ON POSITIVE LAW PROBLEMS IN THE CLASS OF LOCALLY GRADED GROUPS

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Abstract

We consider five known problems, show that three of them are equivalent, and all of them have an affirmative answer in the class of locally graded groups. Outside of this class a counterexample is known to only one problem.

1 Preliminaries

A group G is called *locally graded* if every nontrivial finitely generated subgroup in G has a proper subgroup of finite index. Locally graded groups were introduced to study groups, that do not have finitely generated infinite simple sections, and to avoid the groups which need methods of Adian and Olshanskii. The class of locally graded groups contains for instance all residually finite groups, all locally soluble groups, all infinite locally finite simple groups which are neither locally soluble nor residually finite.

We denote by $\mathfrak{B}_e$ the restricted Burnside variety which consists of all locally finite groups of exponent dividing e (for details see [1]). By $\mathfrak{N}_c$ we denote the nilpotent variety of nilpotency class c.

A relation $u(x, y) = v(x, y)$ is called *positive* if u, v are written without inverses of variables and is *of degree less than or equal to n*, if the length of words u, v is not greater than n ($|u|, |v| \leq n$). If G does not contain a free non-abelian semigroup, we say that G is a $\mathcal{F}$-group. If every pair of elements in G satisfies some positive relation of degree less than or equal to n, then we say that G is a $|\mathcal{F}|$-*group of bound n*. If every pair of elements in G satisfies the same positive relation we say, that G *satisfies a positive law*. We note that each positive law implies a binary positive law and if G satisfies a positive law, then the variety, it generates, has a basis of positive laws [2].

2 Problems

In 1953 Mal'cev [3] proved that nilpotent groups satisfy positive laws. For more then 40 years the following problem was open:

Problem 1: Must every group satisfying positive laws be a periodic extension of a nilpotent group?

An affirmative answer was found for linear, soluble, residually finite groups, however in 1996 Ol'shanskii and Storozhew proved the existence of groups satisfying positive laws, which are not periodic extensions of locally soluble groups [4]. This gives a negative answer to Problem 1 in general, however we show in Theorem 2, that in the class of locally graded groups the answer is affirmative.

A group G is called n-*Engel* if it satisfies the commutator law $[x, y, y, ..., y] = 1$ where y is repeated n times. The following question was posed by Shirshov in 1963 ([6], 2.82):

Problem 2: Does every n-Engel group satisfy a positive law?

The problem is still open in general for $n > 4$, however by combining known results we can see that it has an affirmative answer in the class of locally graded groups.

Theorem 1 *Every locally graded n-Engel group satisfies a positive law.*

Proof It is shown by Kim and Rhemtulla [5], that every locally graded n-Engel group G is locally nilpotent. Then by a result of Burns and Medvedev [10], G is contained in a variety $\mathfrak{N}_c\mathfrak{B}_e \cap \mathfrak{B}_e\mathfrak{N}_c$, where c and e depend on n only. By [3] the nilpotent variety $\mathfrak{N}_c$ satisfies some binary positive law, say $P_c(x, y) = Q_c(x, y)$. Then the variety $\mathfrak{N}_c\mathfrak{B}_e$ satisfies the law $P_c(x^e, y^e) = Q(x^e, y^e)$, which finishes the proof. $\square$

We show that the following open problems are equivalent and have an affirmative answer in the class of locally graded groups.

A group G is called n-*collapsing* if for every n-element subset $S \subseteq G$, the inequality $|S^n| < n^n$ holds. A group is called collapsing if it is n-collapsing for some n. In his paper [7], A. Shalev posed the following question:

Problem 3: Does every collapsing group satisfy a positive law?

An affirmative answer was given for soluble and residually finite groups.

