ARCHIVES of FOUNDRY ENGINEERING

ISSN (1897-3310)
Volume 8
Special Issue 1/2008
187–192

38/1

Published quarterly as the organ of the Foundry Commission of the Polish Academy of Sciences

Explicit and implicit approach of sensitivity analysis in numerical modelling of solidification

E. Majchrzak [a, b]*, G. Kałuża [b]

[a] Department for Strength of Materials and Computational Mechanics
Silesian University of Technology, Konarskiego 18a, 44-100 Gliwice, Poland
[b] Institute of Mathematics and Computer Science
Czestochowa University of Technology, Dabrowskiego 73, 42-200 Czestochowa, Poland
* Corresponding author.E-mail address: ewa.majchrzak@polsl.pl

Received 15.02.2008; accepted in revised form 22.03.2008

Abstract

The explicit and implicit approaches of sensitivity analysis using the boundary element method are presented. In particular, the problem of casting solidification is considered. A perturbation of an input parameter (for example the thermal conductivity of casting material) causes the changes of transient temperature field in the domain analyzed. The methods of sensitivity analysis allows to determine in mathematical way the mutual connections between parameters perturbations and final results. In the paper some significant aspects of computational algorithms associated with explicit and implicit approaches of sensitivity analysis are demonstrated.

Keywords: Application of Information Technology to the Foundry Industry; Solidification Process; Numerical Techniques; Sensitivity Analysis; Boundary Element Method

1. Introduction

The thermal processes proceeding in the casting domain are described by the energy equation (Fourier-Kirchhoff type equation) and boundary initial conditions resulting from the technology considered [1, 2, 3]. The transient temperature field in the casting domain is dependent on the set of thermophysical parameters of material, coefficients appearing in the boundary conditions and initial (pouring) temperature. The perturbations of above selected input data cause the change of the course of solidification process. To analyze the connections between the parameters perturbations and results of numerical simulations the sensitivity methods are applied [2, 4, 5, 6, 7, 8].

There are two basic approaches to sensitivity analysis using boundary element formulation: the continuous approach and the discretized one [9]. In the continuous approach (explicit differentiation method) the analytical expressions for sensitivities are derived and then they are calculated numerically using BEM. They have the form of boundary integrals with integrands that depend only on the variables of the primary and additional problems. The implicit differentiation method, which belongs to the discretized approach, bases on the differentiation of the algebraic boundary element matrix equations. The derivatives of the boundary element system matrices can be calculated either analytically or semi-analytically. In the paper some significant aspects of formulations and computational algorithms associated with both methods are demonstrated and the advantages and disadvantages of both techniques are discussed.

2. Governing equations

A casting-mould-environment system is considered. A transient temperature field in a casting sub-domain is described by the following equation

$$x \in \Omega: \quad c(T)\frac{\partial T}{\partial t} = \nabla\left[\lambda(T)\nabla T\right] + Q \tag{1}$$

where $\lambda(T)$ is the thermal conductivity, $c(T)$ is the volumetric specific heat, $Q = Q(x, t)$ is the source function, $T = T(x, t)$, x, t denote temperature, spatial co-ordinates and time, respectively.

As is well known, the source term $Q(x, t)$ is proportional to the local solidification rate [1, 2, 3], this means

$$Q = L_V \frac{\partial f_S}{\partial t} \tag{2}$$

where f_S is the solid state fraction at the neighborhood of the point considered from casting domain, L_V is the volumetric latent heat.

If one assumes that f_S is the known function of temperature (the scope of f_S is from 0 to 1, of course) then

$$\frac{\partial f_S}{\partial t} = \frac{\mathrm{d}f_S(T)}{\mathrm{d}T}\frac{\partial T}{\partial t} \tag{3}$$

and

$$x \in \Omega: \quad C(T)\frac{\partial T}{\partial t} = \nabla\left[\lambda(T)\nabla T\right] \tag{4}$$

where the parameter

$$C(T) = c(T) - L_V\frac{\mathrm{d}f_S}{\mathrm{d}T} \tag{5}$$

is called a substitute thermal capacity of mushy zone sub-domain [1, 2]. In the case of binary alloys the mushy zone sub-domain corresponds to the temperature interval $[T_S, T_L]$, where T_S, T_L are the border temperatures determining the end and the beginning of the solidification process.

