ON VARIETIES OF GROUPS WITHOUT POSITIVE LAWS

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Abstract

We give negative answers to three questions concerning positive laws and $\mathfrak{A}_p\mathfrak{A}$ varieties.

Let $\mathfrak{A}$ denote the variety of all abelian groups and $\mathfrak{A}_p$ — the variety of all abelian groups of exponent p. By F we denote a free group and by V — a verbal subgroup in F.

We write $G \supseteq \mathcal{F}$ to say that G contains a free nonabelian subsemigroup. A variety generated by G is denoted by $var(G)$.

A law $u(x_1, ..., x_n) = u'(x_1, ..., x_n)$ is called positive if u, u' are positive words, i.e. words written without inverses of variables. Each positive law implies a binary positive law. If a group G satisfies a positive law, then $var(G)$ has a basis of positive laws [3]. By Zorn Lemma there exist minimal varieties without positive laws, so called *just not p.l.* varieties. It follows from [1], that varieties $\mathfrak{A}_p\mathfrak{A}$ for prime p are *just not p.l.* varieties.

For each finitely generated relatively free group G the following questions either have a positive answer or none.

Question 1. Let G contain a free nonabelian subsemigroup. Does $var(G)$ contain $\mathfrak{A}_p\mathfrak{A}$ for some p?

Question 2. Let G contain a free nonabelian subsemigroup. Does G/G'' also contain a free nonabelian subsemigroup?

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Lemma. *For a finitely generated relatively free group G the above Questions are equivalent.*

Proof. We get the required equivalence by proving the following implications:

$$\begin{array}{ccc} G \supseteq \mathcal{F} & \stackrel{1}{\longleftarrow} & var(G) \supseteq \mathfrak{A}_p\mathfrak{A} \\ {}_2\uparrow & & \updownarrow \\ G/G'' \supseteq \mathcal{F} & \longleftrightarrow & var(G/G'') \supseteq \mathfrak{A}_p\mathfrak{A} \end{array}$$

In implication 1 we have that a verbal subgroup $V \subseteq F$, corresponding to $var(G)$ satisfies $V \subseteq F''(F')^p$ for some p. By result of Mal'cev [4], $F/F''(F')^p$ contains a free nonabelian subsemigroup and hence the relatively free group G also contains a free nonabelian subsemigroup, which proves the implication. Implication 2 is clear.

Consider now the horisontal equivalence. By result of Rosenblatt [6] for finitely generated soluble groups: G/G'' either contains a free nonabelian subsemigroup or is nilpotent-by-finite. By result of Groves [1] for finitely generated soluble groups: either $var(G/G'') = \mathfrak{A}_p\mathfrak{A}$ for some p, or G/G'' is nilpotent-by-finite. These imply the horisontal equivalence.

The vertical equivalence follows, because for a verbal subgroup V, the inclusion $V \subseteq F''(F')^p$ is equivalent to $F''V \subseteq F''(F')^p$.

Since our Questions concern the implications inverse to 1, 2, their equivalence follows. □

Corollary. *A negative answer to the above Questions implies the existence of a just not p.l. variety different from $\mathfrak{A}_p\mathfrak{A}$ for any p. This solves the problem posed in* ([7] 19.2): *whether $\mathfrak{A}_p\mathfrak{A}$ are the only just not p.l. varieties?*

Proof. Since in a relatively free group G any relation on free generators is a law, we have that G contains a free nonabelian subsemigroup if and only if G does not satisfy a positive law. So the negative answer to Question 1 means that there exists G such that $var(G)$ does not satisfy positive laws and does not contain $\mathfrak{A}_p\mathfrak{A}$ for any p. This implies that $var(G)$ contains a *just not p.l.* variety different from any of $\mathfrak{A}_p\mathfrak{A}$. □