Let G^ω denote a cartesian power of a group G, and G^* denote an ultrapower of G modulo a fixed nonprincipal ultrafilter over natural numbers. The ultrapower G^* is an image of the cartesian power G^ω under the congruence defined by the nonprincipal ultrafilter. We note the following: *If G^ω has*

no free non-abelian subsemigroup then G satisfies a positive law, because if G does not satisfy any positive law, we enumerate binary relations and for i-th relation find a pair $a_i, b_i \in G$ which does not satisfy this relation. Hence elements $a = (a_1, a_2, ...), b = (b_1, b_2, ...) \in G^\omega$ must generate a free semigroup. A question arises, whether the same is true for G^* instead of G^ω:

Problem 4: Let G^* have no free non-abelian subsemigroup. Must G satisfy a positive law?

A group G satisfies a *finite disjunction of positive relations* if every pair of elements in G satisfies some positive relation in a finite set of relations $\{u_i(x, y) = v_i(x, y), \ i = 1, 2, ..., m\}$. In [8] M. Boffa posed the question, whether a group satisfying a finite disjunction of positive relations, must satisfy a positive law. Since for each n there exists only finitely many binary relations of degree less than or equal to n, we conclude that G satisfies a finite disjunction of positive relations if and only if G is a $|\mathcal{F}|$-group. So the Boffa question can be formulated as:

Problem 5: Does every $|\mathcal{F}|$-group satisfy a positive law?

To prove the equivalence of Problems 3-5 we need

Lemma 1 *The following classes of groups coincide:*
(i) The class of $|\mathcal{F}|$-groups,
(ii) The class of collapsing groups,
(iii) The class of groups, the nonprincipal ultrapowers of which are $\mathcal{F}$-groups.

Proof $(i) \to (ii)$ We show that a $|\mathcal{F}|$-group of bound n is $2n$-collapsing, that is for any set $S = \{s_1, s_2, ..., s_{2n}\}$ in G the inequality $|S^{2n}| < (2n)^{2n}$ holds. To prove the inequality, it suffices to find two equal words $s_{i_1} \cdots s_{i_{2n}}$ and $s_{j_1} \cdots s_{j_{2n}}$ in S^{2n}. By assumption elements s_1, s_2 satisfy some relation $u(x, y) = v(x, y)$ where $|u|, |v| \leq n$, $|u|, |v|$ need not be equal. The relation $u = v$ implies $uv = vu$, where $|uv| = |vu|$. If $|uv| = 2n$, we get two required equal words $u(s_1, s_2)v(s_1, s_2)$ and $v(s_1, s_2)u(s_1, s_2)$ in S^{2n}. If $|uv| < 2n$, ($|uv| = 2n - k$, say), then we use the relation $x^k uv = x^k vu$, which gives equal words $s_1^k u(s_1, s_2)v(s_1, s_2) = s_1^k v(s_1, s_2)u(s_1, s_2)$, and hence $|S^{2n}| < (2n)^{2n}$.

$(ii) \to (i)$ Let G be an n-collapsing group, then for each subset $S = \{s_1, ..., s_n\}$ in G, it holds $|S^n| < n^n$. So there exist two equal words $s_{i_1} \cdots s_{i_n} = s_{j_1} \cdots s_{j_n}$. To find a relation for any fixed $a, b \in G$, we take $S = \{a, ab, ab^2, ..., ab^{n-1}\}$. The relation we obtain for a, b is of degree less or equal to n^2. So G is a $|\mathcal{F}|$-group of bound n^2.

$(i) \to (iii)$ Let G be a $|\mathcal{F}|$-group of bound n. We assume that elements α^*, β^* generate a free 2-generator subsemigroup in an ultraproduct G^* and show that it gives a contradiction. Let $\alpha = (a_1, a_2, ...)$, $\beta = (b_1, b_2, ...)$ be some pre-images of α^*, β^* in G^ω, respectively. There is only finitely many,

say M, binary relations of degree less or equal to n. We enumerate them by naturals $1, 2, ..., M$. Let $A_k \in \mathbb{N}$ denote a subset of indices such that for $i \in A_k$ the pair a_i, b_i satisfies the k-th relation. Obviously $\mathbb{N} = A_1 \cup A_2 \cup ... \cup A_M$. If the semigroup generated by α^*, β^* is free, then no A_k is in the ultrafilter. Thus by the definition, all A'_k are in the ultrafilter, so does their intersection. Hence $\varnothing = A'_1 \cap A'_2 \cap ... \cap A'_M$ is in the ultrafilter, which is the required contradiction.