In literature the several hypotheses concerning the function describing a substitute thermal capacity of the mushy zone are discussed [1, 2]. In this paper the substitute thermal capacity for cast steel is defined as follows

$$C(T) = \begin{cases} c_L, & T > T_L \\ \dfrac{c_L + c_S}{2} + \dfrac{L_V}{T_L - T_S}, & T_S \le T \le T_L \\ c_S, & T < T_S \end{cases} \tag{6}$$

where c_L, c_S are the constant volumetric specific heats of liquid and solid state.

Because the solidification process proceeds in a rather small interval of temperature one can assume the constant value of thermal conductivity of cast steel and then the equation (4) takes a form

$$x \in \Omega: \quad C(T)\frac{\partial T}{\partial t} = \lambda\nabla^2 T \tag{7}$$

On a casting surface Γ the Robin condition is given

$$x \in \Gamma: \quad -\lambda\mathbf{n}\cdot\nabla T = \alpha\left(T - T_a\right) \tag{8}$$

where α is a substitute heat transfer coefficient (influence of mould), T_a is a conventionally assumed ambient temperature.

For the moment $t = 0$ the initial temperature distribution (pouring temperature) is known, namely

$$t = 0: \quad T = T_p \tag{9}$$

3. Boundary element method

To solve the equation (7) the boundary element method has been used. If the thermal diffusivity is constant, this means $a(T) = \lambda/C(T) = a = \text{const}$, then for the partial differential equation (7) a fundamental solution is available [10, 11, 12]

$$T^*\left(\xi, x, t^F, t\right) = \frac{1}{\left[4\pi a\left(t^F - t\right)\right]^{m/2}}\exp\left[-\frac{r^2}{4a\left(t^F - t\right)}\right] \tag{10}$$

where m is the problem dimension ($m = 1, 2, 3$ corresponds to 1D, 2D, 3D problem, respectively), $[0, t^F]$ is the time interval under consideration, ξ is the observation point, r is the distance between the points ξ and x.

In the case of non-constant thermal diffusivity the use of the fundamental solution (10) should be accompanied by a time marching technique in which $a(T)$ is assumed constant at the beginning of each time step [13]. So, the time grid is introduced

$$0 = t^0 < t^1 < \ldots < t^{f-1} < t^f < \ldots < t^F \tag{11}$$

with constant time step $\Delta t = t^f - t^{f-1}$.

Starting from the initial time t^0 over each time step $[t^{f-1}, t^f]$, the value of a is taken as the mean average, namely

$$a^f = \frac{\lambda}{C^f} = \frac{\lambda}{\int\limits_\Omega C\left[T\left(x, t^{f-1}\right)\right]\mathrm{d}\Omega} \tag{12}$$

Basing on the approximation (12), the equation (7) can be transformed into the following integral equation for each time step $[t^{f-1}, t^f]$

$$\begin{gathered} B(\xi)T\left(\xi, t^f\right) + \frac{1}{C^f}\int\limits_{t^{f-1}}^{t^f}\int\limits_\Gamma T^*\left(\xi, x, t^f, t\right)q\left(x, t\right)\mathrm{d}\Gamma\,\mathrm{d}t = \\ \frac{1}{C^f}\int\limits_{t^{f-1}}^{t^f}\int\limits_\Gamma q^*\left(\xi, x, t^f, t\right)T\left(x, t\right)\mathrm{d}\Gamma\,\mathrm{d}t + \\ \int\limits_\Omega T^*\left(\xi, x, t^f, t^{f-1}\right)T\left(x, t^{f-1}\right)\mathrm{d}\Omega \end{gathered} \tag{13}$$

where for $\xi \in \Omega$: $B(\xi) = 1$ and for $\xi \in \Gamma$: $B(\xi) \in (0, 1)$, $q(x, t) = -\lambda\mathbf{n}\cdot\nabla T(x, t)$, $q^*(\xi, x, t^f, t) = -\lambda\mathbf{n}\cdot\nabla F^*(\xi, x, t^f, t)$.

