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Identification of boundary heat flux on the continuous casting surface

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Abstract

In the paper the numerical solution of the inverse problem consisting in the identification of the heat flux on the continuous casting surface is presented. The additional information results from the measured surface or interior temperature histories. In particular the sequential function specification method using future time steps is applied. On the stage of numerical computations the 1st scheme of the boundary element method for parabolic equations is used. Because the problem is strongly non-linear the additional procedure 'linearizing' the task discussed is introduced. This procedure is called the artificial heat source method. In the final part of the paper the examples of computations are shown.

Keywords: Application of information technology to the foundry industry, Solidification process, Numerical techniques, Inverse problems, Boundary element method

1. Introduction

The vertical, rectangular cast slab is considered (Figure 1). Neglecting the convection proceeding in the molten metal sub-domain, the thermal processes in the continuous casting volume are described by the following equation [1]

$$C(T)\left[\frac{\partial T(x',t)}{\partial t}+\mathbf{w}\cdot \operatorname{grad} T(x',t)\right]=\operatorname{div}\left[\lambda(T)\operatorname{grad} T(x',t)\right] \tag{1}$$

where $x'=\{x_1', x_2', x_3'\}$. The casting shifts in axis x_3' direction and its pulling rate is equal to **w** (more precisely, the velocity field in domain considered: $\mathbf{w}=[0,0,w]$). The same mathematical description can be used in a case of so-called radial plants, because a large radius of plant curvature (in comparison to the casting dimensions) allows to treat the radial installation as a vertical one.

On the upper surface of the casting (free surface of molten metal) the boundary condition of the 1st type (pouring temperature) can be taken into account. On the conventionally assumed bottom surface limiting the domain considered (it is a region of final cooling zone) we can put $\partial T/\partial n=0$, this means the adiabatic condition. On the lateral surface the Neumann condition is assumed (the data concerning the boundary heat fluxes are collected in [2]).

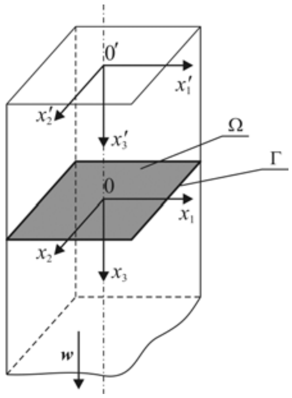

Fig. 1. Rectangular cast slab

The initial condition resolves itself into the assumption, that a certain layer of molten metal directly over the starter bar has a pouring temperature. The starter bar allows to shut the continuous casting mould during the plant starting.

The numerous experiments show that conductional component of heat transfer corresponding to the direction of casting displacement is very small (this component constitutes about 5% of the heat conducted from the axis to the lateral surfaces), this means that the component $\mathrm{div}\left[\lambda(T)\,\mathrm{grad}\,T\right]$ can be simplified to the form

$$\mathrm{div}\left[\lambda(T)\,\mathrm{grad}\,T\right]=\frac{\partial}{\partial x_1'}\left[\lambda(T)\frac{\partial T}{\partial x_1'}\right]+\frac{\partial}{\partial x_2'}\left[\lambda(T)\frac{\partial T}{\partial x_2'}\right] \tag{2}$$

A pretty interesting and effective in numerical simulation variant of mathematical approach to the continuous casting problem was presented by the authors of this paper in [3, 4]. The algorithm has been called 'a wandering cross section method'.

Let we rewrite the equation (2) in coordinate system 'tied' to a certain section Ω of shifting casting, namely $x_1 = x_1'$, $x_2 = x_2'$, $x_3 = x_3' - w\,t$. We assume, as previously, that the heat conduction in x_3' direction can be neglected. It is easy to check up that we 'lose' in energy equation the component $w\cdot\mathrm{grad}\,T$

$$(x_1,x_2)\in\Omega:\; C(T)\frac{\partial T}{\partial t}=\frac{\partial}{\partial x_1}\left[\lambda(T)\frac{\partial T}{\partial x_1}\right]+\frac{\partial}{\partial x_2}\left[\lambda(T)\frac{\partial T}{\partial x_2}\right] \tag{3}$$

The last equation corresponds to typical thermal diffusion equation for a 2D object oriented in cartesian coordinate system (but finally we find the 3D solution). It should be solved for initial condition $T(x_1, x_2, 0) = T_0$ (pouring temperature), while the boundary heat flux on the perimeter Γ of the section considered is the function of time.

