau)}{\partial y} = \vartheta(\tau)\,\xi(\tau), \quad \text{on } \overline{\Gamma}_1, \tag{18}$$

$$u(1,\tau) = u^*, \quad \text{on } \overline{\Gamma}_g. \tag{19}$$

In the second step, to the above system of equations we apply the VIM method. The correction functionals for equations (14)-(15) are of the form:

$$u_{n+1}(y,\tau) = u_n(y,\tau) + \int_0^\tau \lambda_1(s)\left(\frac{\partial u_n(y,s)}{\partial s} - N_1(\widetilde{u}_n, \overline{\xi}_n; y, s)\right) ds,$$

$$\xi_{n+1}(\tau) = \xi_n(\tau) + \int_0^\tau \lambda_2(s)\left(\frac{\partial \xi_n(s)}{\partial s} - N_2(\widetilde{u}_n, \overline{\xi}_n; s)\right) ds,$$

where, for simplifying the notation, we introduce the functions:

$$N_1(u,\xi; y,\tau) := \frac{y}{\xi(\tau)}\frac{d\xi(\tau)}{d\tau}\frac{\partial u(y,\tau)}{\partial y} + \frac{a}{(\xi(\tau))^2}\frac{\partial^2 u(y,\tau)}{\partial y^2},$$

$$N_2(u,\xi;\tau) := \frac{\lambda}{\kappa \xi(\tau)}\frac{\partial u(y,\tau)}{\partial y}\bigg|_{y=1}.$$

For making the correction functionals stationary, we determine the general Lagrange multipliers as follows:

$$\lambda_1(s) = -1, \qquad \lambda_2(s) = -1. \tag{20}$$

In this way, we receive the following iteration formulas for functions $u(y,\tau)$ and $\xi(\tau)$, respectively:

$$u_{n+1}(y,\tau) = u_n(y,\tau) - \int_0^\tau \left(\frac{\partial u_n(y,s)}{\partial s} - N_1(u_n, \xi_n; y, s)\right) ds, \tag{21}$$

$$\xi_{n+1}(\tau) = \xi_n(\tau) + \int_0^\tau \left(\frac{\partial \xi_n(s)}{\partial s} - N_2(u_n, \xi_n; s)\right) ds. \tag{22}$$

In the third step of our procedure, we need do derive the initial approximations of the searched functions. For using the boundary condition (18) of the second kind, we dispose, we integrate the equation (14) for reducing the second derivative of the sought function $u(y,\tau)$. We obtain the equation:

$$\int_0^y \frac{\partial u(y,\tau)}{\partial \tau} dy = \frac{1}{\xi(\tau)}\frac{\partial \xi(\tau)}{\partial \tau}\left(y\,u(y,\tau) - \int_0^y u(y,\tau) dy\right) + \\ + \frac{a}{(\xi(\tau))^2}\frac{\partial u(y,\tau)}{\partial y} - \frac{a}{\xi(\tau)}\vartheta(\tau). \tag{23}$$

Let us assume, that as the initial approximation $\xi_0(\tau)$ we take the function describing the initial condition (16):

$$\xi_0(\tau) = s, \tag{24}$$

while the initial approximation $u_0(y,\tau)$ we introduce in the form:

$$u_0(y,\tau) = \overline{a}\ y + \overline{b}\ \tau + \overline{c}\ y\,\tau + \widetilde{d}, \tag{25}$$

where the coefficients $\overline{a}, \overline{b}, \overline{c}$ and function $\widetilde{d}$ will be determined form the equation (23). Functions (24) and (25) should satisfy equation (23), therefore let's substitute both of those functions into the equation (23):

$$\int_0^y \frac{\partial u_0(y,\tau)}{\partial \tau} dy = \frac{1}{\xi_0(\tau)} \frac{\partial \xi_0(\tau)}{\partial \tau} \left(y u_0(y,\tau) - \int_0^y u_0(y,\tau) dy \right) +$$

$$+ \frac{a}{(\xi_0(\tau))^2} \frac{\partial u_0(y,\tau)}{\partial y} - \frac{a}{\xi_0(\tau)} \vartheta(\tau).$$

After some basic calculations we get the relationship:

$$-\frac{a}{s^2}\overline{a} + \overline{b}\, y + \left(\frac{1}{2} y^2 - \frac{a}{s^2} \tau \right) \overline{c} + \frac{a}{s} \vartheta(\tau) = 0.$$

Assuming the exact accuracy of the received approximation at the particular points, such that $(y,\tau) = (0,0)$, $(y,\tau) = (1,0)$, $(y,\tau) = (0,t^*)$, and including the initial condition (17), we obtain the following form of the initial approximation:

$$\xi_0(\tau) = s,$$
$$u_0(y,\tau) = s \frac{\vartheta(t^*) - \vartheta(0)}{t^*} \tau \left(y - \frac{1}{2} \right) + \varphi(s\, y), \tag{26}$$

being the grounds for calculating the successive approximations $\xi_n(\tau)$ and $u_n(y,\tau)$ of functions $\xi(\tau)$ and $u(y,\tau)$. Those approximations are determined for the domain $\overline{D}$, but by applying the relation reciprocal to (13), we designate the searched approximate solution of the problem (4)-(9).