Fundamental solution $T^*(\xi, x, t^f, t)$ has the following form

$$T^* = \frac{1}{\left[4\pi a^f\left(t^f - t\right)\right]^{m/2}}\exp\left[-\frac{r^2}{4a^f\left(t^f - t\right)}\right] \tag{14}$$

The function $q^*(\xi, x, t^f, t)$ can be calculated analytically [11]

$$q^* = \frac{\lambda d}{2(4\pi)^{m/2}\left[a^f\left(t^f - t\right)\right]^{(m+1)/2}} \exp\left[-\frac{r^2}{4a^f\left(t^f - t\right)}\right] \tag{15}$$

where

$$d = \sum_{k=1}^{m}(x_k - \xi_k)\cos\alpha_k \tag{16}$$

and $\cos\alpha_k$ are directional cosines of the normal vector $\mathbf{n}$.

For constant elements with respect to time [10, 11], the equation (13) can be written in the form

$$\begin{gathered} B(\xi)T\left(\xi,t^f\right) + \int_\Gamma q\left(x,t^f\right)g\left(\xi,x\right)\mathrm{d}\Gamma = \\ \int_\Gamma T\left(x,t^f\right)h\left(\xi,x\right)\mathrm{d}\Gamma + \int_\Omega T^*\left(\xi,x,t^f,t^{f-1}\right)T\left(x,t^{-1}\right)\mathrm{d}\Omega \end{gathered} \tag{17}$$

where

$$h\left(\xi,x\right) = \frac{1}{C^f}\int_{t^{f-1}}^{t^f} q^*\left(\xi,x,t^f,t\right)\mathrm{d}t \tag{18}$$

and

$$g\left(\xi,x\right) = \frac{1}{C^f}\int_{t^{f-1}}^{t^f} T^*\left(\xi,x,t^f,t\right)\mathrm{d}t \tag{19}$$

Functions $h(\xi, x)$, $g(\xi, x)$ are determined in analytical way [11] and then

$$h\left(\xi,x\right) = \begin{cases} \dfrac{\mathrm{sgn}(x-\xi)}{2}\mathrm{erfc}\left(\dfrac{|x-\xi|}{2\sqrt{a^f\Delta t}}\right), & \text{1D problem} \\ \dfrac{d}{2\pi r^2}\exp\left(-\dfrac{r^2}{4a^f\Delta t}\right), & \text{2D problem} \\ \dfrac{d}{4\pi^{3/2}r^3}\left[\sqrt{\pi}\,\mathrm{erfc}\left(\dfrac{r}{2\sqrt{a^f\Delta t}}\right) + \right. \\ \left.\dfrac{r}{\sqrt{a^f\Delta t}}\exp\left(-\dfrac{r^2}{4a^f\Delta t}\right)\right], & \text{3D problem} \end{cases} \tag{20}$$

and

$$g\left(\xi,x\right) = \begin{cases} \sqrt{\dfrac{\Delta t}{\pi\lambda C^f}}\exp\left(-\dfrac{(x-\xi)^2}{4a^f\Delta t}\right) - \\ \dfrac{|x-\xi|}{2\lambda}\mathrm{erfc}\left(\dfrac{|x-\xi|}{2\sqrt{a^f\Delta t}}\right), & \text{1D problem} \\ \dfrac{1}{4\pi\lambda}\mathrm{Ei}\left(\dfrac{r^2}{4a^f\Delta t}\right), & \text{2D problem} \\ \dfrac{1}{4\pi\lambda r}\mathrm{erfc}\left(\dfrac{r}{2\sqrt{a^f\Delta t}}\right), & \text{3D problem} \end{cases} \tag{21}$$

In order to solve equation (14), the boundary Γ is divided into N linear boundary elements Γ_j, the interior Ω is divided into L linear internal cells Ω_l and then we obtain the following system of algebraic equations ($i = 1, 2, ..., N$)

$$\sum_{j=1}^{N} G_{ij}\, q_j^f = \sum_{j=1}^{N} H_{ij}\, T_j^f + \sum_{l=1}^{L} P_{il}\, T_{N+l}^{f-1} \tag{22}$$

where

$$G_{ij} = \int_{\Gamma_j} g\left(\xi^i, x\right)\mathrm{d}\Gamma_j \tag{23}$$

and

$$H_{ij} = \begin{cases} \displaystyle\int_{\Gamma_j} h\left(\xi^i, x\right)\mathrm{d}\Gamma_j, & i \neq j \\ -1/2, & i = j \end{cases} \tag{24}$$

while

$$P_{il} = \int_{\Omega_l} T^*\left(\xi^i, x, t^f, t^{f-1}\right)\mathrm{d}\Omega_l \tag{25}$$