In the paper the 2D task concerning the continuous casting technology is discussed. On the basis of the knowledge of temperature history at the selected set of the points from the casting domain the boundary heat flux is identified [5, 6, 7]. The identification of the boundary heat flux in the primary cooling zone (continuous casting mould) has been done using sequential function specification method [6, 7]. The results of computations are presented in the final part of the paper.

2. Identification of boundary heat flux

In this chapter the sequential function specification method using future temperature information is presented [6, 7]. This method allows to estimate the boundary heat flux on the basis of temperature measurements in the casting domain.

If we assume the constant value of thermal conductivity $\lambda(T)=\lambda$ (this assumption in the case of typical alloys is acceptable) then the problem discussed is of the form (c.f. equation (3))

$$\begin{cases} x\in\Omega: & C(T)\dfrac{\partial T(x,t)}{\partial t}=\lambda\nabla^2 T(x,t)\\ x\in\Gamma: & q(x,t)=-\lambda\dfrac{\partial T(x,t)}{\partial n}\\ t=0: & T(x,t)=T_0 \end{cases} \tag{4}$$

where $x = \{x_1, x_2\}$ and

$$\nabla^2 T(x,t)=\frac{\partial^2 T(x,t)}{\partial x_1^2}+\frac{\partial^2 T(x,t)}{\partial x_2^2} \tag{5}$$

We define the substitute thermal capacity $C(T)$ for $T\in[T_S, T_L]$ in the form of polynomial

$$C(T)=c_0+c_1T+c_2T^2+c_3T^3+c_4T^4 \tag{6}$$

which fulfills the conditions:

i. For $T = T_S$: $C(T_S) = c_S$ and for $T = T_L$: $C(T_L) = c_L$, where c_S, c_L are the volumetric specific heats of solid and liquid states, respectively.

ii. For $T = T_S$ and $T = T_L$ $\mathrm{d}C(T)/\mathrm{d}T = 0$.

iii. The change of enthalpy connected with the solidification equals

$$\int_{T_S}^{T_L} C(T)\,\mathrm{d}T = L + c_P(T_L - T_S) \tag{7}$$

where c_p is the volumetric specific heat of mushy zone.

For direct problems the boundary - initial conditions as well as the parameters λ and $C(T)$ are known and we determine the temperature distribution $T(x, t)$. For the inverse boundary problem analyzed we assume that the heat flux $q(x, t)$ on Γ is unknown. Additionally, measured temperature histories at the boundary or interior points x^i for times t^f, $f = 1, 2, ..., F$ are given

$$T_{d\,i}^{f} = T_d\left(x_1^i, x_2^i, t^f\right), \quad i = 1, 2, ..., M, \quad f = 1, 2, ..., F \tag{8}$$

where M is the number of sensors.

In sequential function specification method [6, 7] it is assumed that the heat flux is known at the set of boundary points $x^j \in \Gamma$, $j = 1, 2, ..., J$ for times $t^1, t^2, ..., t^{f-1}$, and we want to determine the heat flux $q_j^f = q\left(x_1^j, x_2^j, t^f\right)$ at time t^f. Additionally, the temperature values are known for R future intervals, namely ($r = 1, 2, ..., R$)

$$T_{d\,i}^{f+r-1} = T_d\left(x^i, t^{f+r-1}\right), \quad i = 1, 2, ..., M \tag{9}$$

and we assume that the heat flux is constant over R future steps and equal to the heat flux at time t^f

$$q_j^f = q_j^{f+1} = ... = q_j^{f+R-1} \tag{10}$$

In order to solve the inverse problem, the least squares method is applied [5, 6, 8]

$$S = \sum_{i=1}^{M}\sum_{r=1}^{R}\left(T_i^{f+r-1} - T_{d\,i}^{f+r-1}\right)^2 \tag{11}$$

Function $T_i^{f+r-1} = T\left(x^i, T^{f+r-1}\right)$ is expanded in a Taylor series about arbitrary but known value of heat flux q_j^{*f}

$$T_i^{f+r-1} = T_i^{*f+r-1} + \sum_{j=1}^{J}\left.\frac{\partial T_i^{f+r-1}}{\partial q_j^f}\right|_{q_j^f = q_j^{*f}}\left(q_j^f - q_j^{*f}\right) \tag{12}$$

where T_i^{*f+r-1} denotes the calculated temperature at point x^i for time t_i^{f+r-1} obtained under the assumption that for $t \in [t^{f-1}, t^{f+r-1}]$ the heat fluxes equal $q_j^f = q_j^{f+1} = ... = q_j^{f+R-1} = q_j^{*f}$ We introduce the sensitivity coefficients [3, 4] and then

$$T_i^{f+r-1} = T_i^{*f+r-1} + \sum_{j=1}^{J} Z_{j,t}^{f+r-1}\left(q_j^f - q_j^{*f}\right) \tag{13}$$