$(iii) \leftrightarrow (i)$ Let G^* be a $\mathfrak{X}$-group. If assume that G is not $|\mathfrak{X}|$-group, then for every natural k there exists a pair a_k, b_k of elements in G which does not satisfy any relation of degree $\leq k$. We use these pairs to define $\alpha = (a_1, a_2, ...), \beta = (b_1, b_2, ...)$ in G^ω. Let α^*, β^* be images of α, β in the group G^*, and let $u = v$ be any fixed binary relation. By construction, the subset I of indices i for which a_i, b_i satisfy the relation $u = v$ is finite, and hence I is not in the ultrafilter. So $u(\alpha, \beta)$ and $v(\alpha, \beta)$ are not equal modulo the ultrafilter and then $u(\alpha^*, \beta^*) \neq v(\alpha^*, \beta^*)$. Hence α^*, β^* do not satisfy any relation and must generate a free subsemigroup in G^*, which is a contradiction. $\square$

Corollary 1 *It follows from Lemma 1, that Problems 3-5 are equivalent.*

We show now that in the class of locally graded groups Problem 1 has an affirmative answer.

Theorem 2 *Every locally graded group satisfying positive laws is a periodic extension of a nilpotent group. In particular, every locally graded group G satisfying a positive law of degree n lies in the variety $\mathfrak{N}_c \mathfrak{B}_e$, where c and e depend on n only.*

Proof Let G be a locally graded group. If G satisfies a positive law of degree n, then G is a $|\mathfrak{X}|$-group of bound n, and hence, by Lemma 1, G is collapsing. It follows now from results of Kim and Rhemtulla ([5] Theorem A and Lemma 3), that G must be locally - (polycyclic-by-finite). Since G is soluble-by-finite and satisfies a positive law of degree n, then by ([9], Theorem B), G must be periodic extension of a nilpotent group. In particular $G \in \mathfrak{N}_c \mathfrak{B}_e$, where c and e depend on n only, which finishes the proof. $\square$

To show that Problems 3-5 have affirmative answers in the class of locally graded groups it suffices, in view of Corollary 1, to prove:

Theorem 3 *Every locally graded $|\mathfrak{X}|$-group G of bound n satisfies a positive law. Moreover G lies in the variety $\mathfrak{N}_c \mathfrak{B}_e$, where c and e depend on n only.*

Proof Let G be a locally graded $|\mathfrak{X}|$-group of bound n, then by Lemma 1, G is $2n$-collapsing. Let H be a two-generator subgroup in G. Since H also is locally graded and collapsing, then by ([5] Theorem A and Lemma 3), H contains a polycyclic normal subgroup N of finite index. Then by the result of Rosenblatt ([11], 4.12), the finitely generated soluble $\mathfrak{X}$-group N must be

nilpotent-by-finite. Hence H is also nilpotent-by-finite, and then by [12], H is residually finite.

Since H is $2n$-collapsing and residually finite, then by the result of Shalev ([7], Theorem A′), H belongs to the product-variety $\mathfrak{N}_c\mathfrak{B}_e$, where c, e depend on $2n$ only. By [3], every group in this variety satisfies some positive law $P_c(x^e, y^e) = Q_c(x^e, y^e)$ whose degree depends on c, e only, and hence depends on n only. Since H is an arbitrary two-generator subgroup, then the whole group G must satisfy the positive law $P_c(x^e, y^e) = Q_c(x^e, y^e)$. Now by Theorem 2 the statement follows. □

From above Theorems we have

Corollary 2 *Let G be a locally graded group with one of the properties: G is an n-Engel group; G is a collapsing group; a nonprincipal ultrapower of G does not contain a free non-abelian semigroup; G is a $|\mathfrak{F}|$-group . Then G is nilpotent-by-(locally finite of finite exponent).*

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