The system of equations (22) can be written in the matrix form

$$\mathbf{G}\,\mathbf{q}^f = \mathbf{H}\mathbf{T}^f + \mathbf{P}\mathbf{T}^{f-1} \tag{26}$$

After determining the 'missing' boundary values (T_j^f or q_j^f), the temperatures T_i^f at internal nodes $x^i \in \Omega$ for time t^f are calculated using the formula ($i = N+1, N+2, ..., N+L$)

$$T_i^f = \sum_{j=1}^{N} H_{ij}\, T_j^f - \sum_{j=1}^{N} G_{ij}\, q_j^f + \sum_{l=1}^{L} P_{il}\, T_{N+l}^{f-1} \tag{27}$$

4. Sensitivity analysis - explicit differentiation method

Let us consider the sensitivity of the solidification problem solution with respect to the parameter p (e.g. p corresponds to the thermal conductivity or heat transfer coefficient or ambient temperature).

The explicit differentiation method, which belongs to the continuous approach, bases on the differentiation of governing equations with respect to the parameter p [14, 15, 16]. So, the differentiation of equation (7) leads to the following equation

$$\begin{gathered} x \in \Omega: \quad \frac{\partial C(T)}{\partial p}\frac{\partial T}{\partial t} + C(T)\frac{\partial}{\partial p}\left(\frac{\partial T}{\partial t}\right) = \\ \frac{\partial \lambda}{\partial p}\nabla^2 T + \lambda\frac{\partial}{\partial p}\left(\nabla^2 T\right) \end{gathered} \tag{29}$$

or

$$x \in \Omega: \quad C(T)\frac{\partial U}{\partial t} = \lambda\nabla^2 U + \frac{\partial\lambda}{\partial p}\nabla^2 T - \frac{\partial C(T)}{\partial p}\frac{\partial T}{\partial t} \tag{30}$$

where $U = \partial T/\partial p$ is the sensitivity function.

Taking into account the formula (7) the equation (30) can be written in the form

$$x \in \Omega: \quad C(T)\frac{\partial U}{\partial t} = \lambda \nabla^2 U + \frac{\partial \lambda}{\partial p}\frac{C(T)}{\lambda}\frac{\partial T}{\partial t} - \frac{\partial C(T)}{\partial p}\frac{\partial T}{\partial t} \quad (31)$$

The boundary condition (8) is also differentiated with respect to p and then

$$x \in \Gamma: \quad -\frac{\partial \lambda}{\partial p}\mathbf{n}\cdot\nabla T - \lambda\frac{\partial}{\partial p}(\mathbf{n}\cdot\nabla T) = \frac{\partial \alpha}{\partial p}(T - T_a) + \alpha\left(\frac{\partial T}{\partial p} - \frac{\partial T_a}{\partial p}\right) \quad (32)$$

or

$$x \in \Gamma: \quad -\lambda\,\mathbf{n}\cdot\nabla U = \frac{\partial \lambda}{\partial p}\,\mathbf{n}\cdot\nabla T + \frac{\partial \alpha}{\partial p}(T - T_a) + \alpha\left(U - \frac{\partial T_a}{\partial p}\right) \quad (33)$$

Differentiation of initial condition (9) gives

$$t = 0: \quad U = \frac{\partial T_p}{\partial p} = 0 \quad (34)$$

In this way one obtains the additional boundary initial problem connected with the sensitivity function U described by equations (31), (33), (34). This problem has been solved by means of the boundary element method. So, one obtains the following system of algebraic equations (c.f. equation (22))

$$\sum_{j=1}^{N} G_{ij} W_j^f = \sum_{j=1}^{N} H_{ij} U_j^f + \sum_{l=1}^{L} P_{il} U_l^{f-1} + \sum_{l=1}^{L} Z_{il} Q_l^f \quad (35)$$

where

$$Z_{il} = \int_{\Omega_l} g(\xi^i, x)\,\mathrm{d}\Omega_l \quad (36)$$

and

$$Q_l^f = \frac{1}{a^f}\left(\frac{\partial T}{\partial t}\right)_l^f = \frac{1}{a^f}\frac{T_l^f - T_l^{f-1}}{\Delta t} \quad (37)$$