Putting (13) into (11) one has

$$S = \sum_{i=1}^{M}\sum_{r=1}^{R}\left[T_i^{*f+r-1} + \sum_{j=1}^{J} Z_{j,t}^{f+r-1}\left(q_j^f - q_j^{*f}\right) - T_{d\,i}^{f+r-1}\right]^2 \tag{14}$$

Differentiating the criterion (14) with respect to the unknown heat fluxes q_j^f and using the necessary condition of minimum, one obtains the following system of equations

$$\sum_{i=1}^{M}\sum_{r=1}^{R}\sum_{j=1}^{J} Z_{l,i}^{f+r-1} Z_{j,i}^{f+r-1}\left(q_j^f - q_j^{*f}\right) = \sum_{i=1}^{M}\sum_{r=1}^{R} Z_{l,t}^{f+r-1}\left(T_{d\,i}^{f+r-1} - T_i^{*f+r-1}\right) \tag{15}$$

where $l = 1, 2, ..., J$.

The system f equations (15) can be written in the matrix form

$$\left(\mathbf{Z}^f\right)^{\mathrm{T}}\mathbf{Z}^f\mathbf{q}^f = \left(\mathbf{Z}^f\right)^{\mathrm{T}}\mathbf{Z}^f\mathbf{q}^{*f} + \left(\mathbf{Z}^f\right)^{\mathrm{T}}\left(\mathbf{T}_d^f - \mathbf{T}^{*f}\right) \tag{16}$$

where

$$\mathbf{q}^f = \begin{bmatrix} q_1^f \\ q_2^f \\ ... \\ q_J^f \end{bmatrix}, \quad \mathbf{q}^{*f} = \begin{bmatrix} q_1^{*f} \\ q_2^{*f} \\ ... \\ q_J^{*f} \end{bmatrix} \tag{17}$$

and

$$\mathbf{Z}^f = \begin{bmatrix} Z_{1,1}^f & Z_{2,1}^f & ... & Z_{J,1}^f \\ ... & ... & ... & ... \\ Z_{1,1}^{f+R-1} & Z_{2,1}^{f+R-1} & ... & Z_{J,1}^{f+R-1} \\ Z_{1,2}^f & Z_{2,2}^f & ... & Z_{J,2}^f \\ ... & ... & ... & ... \\ Z_{1,2}^{f+R-1} & Z_{2,2}^{f+R-1} & ... & Z_{J,2}^{f+R-1} \\ Z_{1,M}^f & Z_{2,M}^f & ... & Z_{J,M}^f \\ ... & ... & ... & ... \\ Z_{1,M}^{f+R-1} & Z_{2,M}^{f+R-1} & ... & Z_{J,M}^{f+R-1} \end{bmatrix} \tag{18}$$

while

$$\mathbf{T}_d^f = \begin{bmatrix} T_{d1}^f \\ ... \\ T_{d1}^{f+R-1} \\ T_{d2}^f \\ ... \\ T_{d2}^{f+R-1} \\ T_{dM}^f \\ ... \\ T_{dM}^{f+R-1} \end{bmatrix}, \quad \mathbf{T}^{*f} = \begin{bmatrix} T_1^{*f} \\ ... \\ T_{d1}^{*f+R-1} \\ T_2^{*f} \\ ... \\ T_2^{*f+R-1} \\ T_M^{*f} \\ ... \\ T_M^{*f+R-1} \end{bmatrix} \tag{19}$$

The system of equations (16) allows to find the values of heat fluxes q_j^f at boundary nodes x^j, $j = 1, 2, ..., J$ at time t^f.

The idea of the sequential function specification method using future temperature information for transition $t^{f-1} \to t^f$ is shown in Figure 2.