5. Example

The proposed method of solution, presented in previous section, will be illustrated with the theoretical example, for which the exact solution is known. Let us take the data:

$$a = 0.1,\ u^* = 1,\ s = 1,\ \lambda = 1,\ \kappa = -\frac{\lambda}{a},$$
$$\varphi(x) = \exp(1 - x),\ \vartheta(t) = -\exp(a\, t + 1).$$

Under those assumptions, the exact solutions of the problem (4)-(9) are the functions:

$$u(x,t) = \exp(a\, t - x + 1), \tag{27}$$

$$\xi(t) = a\, t + 1. \tag{28}$$

Figure 2 presents the comparison between the exact function $\xi(t)$, describing the position of the moving interface (solid line) and its second approximation $\xi_2(t)$ (dots).

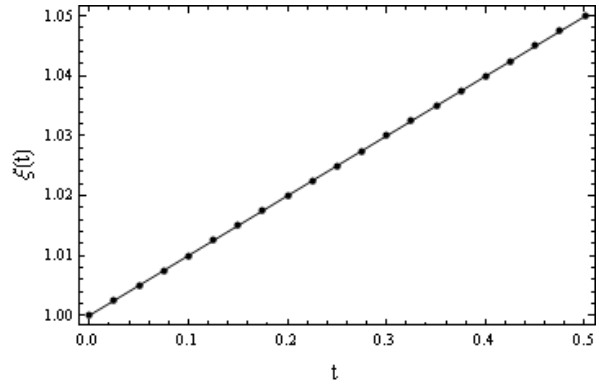

Fig. 2. Comparison between the position of the moving interface $\xi(t)$ (solid line) and its approximation $\xi_2(t)$ (dots)

The absolute error of this approximation, calculated in particular points from range $[0,t^*)$, is displayed in the Figure 3.

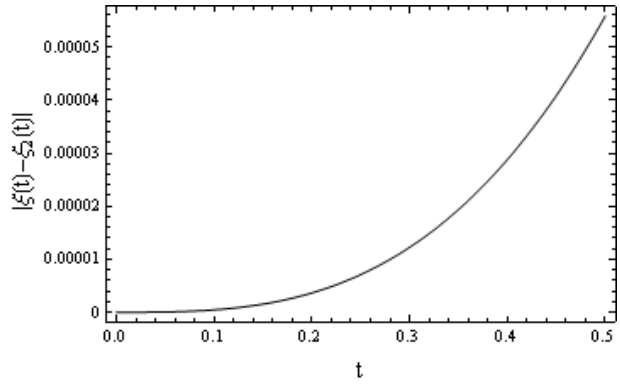

Fig. 3. Error distribution in the second approximation of the position of the moving interface

The absolute (δ) and relative percentage errors (Δ) of the successive approximations of functions $\xi(t)$ and $u(x,t)$, received with the presented procedure, are compiled in the Table 1. The absolute errors are calculated according the following formulas:

$$\delta_\xi = \left(\frac{1}{t^*} \int_0^{t^*} (\xi(t) - \xi_n(t))^2 dt \right)^{1/2}, \tag{29}$$

$$\delta_u = \left(\frac{1}{|D|} \iint_D (u(x,t) - u_n(x,t))^2 dx\, dt \right)^{1/2}, \tag{30}$$

where:

$$|D| = \iint_D 1\, dx\, dt,$$

while the relative percentage errors are derived by using the formulas:

$$\Delta_{\xi} = \delta_{\xi}\left(\frac{1}{t^*}\int_0^{t^*}(\xi(t))^2\,dt\right)^{-1/2} 100\%, \tag{31}$$

$$\Delta_{u} = \delta_{u}\left(\frac{1}{|D|}\iint_D (u(x,t))^2\,dx\,dt\right)^{-1/2} 100\%. \tag{32}$$

Except checking the quality of the received approximations, by comparing them with the exact solutions, it is also important to verify, whether the approximate solutions are compatible with the assumed initial (5)-(6) and boundary conditions (7)-(8).

Table 1.
Values of the errors in the reconstruction of the position of the moving interface and the temperature distribution.

n	δ_{ξ}	Δ_{ξ}	δ_{u}	Δ_{u}
0	0.0288675	0.0281606	0.0513521	0.0282556
1	0.0015582	0.0015200	0.0234418	0.0128984
2	0.0000212	0.0000207	0.0025637	0.0014106

The initial conditions (5)-(6) in each calculated iteration of the procedure are satisfied without errors.