The system of equations (35) can be written in the matrix form

$$\mathbf{G}\mathbf{W}^f = \mathbf{H}\mathbf{U}^f + \mathbf{P}\mathbf{U}^{f-1} + \mathbf{Z}\mathbf{Q}^f \quad (38)$$

After determining the 'missing' boundary values (U_j^f or W_j^f), the values of function U_i^f at internal nodes $x^i \in \Omega$ for time t^f are calculated using the formula ($i = N+1, N+2, ..., N+L$)

$$U_i^f = \sum_{j=1}^{N} H_{ij} U_j^f - \sum_{j=1}^{N} G_{ij} W_j^f + \sum_{l=1}^{L} P_{il} U_l^{f-1} + \sum_{l=1}^{L} Z_{il} Q_l^f \quad (39)$$

5. Sensitivity analysis - implicit differentiation method

The implicit differentiation method, which belongs to the discretized approach, bases on the differentiation of the algebraic boundary element matrix equations [9].

So, the system of equations (26) has been differentiated with respect to the parameter p and then

$$\frac{\partial \mathbf{G}}{\partial p}\mathbf{q}^f + \mathbf{G}\frac{\partial \mathbf{q}^f}{\partial p} = \frac{\partial \mathbf{H}}{\partial p}\mathbf{T}^f + \mathbf{H}\frac{\partial \mathbf{T}^f}{\partial p} + \frac{\partial \mathbf{P}}{\partial p}\mathbf{T}^{f-1} + \mathbf{P}\frac{\partial \mathbf{T}^{f-1}}{\partial p} \quad (40)$$

or

$$\mathbf{G}\mathbf{W}^f = \mathbf{H}\mathbf{U}^f + \frac{\partial \mathbf{H}}{\partial p}\mathbf{T}^f - \frac{\partial \mathbf{G}}{\partial p}\mathbf{q}^f + \frac{\partial \mathbf{P}}{\partial p}\mathbf{T}^{f-1} + \mathbf{P}\mathbf{U}^{f-1} \quad (41)$$

The equation (27) concerning internal nodes is also differentiated with respect to p, namely

$$U_i^f = \sum_{j=1}^{N} H_{ij} U_j^f - \sum_{j=1}^{N} G_{ij} W_j^f + \sum_{j=1}^{N}\frac{\partial H_{ij}}{\partial p} T_j^f - \sum_{j=1}^{N}\frac{\partial G_{ij}}{\partial p} q_j^f + \sum_{l=1}^{L} P_{il} U_l^{f-1} + \sum_{l=1}^{L}\frac{\partial P_{il}}{\partial p} T_l^{f-1} \quad (42)$$

6. 1D problem

The casting of thickness $2D$ is analyzed (1D problem). Taking into account the symmetry, the following boundary initial problem is considered

$$\begin{cases} 0 < x < D: & C(T)\dfrac{\partial T}{\partial t} = \lambda\dfrac{\partial^2 T}{\partial x^2} \\ x = 0: & q = \lambda\dfrac{\partial T}{\partial x} = 0 \\ x = D: & q = -\lambda\dfrac{\partial T}{\partial x} = \alpha(T - T_a) \\ t = 0: & T = T_p \end{cases} \quad (43)$$

In this case the boundary is reduced to the two points, namely point 1 ($x = 0$) and point 2 ($x = D$). The domain $[0, D]$ is divided into L linear internal cells of thickness h. The central points of these cells are numbered as 3, 4, ..., L+2. The system of equations (26) resulting from the BEM application has the following form

$$\begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}\begin{bmatrix} q_1^f \\ q_2^f \end{bmatrix} = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}\begin{bmatrix} T_1^f \\ T_2^f \end{bmatrix} + \begin{bmatrix} P_{11} & P_{12} & \dots & P_{1L} \\ P_{21} & P_{22} & \dots & P_{2L} \end{bmatrix}\begin{bmatrix} T_3^{f-1} \\ T_4^{f-1} \\ \dots \\ T_{2+L}^{f-1} \end{bmatrix} \quad (44)$$

where (c.f. equations (23), (24))

$$G_{i1} = g(\xi^i, 0), \qquad G_{i2} = g(\xi^i, D) \quad (45)$$

and

$$H_{i1} = \begin{cases} h(\xi^i, 0), & i \neq j \\ -1/2, & i = j \end{cases}, \quad H_{i2} = \begin{cases} h(\xi^i, D), & i \neq j \\ -1/2, & i = j \end{cases} \quad (46)$$