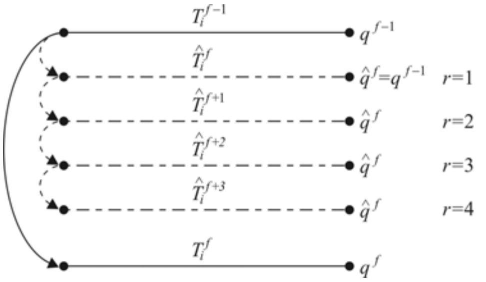

Fig. 2. Transition $t^{f-1} \rightarrow t^f$ using future time steps ($R = 4$)

In the sequential function specification method the sensitivity coefficients are used (c.f. matrix (18)). In order to determine them, the governing equations (4) are differentiated with respect to the unknown heat flux $q_j = q\left(x^j, t\right)$ at boundary point $x^j \in \Gamma$, and then [6, 8, 9, 10]

$$\begin{cases} x \in \Omega: & C(T)\dfrac{\partial Z_j(x,t)}{\partial t} = \lambda \nabla^2 Z_j\,(x,t) \\ & -\dfrac{\mathrm{d}\,C(T)}{\mathrm{d}\,T} Z_j\,(x,t)\dfrac{\partial T(x,t)}{\partial t} \\ x \in \Gamma: & W_j\,(x,t) = -\lambda \dfrac{\partial Z_j(x,t)}{\partial n} = \begin{cases} 1, & x = x^j \\ 0, & x \neq x^j \end{cases} \\ t = 0: & Z_j\,(x,t) = 0 \end{cases} \tag{20}$$

where

$$Z_j\,(x,t) = \frac{\partial T(x,t)}{\partial q_j} \tag{21}$$

So, in the case considered the additional boundary-initial problems (20) for $j = 1, 2, ..., J$ should be solved.

In equation (20) the derivative of substitute thermal capacity with respect to temperature, this means $\mathrm{d}C(T)/\mathrm{d}T$ appears.

3. Boundary element method

The basic problem (4) for the arbitrary assumed value of boundary heat flux $q(x, t)$ and the additional problems (20) associated with the sequential function specification method have been solved using the boundary element method for 2D parabolic equations.

So, we consider the following equation

$$x \in \Omega: \quad C(T)\frac{\partial F(x,t)}{\partial t} = \lambda \nabla^2 F(x,t) + R(x,t) \tag{22}$$

where for primary problem (4): $F(x, t) = T(x, t)$, $R(x, t) = 0$ and for additional problems (20): $F(x,t) = Z_j(x,t)$, $R(x,t) = \mathrm{d}C(T)/\mathrm{d}T\, Z_j(x, t)\, \partial T(x, t)/\partial t$.

It should be pointed out that taking into account the course of functions $C(T)$ and $\mathrm{d}C(T)/\mathrm{d}T$ and the form of source functions appearing in additional problems, the equation (22) is strongly nonlinear. In order to solve it, the artificial heat source method has been applied [11, 12]. This method is a very effective supplementary algorithm first of all in a case of the BEM application for the non-linear problems solution.

We express the function $C(T)$ as a sum of two components, this means a constant part C_0 and a certain increment $\Delta C(T)$

$$C(T) = C_0 + \Delta C(T) \tag{23}$$

The equation (22) can be written in the form

$$C_0 \frac{\partial F(x,t)}{\partial t} = \lambda \nabla^2 F(x,t) + R(x,t) - \Delta C(T)\frac{\partial F(x,t)}{\partial t} \tag{24}$$

or

$$C_0 \frac{\partial F(x,t)}{\partial t} = \lambda \nabla^2 F(x,t) + S(x,t) \tag{25}$$

where

$$S(x,t) = R(x,t) - \Delta C(T)\frac{\partial F(x,t)}{\partial t} \tag{26}$$

is the artificial heat source term. The essential feature of equation (25) consists in a fact, that leaving out the last term we obtain the linear form of energy equation. Taking into account the possibilities of the boundary element method application in the range of non-steady problems modelling, this is a very convenient form of basic differential equation (a non-linearity appears only in the component determining the internal heat sources, and the function describing the fundamental solution for the problem considered is well known). The calculation of a source function requires, of course, the introduction of a certain iterative procedure [11, 12]. As was mentioned above, in order to solve the equation (25) the 1st scheme of boundary element method for 2D parabolic equations has been used. The boundary integral equation corresponding to the transition $t^{f-1} \rightarrow t^f$ takes a form [13,14,15,16]