To show that the questions have negative answer, we consider the relatively free group G with two or more free generators, defining a pseudo-abelian variety (without non-abelian metabelian subgroups) which is studied in Chapter 9 of [5]. Let

$$w(x,y) = [x,y]v(x,y)^n[x,y]^{e_1}v(x,y)^{n+1}\dots[x,y]^{e_{h-1}}v(x,y)^{n+h-1},$$

where $v(x,y) = [[x^d, y^d]^d, [y^d, x^{-d}]^d]$, $h \equiv 1 \pmod{10}$, $e_{10k+1} = e_{10k+2} = e_{10k+3} = e_{10k+5} = e_{10k+6} = 1$, $e_{10k+4} = e_{10k+7} = e_{10k+8} = e_{10k+9} = e_{10k+10} = -1$, $k = 0, 1, \dots, (h-1)/10$ and d, n, h are sufficiently large natural numbers chosen with respect to the restrictions that are introduced in Chapter 7 of [5]. Note that $e_1 + \dots + e_{h-1} = 0$. G is defined by the law $w(x,y) = 1$.

Theorem. *The group G defined above contains a free non-abelian subsemigroup, while $var(G)$ does not contain $\mathfrak{A}_p\mathfrak{A}$ for any p.*

Proof Assume that $var(G)$ contains $\mathfrak{A}_p\mathfrak{A}$ for some p, then $G \cong F/V$ and $V \subseteq F''(F')^p$. This implies $G/G''(G')^p \cong F/F''(F')^p$, which is not true, because every metabelian group in $var(G)$ is abelian.

To show that G contains a free non-abelian subsemigroup we use the technique of graded diagrams developed in [5]. All references below concern this book. Let a, b be two distinct free generators in G. We are going to prove that the subsemigroup generated by a and b is a free semigroup. Suppose to the contrary that an equality $u(a,b) = u'(a,b)$ holds in G, for some distinct positive words u, u'. Without loss of generality we can assume that u and u' have distinct leading and ending letters; thus we can assume that the word $u^{-1}u'$ is a cyclically reduced non-empty word (one of the words u, u' could be an empty word).

Now for the group G, defined by the word w we consider a reduced diagram Δ of the equality $u^{-1}u' = 1$. The contour of Δ is presented in the form pq, where the section p has the label $\varphi(p) = u^{-1}$ and the section q has the label $\varphi(q) = u'$. Since the equality $u^{-1}u' = 1$ does not hold in a free group, the rank of Δ is greater than 0 (see § 11).

By Theorem 22.2, in Δ there is an $\mathcal{R}$-cell Π and a subdiagram Γ of rank 0 satisfying the following conditions: 1) Γ is a contiguity subdiagram of a long section t of the contour of Π to one of the sections p or q; 2) Γ-contiguity degree of t to p (or q) is not less than ε (see Chapter 7, §20). This implies (by reasoning similar to that in Theorem 19.1) that one of the labels $\varphi(p)$, $\varphi(q)$ has a common subword of length $|A|^{[\varepsilon n]}$ with the section t, where A is a period corresponding to the cell Π. In particular, one of the words $\varphi(p)$, $\varphi(q)$ contains either the word A or the word A^{-1} as a subword. Since $\varphi(p)$ is a negative (i.e. containing only negative powers of a and b) word and $\varphi(q)$ is a positive word, we see that the word A is either positive or negative. From the definition of periods (see § 29.3) it follows that A^f for some integer f ($f \neq 0$ by Lemma 30.3) is conjugate in G to a word $v(X,Y)$ for some words X, Y and the word v defined above. The word v is a commutator word (i.e. $v(X,Y)$ belongs to the commutator subgroup of a free group with free generators X, Y), hence $A^f \in G'$. The variety $var(G)$ is defined by the law $w(x, y) = 1$, where w is a commutator word; hence, $\mathfrak{A} \subseteq var(G)$. Therefore A^f is a commutator word (i.e. A^f belongs to the commutator subgroup F' of the free group $F = F(a, b)$ freely generated by a and b). Since the factor-group F/F' has no torsion and $f \neq 0$, the word A belongs to F'. This means that the number of occurrences of the letter a in the word A is equal to the number of occurrences of the letter a^{-1}, and the same is true for the letters b and b^{-1}. Hence the word A is neither positive nor negative. We get the contradiction which ends the proof. □

Corollary. *There exists continuously many just not p.l. varieties.*

Proof. It follows from the results obtained in [2], that there exists a continuous set of infinitely based pseudo-abelian varieties with pairwise intersections equal to $\mathfrak{A}$. Since the Theorem holds if we replace $var(G)$ by any of these varieties, the statement follows. □

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