In the case of the boundary condition (7) of the second kind, the comparison between the exact values of the function $g(t)$ $\frac{\partial u_2(0,t)}{\partial x}$ (solid line) and its second approximation (dots) is plotted in the Figure 4. The absolute error of this approximation is shown in Figure 5 and it does not exceed the value of 0.00454.

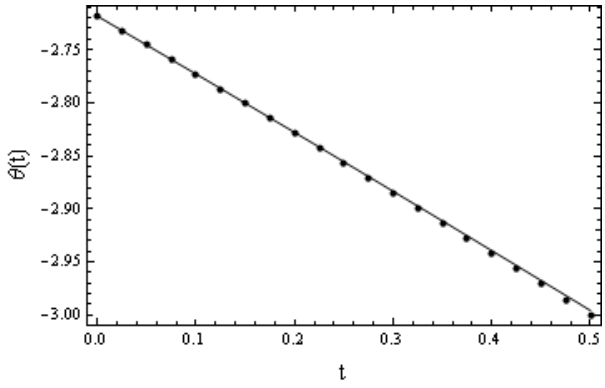

Fig. 4. Comparison between the exact values of the boundary condition Γ_1 (solid line) and its second approximation (dots)

The temperature of solidification u^* at the moving interface (condition (8)) is reconstructed with the error, systematically decreasing for the successive iterations. The compilation of the exact value of u^* (solid line) and its approximate value, obtained from the second iteration (dots), is presented in the Figure 6. However, figure 7 shows the absolute error of this approximation, which is not bigger than 0.00474.

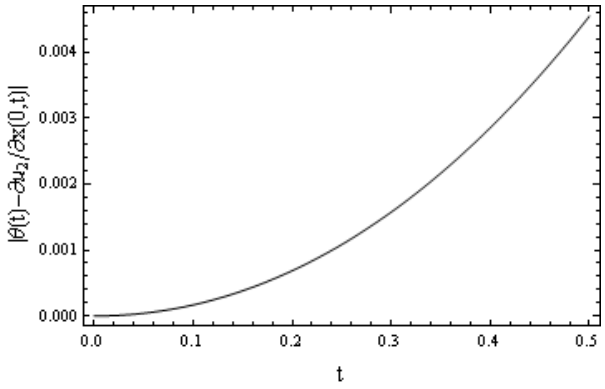

Fig. 5. Error distribution in the second approximation of the boundary condition on boundary Γ_1

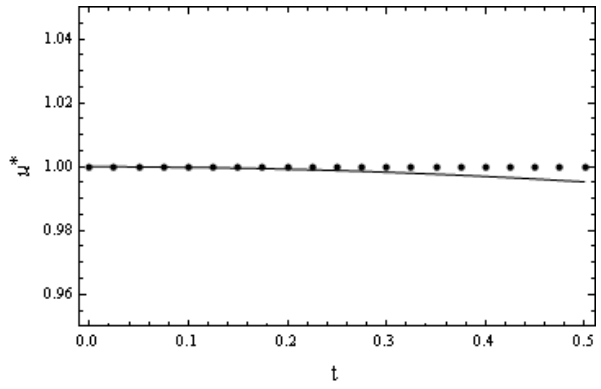

Fig. 6. Comparison between the phase change temperature u^* (solid line) and its second approximation (dots)

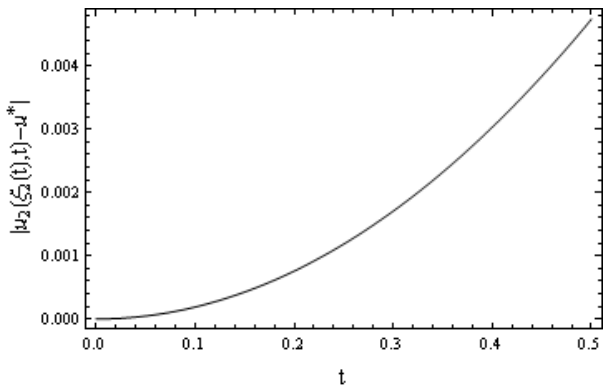

Fig. 7. Error distribution in the second approximation of the phase change temperature

6. Conclusions

This paper presents the application of the variational iteration method for finding the approximate solution for the one-phase one-dimensional Stefan problem, with the boundary condition of the second kind. The proposed approach consist of describing the

considered problem with the aid of the system of two differential equations in the domain of the known boundary, and solving the received system by using the variational iteration method. In constructing the initial approximation, the assumed initial and boundary conditions of the problem are included.

The advantage of such approach is, that it does not need the discretization of the considered domain, which is required in the classical methods, based for example on the finite-difference method or finite-element method. As a result of the proposed method we receive the sequence of the successive approximations, convergent to the exact solution, if it exists. Another advantage of the current approach is, that using any additional optimization algorithms is not required, like it was in the previous procedures, described in [9].

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