$$\begin{aligned} & B(\xi)F\left(\xi, t^f\right) + \frac{1}{C_0}\int_{t^{f-1}}^{t^f}\int_\Gamma F^*\left(\xi, x, t^f, t\right) J(x,t)\,\mathrm{d}\Gamma\,\mathrm{d}t = \\ & \frac{1}{C_0}\int_{t^{f-1}}^{t^f}\int_\Gamma J^*\left(\xi, x, t^f, t\right) F(x,t)\,\mathrm{d}\Gamma\,\mathrm{d}t + \\ & \int_\Omega F^*\left(\xi, x, t^f, t^{f-1}\right) F(x,t)\,\mathrm{d}\Omega + \frac{1}{C_0}\int_\Omega\int_{t^{f-1}}^{t^f} S(x,t) F^*\left(\xi, x, t^f, t\right)\mathrm{d}t\,\mathrm{d}\Omega \end{aligned} \tag{27}$$

where $\xi = (\xi_1, \xi_2)$ is the observation point, $B(\xi) = 1$ for $\xi \in \Omega$ and $B(\xi) \in (0, 1)$ for $\xi \in \Gamma$, $F^*\left(\xi, x, t^f, t\right)$ is the fundamental solution [11, 12]

$$F^*\left(\xi, x, t^f, t\right) = \frac{1}{4\pi a\left(t^f - t\right)} \exp\left[-\frac{r^2}{4a\left(t^f - t\right)}\right] \tag{28}$$

where r is the distance between the points ξ and x, $a = \lambda / C_0$, while $J\left(x, t\right) = -\lambda \partial F\left(x, t\right) / \partial n$ and $J^*\left(\xi, x, t^f, t\right) = -\lambda\, \partial F^*\left(\xi, x, t^f, t\right) / \partial n.$

For constant elements with respect to time [13, 14], the equation (27) can be written in the form

$$B(\xi) F\left(\xi, t^f\right) + \int_\Gamma J\left(x, t^f\right) g\left(\xi, x\right) \mathrm{d}\Gamma = \int_\Gamma F\left(x, t^f\right) h\left(\xi, x\right) \mathrm{d}\Gamma + \iint_\Omega \left[F^*\left(\xi, x, t^f, t^{f-1}\right) F\left(x, t^{f-1}\right) S\left(x, t\right) g\left(\xi, x\right)\right] \mathrm{d}\Omega \tag{29}$$

where

$$h\left(\xi, x\right) = \frac{1}{C_0} \int_{t^{f-1}}^{t^f} J^*\left(\xi, x, t^f, t\right) \mathrm{d}\, t \tag{30}$$

and

$$g\left(\xi, x\right) = \frac{1}{C_0} \int_{t^{f-1}}^{t^f} F^*\left(\xi, x, t^f, t\right) \mathrm{d}\, t \tag{31}$$

In order to solve equation (29), the boundary Γ is divided into N constant boundary elements Γ_j, the interior Ω is divided into L constant internal cells Ω_l and then we obtain the following system of algebraic equations ($i = 1, 2, ..., N$)

$$\sum_{j=1}^{N} G_{ij} J\left(x^j, t^f\right) = \sum_{j=1}^{N} H_{ij} F\left(x^j, t^f\right) + \sum_{l=1}^{L} P_{il} F\left(x^l, t^{f-1}\right) + \sum_{l=1}^{L} D_{il} S\left(x^l, t^f\right) \tag{32}$$

where

$$G_{ij} = \int_{\Gamma_j} g\left(\xi^i, x\right) \mathrm{d}\Gamma_j \quad \text{and} \quad H_{ij} = \begin{cases} \int_{\Gamma_j} h\left(\xi^i, x\right) \mathrm{d}\Gamma_j, & i \neq j \\ -1/2 & i = j \end{cases} \tag{33}$$

while

$$P_{il} = \int_{\Omega_l} F^*\left(\xi^i, x, t^f, t^{f-1}\right) \mathrm{d}\Omega_l \tag{34}$$

at the same time

$$D_{il} = \iint_{\Omega_l} g\left(\xi^i, x\right) \mathrm{d}\Omega_l \tag{35}$$

After determining the 'missing' boundary values ($F\,(x^j, t^f)$ or $J\,(x^j, t^f)$), the values of function $F\,(x^i, t^f)$ at internal nodes $x^i \in \Omega$ for time t^f are calculated using the formula ($i = N+1, N+2, ..., N+L$)

$$F\left(x^i, t^f\right) = \sum_{j=1}^{N} \left[H_{ij} F\left(x^j, t^f\right) - G_{ij} J\left(x^j, t^f\right)\right] + \sum_{l=1}^{L} \left[P_{il} F\left(x^l, t^{f-1}\right) + D_{il} S\left(x^l, t^f\right)\right] \tag{36}$$