Let us assume that the parameter p corresponds to the thermal conductivity of casting material. Using the explicit approach of sensitivity analysis one obtains the following additional problem ($\partial C(T)/\partial\lambda = 0$ - c.f. equation (6))

$$\begin{cases} 0 < x < D: & C(T)\dfrac{\partial U}{\partial t} = \lambda \dfrac{\partial^2 U}{\partial x^2} + \dfrac{1}{\lambda}\dfrac{\partial T}{\partial t} \\ x = 0: & W = \lambda \dfrac{\partial U}{\partial x} = 0 \\ x = D: & W = -\lambda \dfrac{\partial U}{\partial x} = \alpha\left(U - \dfrac{q}{\lambda \alpha}\right) \\ t = 0: & U = 0 \end{cases} \tag{47}$$

The problem (47) is coupled with the primary one (equation (43)) because in order to solve it the derivative $\partial T/\partial t$ and heat flux q should be known.

In the case of implicit differentiation method application the derivatives $\partial G_{ij}/\partial\lambda$, $\partial H_{ij}/\partial\lambda$ and $\partial P_{il}/\partial\lambda$ should be calculated. Because

$$\frac{\partial g(\xi,x)}{\partial\lambda} = -\frac{1}{2\lambda}\sqrt{\frac{\Delta t}{\pi\lambda C^f}}\exp\left(-\frac{(x-\xi)^2}{4a^f\Delta t}\right) + \frac{|x-\xi|}{2\lambda^2}\operatorname{erfc}\left(\frac{|x-\xi|}{2\sqrt{a^f\Delta t}}\right) \tag{48}$$

and

$$\frac{\partial h(\xi,x)}{\partial\lambda} = \frac{(x-\xi)}{4\lambda\sqrt{\pi a^f \Delta t}}\exp\left(-\frac{(x-\xi)^2}{4a^f\Delta t}\right) \tag{49}$$

so

$$\frac{\partial G_{i1}}{\partial\lambda} = \left.\frac{\partial g(\xi^i,x)}{\partial\lambda}\right|_{x=0}, \quad \frac{\partial G_{i2}}{\partial\lambda} = \left.\frac{\partial g(\xi^i,x)}{\partial\lambda}\right|_{x=D} \tag{50}$$

and

$$\frac{\partial H_{i1}}{\partial\lambda} = \begin{cases} \left.\dfrac{\partial h(\xi^i,x)}{\partial\lambda}\right|_{x=0}, \\ 0, \end{cases} \quad \frac{\partial H_{i2}}{\partial\lambda} = \begin{cases} \left.\dfrac{\partial h(\xi^i,x)}{\partial\lambda}\right|_{x=D}, & i \neq j \\ 0, & i = j \end{cases} \tag{51}$$

Finally, the derivative $\partial P_{il}/\partial\lambda$ is calculated

$$\frac{\partial P_{il}}{\partial\lambda} = \frac{1}{4\lambda\sqrt{\pi a^f\Delta t}}\int_{x_{l-1}}^{x_l}\left[\frac{(x-\xi^i)^2}{2a^f\Delta t} - 1\right]\exp\left(-\frac{(x-\xi^i)^2}{4a^f\Delta t}\right)\mathrm{d}x \tag{52}$$

The system of equations connected with the sensitivity function has the following form

$$\begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22}\end{bmatrix}\begin{bmatrix} W_1^f \\ W_2^f\end{bmatrix} = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22}\end{bmatrix}\begin{bmatrix} U_1^f \\ U_2^f\end{bmatrix} + \begin{bmatrix} \dfrac{\partial H_{11}}{\partial\lambda} & \dfrac{\partial H_{12}}{\partial\lambda} \\ \dfrac{\partial H_{21}}{\partial\lambda} & \dfrac{\partial H_{22}}{\partial\lambda}\end{bmatrix}\begin{bmatrix} T_1^f \\ T_2^f\end{bmatrix} -$$

$$\begin{bmatrix} \dfrac{\partial G_{11}}{\partial\lambda} & \dfrac{\partial G_{12}}{\partial\lambda} \\ \dfrac{\partial G_{21}}{\partial\lambda} & \dfrac{\partial G_{22}}{\partial\lambda}\end{bmatrix}\begin{bmatrix} q_1^f \\ q_2^f\end{bmatrix} + \begin{bmatrix} \dfrac{\partial P_{11}}{\partial\lambda} & \dfrac{\partial P_{12}}{\partial\lambda} & \cdots & \dfrac{\partial P_{1L}}{\partial\lambda} \\ \dfrac{\partial P_{21}}{\partial\lambda} & \dfrac{\partial P_{22}}{\partial\lambda} & \cdots & \dfrac{\partial P_{2L}}{\partial\lambda}\end{bmatrix}\begin{bmatrix} T_3^{f-1} \\ T_4^{f-1} \\ \cdots \\ T_{2+L}^{f-1}\end{bmatrix} + \begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1L} \\ P_{21} & P_{22} & \cdots & P_{2L}\end{bmatrix}\begin{bmatrix} U_3^{f-1} \\ U_4^{f-1} \\ \cdots \\ U_{2+L}^{f-1}\end{bmatrix} \tag{53}$$

7. Example of computations

In numerical computations the following data have been introduced: $2D = 0.02$ m, $\lambda = 35$ [W/(mK)], $c_L = 5.74$ [MJ/(m^{3}K)], $c_S = 5.175$ [MJ/m^3 K], $L_V = 1957.5$ [MJ/m^3], pouring temperature $T_0 = 1570\,^\circ$C, liquidus temperature $T_L = 1505\,^\circ$C, solidus temperature $T_S = 1470\,^\circ$C, heat transfer coefficient $\alpha = 250$ [W/(m^2 K)], ambient temperature $T_a = 600\,^\circ$C. The domains has been divided into 20 internal cells, time step $\Delta t = 0.5$ s.

In Figure 1 the temperature distribution in the domain considered for times 10, 20, ..., 100 s is shown.

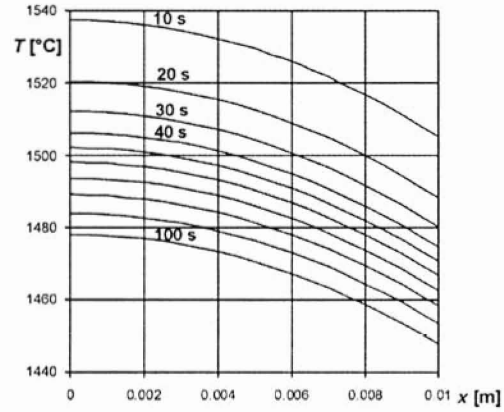

Fig. 1. Temperature distribution

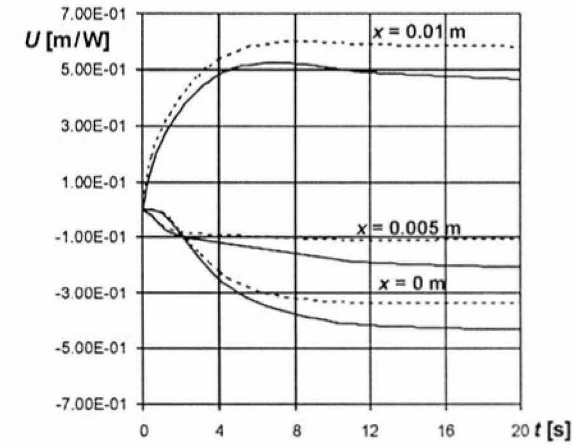

Fig. 2. Courses of sensitivity function U
(explicit approach - solid line, implicit approach - dotted line)

Figure 2 illustrates the courses of function $U = \partial T/\partial\lambda$ at the points $x = 0$ (axis of symmetry), $x = 0.005$ m and $x = 0.01$ m (boundary) obtained by explicit and implicit differentiation method. It is visible that these courses are similar but not identical.