4. Results of computations

The lateral section of steel ingot 0.1×0.1 [m] has been considered - c.f. Figure 1. The following input data have been assumed: thermal conductivity $\lambda = 35$ [W/mK], constant $C_0 = 54.243 \cdot 10^6$ [J/m^3 K] (c.f. equation (25)), liquidus temperature $T_L = 1505$ °C, solidus temperature $T_S = 1470$ °C, pouring temperature $T_0 = 1550$ °C, volumetric specific heats of liquid and solid states $c_L = 5.904 \cdot 10^6$ [J/m^3 K], $c_S = 4.875 \cdot 10^6$ [J/m^3 K], volumetric specific heat of mushy zone sub-domain $c_P = 0.5\,(c_L + c_S)$, volumetric latent heat $L = 1.9845 \cdot 10^9$ [J/m^3], pulling rate $w = 0.0183$ [m/s].

In Figure 3 the discretization of the domain considered is shown. The boundary is divided into 40 constant boundary elements, while the interior is divided into 100 constant internal cells. Time step equals $\Delta t = 0.25$ [s].

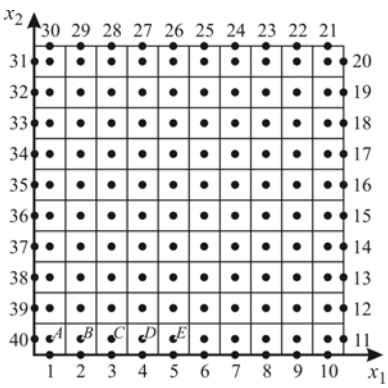

Fig. 3. Discretization

In order to estimate the course of boundary heat flux we assume, that the temperatures at the internal nodes A, B, C, D, E for successive cross sections $x_3 = f\,w\,\Delta t$, $f = 0, 1, ..., 14$ of the cast slab (c.f. Figure 1) are known. In Figure 4 the values of temperature at these nodes are shown. The information concerning the temperature distribution has been obtained from the direct problem solution under the assumption that the boundary heat flux changes according the formula $q\left(t\right) = b_1 + b_2\sqrt{t}$, where $b_1 = 2.7 \cdot 10^6$ and $b_2 = -3.35 \cdot 10^5$.

The inverse problem has been solved using the sequential function specification method for $R=1$, $R=2$ future time steps and $q_j^{*1}=10^4$, $j=1, 2, ..., 5.$ In Figure 5 the real course of boundary heat flux and the identified heat flux are shown.

Summing up, the sequential function specification method coupled with the boundary element method allows to identify the boundary heat flux.

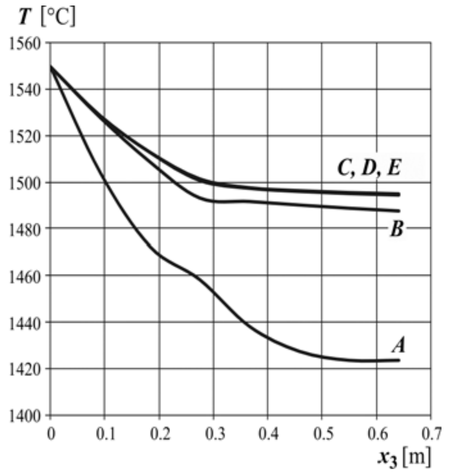

Fig. 4. Temperature at the points *A*, *B*, *C*, *D*, *E*

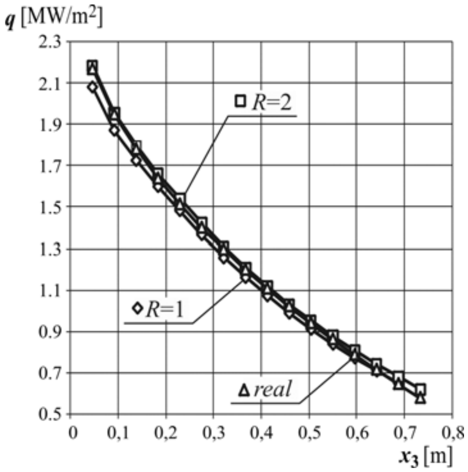

Fig. 5. Real and identified boundary heat fluxes

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