8. Conclusions

Sensitivity analysis is the very effective tool in numerical modelling of solidification problem. It allows to rebuilt the basic solution on the solution concerning the other disturbed value of parameter. In the paper the explicit and implicit approaches using the boundary element method have been presented. From the mathematical point of view the explicit approach is simpler, but the implicit approach gives more exact results.

Acknowledgements

This work was funded by Grant No N N507 3592 33.

References

[1] B. Mochnacki, J. S. Suchy, Numerical methods in computations of foundry processes, PFTA, Cracow (1995).

[2] R. Szopa, Sensitivity analysis and inverse problems in the thermal theory of foundry, Publ. of the Czest. Univ. of Techn., Monographs, 124, Czestochowa (2006).

[3] B. Mochnacki, E. Majchrzak, R. Szopa, J. S. Suchy, Inverse problems in the thermal theory of foundry, Scientific Research of the Institute of Mathematics and Computer Science, Czestochowa 1(5) (2006) 154–179.

[4] M. Kleiber, Parameter sensitivity, J.Wiley & Sons Ltd., Chichester (1997).

[5] K. Dems, B. Rousselet, Sensitivity analysis for transient heat conduction in a solid body, Structural Optimization, 17 (1999), 36–45.

[6] B. Mochnacki, J. S. Suchy, Identification of alloy latent heat on the basis of mould temperature (Part 1), Archives of Foundry 6, 22 (2006) 324–330.

[7] A. Służalec, M. Kleiber, Shape sensitivity analysis for nonlinear steady-state heat conduction problems, International Journal of Heat and Mass Transfer, Vol. 39 No 12 (1996) 2609–2613.

[8] B. Mochnacki, A. Metelski, Identification of internal heat source capacity in the heterogeneous domain, Scientific Research of the Institute of Mathematics and Computer Science, Czestochowa University of Technology, 1(4) (2005) 182–187.

[9] T. Burczyński, Sensitivity analysis, optimization and inverse problems, 245-307, w: D.Beskos, G.Maier, Boundary element advances in solid mechanics, Springer Verlag, Wien, New York (2003).

[10] C. A. Brebbia, J. Dominguez, Boundary elements, an introductory course, Computational Mechanics Publications, McGraw-Hill Book Company, London (1992).

[11] E. Majchrzak, Boundary element method in heat transfer, Publ. of the Czest. Univ. of Techn., Czestochowa (2001) (in Polish).

[12] A. Bokota, S. Iskierka, An analysis of the diffusion-convection problem by the BEM, Engineering Analysis with Boundary Elements, 15 (1995) 267–275.

[13] E. Majchrzak, K. Freus, S. Freus, Application of the BEM for numerical solution of nonlinear diffusion equation, Scientific Research of the Institute of Mathematics and Computer Science, Czestochowa (2007).

[14] E. Majchrzak, M. Jasiński, G. Kałuża, Application of shape sensitivity analysis in numerical modelling of solidification process, Archives of Foundry, 5, 15 (2005) 259–264.

[15] R. Szopa, The parametric sensitivity analysis of solidification process, Archives of Foundry 5, 15 (2005) 395–404.

[16] E. Majchrzak, B. Mochnacki, G. Kałuża, Shape sensitivity analysis in numerical modelling of solidification, Archives of Foundry Engineering, Vol. 7, 4 (2007) 115–120.

Jawna i niejawna metoda analizy wrażliwości w numerycznym modelowaniu procesu krzepnięcia

Streszczenie

W artykule przedstawiono jawne i niejawne podejście analizy wrażliwości z zastosowaniem metody elementów brzegowych. W szczególności rozpatrywano proces krzepnięcia. Zaburzenie parametru wejściowego (np. współczynnika przewodzenia ciepła) powoduje zmiany pola temperatury w rozważanym obszarze. Metody analizy wrażliwości pozwalają w matematyczny sposób przedstawić wzajemne zależności między zaburzeniami parametrów a końcowymi wynikami. W pracy pokazano najważniejsze aspekty algorytmów obliczeniowych związanych z jawną i niejawną metodą analizy wrażliwości.