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Varieties of t-groups

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A test for checking whether a group variety is a variety of t-groups is established and it is shown that any variety of t-groups is pseudoabelian whereas there exists a non-abelian torsion-free variety of t-groups. Also conditions under which all groups in a pseudoabelian variety are t-groups are discussed.

A group G is called a t-group if for any two subgroups A and B the fact that A is normal in G and B is normal in A implies that B is normal in G. A wide range of t-groups that satisfy extra conditions are studied in a number of works; for example, see [1] and [8]. Here we shall deal with varieties of t-groups.

A group variety is a variety of t-groups if all its groups are t-groups. A variety is called pseudoabelian if it is nonabelian but all its finite groups are abelian. Answering a question of H. Neumann (problem 5 of [5]), A. Yu. Ol'shanskii [6] proved the existence of pseudoabelian varieties of groups. In [4], the following question was raised: is it true that all groups in a pseudobelian variety are t-groups? In this paper we set up a test for checking whether a variety consists of t-groups only. This enables us to show that any variety of t-groups is pseudoabelian, and that there exists a non-abelian torsion-free variety of t-groups. Quite a while ago, L. G. Kovacs and P. M. Neumann independently obtained the following result (unpublished, see [4]): if an element of a squarefree order fails to normalize a subnormal subgroup of a group G, then G has a metabelian, nonabelian factor and so cannot belong to any pseudoabelian variety. In this paper we give a proof of a more general result. However, the question as to whether all groups in a pseudoabelian variety are t-groups is still open.

Given two elements x and y of a group, let X and Y stand for the cyclic subgroups generated by x and y respectively. We shall also use the notation $[a, b]$ for $aba^{-1}b^{-1}$.

Lemma. *Let x and y be any two elements of an arbitrary group. Then $[[X, Y], Y]$ and $[[Y, X], X]$ are normal subgroups of $[X, Y]$.*

Proof. Clearly, $[[X, Y], Y]$ and $[[Y, X], X]$ are subgroups of $[X, Y]$. Let f and g be elements of $[X, Y]$. Then $[y^k, fg] = [y^k, f]f[y^k, g]f^{-1}$ for any integer k. Hence for any f, g, the element $f[y^k, g]f^{-1}$ belongs to $[[X, Y], Y]$ which proves the lemma.

Theorem 1. *A group variety is a variety of t-groups if and only if it satisfies a law of the form*

$$[x, y] = u(x, y),$$

where $u(x, y)$ belongs to $[[X, Y], Y]$.

Proof. First, we assume that H is a group satisfying a law of the form $[x, y] = u(x, y)$ where $u(x, y)$ belongs to $[[X, Y], Y]$. Let F be a normal subgroup of G and G a normal subgroup of H. Also let f be an element of F and h an element of H. Then $[h, f] = u(h, f)$ where $u(h, f)$ belongs to $[[H, F], F]$. Since F is a subgroup of G and G is a normal subgroup of H, $[H, F]$ is a subgroup of G. Hence $[[H, F], F]$ is a subgroup of F as F is a normal subgroup of G. Therefore $[h, f]$ belongs to F which means that F is a normal subgroup of H. Thus H is a t-group.

Next we assume that Q is a relatively free group of infinite rank in a variety of t-groups. Let x and y be in Q and let P denote the subgroup of Q that is generated by x and y. Then $[[X, Y], Y]$ is a normal subgroup of $[X, Y]$ by the Lemma. Therefore $Y[[X, Y], Y]$ is a normal subgroup of $Y[X, Y]$. Hence $Y[[X, Y], Y]$ is a normal subgroup of P as $Y[X, Y]$ is a normal subgroup of P, and Q is in a variety of t-groups. Hence $Y[[X, Y], Y]$ contains $[x, y]$ and therefore $[x, y] = y^k v(x, y)$ where k is an integer and $v(x, y)$ belongs to $[[X, Y], Y]$. Since Q is relatively free, we can consider the equation $[x, y] = y^k v(x, y)$ when $x = 1$ and obtain $y^k = 1$. Thus $[x, y] = v(x, y)$ in Q.

Corollary 1. *A variety of t-groups is pseudoabelian, that is, all its finite groups are abelian.*

Proof. The fact that any variety of t-groups is pseudoabelian can be proved by using the arguments in the proofs of Theorem 3.1 of [2] and Lemma 29.1 of [7]. It is well known (for example, see Corollary 6.1 of [7]) that if a variety $\mathfrak{M}$ of t-groups contains a nonabelian finite group, then it contains a metabelian nonabelian group. But according to the Theorem, $\mathfrak{M}$ satisfies $[x, y] = u(x, y)$ where $u(x, y)$ belongs to $[[X, Y], Y]$. Therefore by substituting $[y, z]$ for y in the equation $[x, y] = u(x, y)$, we obtain that all metabelian groups in $\mathfrak{M}$ are nilpotent of class at most 2. Hence we arrive at a contradiction since all nilpotent groups of class at most 2 in $\mathfrak{M}$ are abelian in view of $[x, y] = u(x, y)$.

Corollary 2. *There exists a nonabelian torsion-free variety of t-groups.*

PROOF. To prove the existence of a nonabelian torsion-free variety of t-groups, we shall consider the variety which is studied in Chapter 9 of [7]. So let

$$v(x,y) = [[x^d, y^d]^d, [y^d, x^{-d}]^d]$$
$$u(x,y) = v(y,x)^n [y,x]^{\varepsilon_1} v(y,x)^{n+1} [y,x]^{\varepsilon_2} v(y,x)^{n+2} \cdots [y,x]^{\varepsilon_{h-1}} v(y,x)^{n+h-1},$$

where $h \equiv 1 \,(\mathrm{mod}\ 10)$, $\varepsilon_{10k+1} = \varepsilon_{10k+2} = \varepsilon_{10k+3} = \varepsilon_{10k+5} = \varepsilon_{10k+6} = 1$, $\varepsilon_{10k+4} = \varepsilon_{10k+7} = \varepsilon_{10k+8} = \varepsilon_{10k+9} = \varepsilon_{10k+10} = -1$, $k = 0, 1, \cdots, (h-1)/10$ and d, n, h are sufficiently large natural numbers chosen with respect to the restrictions that are introduced in Chapter 7 of [7]. Note that $\varepsilon_1 + \cdots + \varepsilon_{h-1} = 0$.

Let $\mathfrak{M}$ denote the variety defined by the law $[x,y] = u(x,y)$. It is proved in [7] that $\mathfrak{M}$ is nonabelian and torsion-free. Since

$$[[y^d, x^d]^d, [x^d, y^{-d}]^d] = [y^d [x^d, y^{-d}]^d y^{-d}, [x^d, y^{-d}]^d]$$
$$= [[y^d, [x^d, y^{-d}]^d][x^d, y^{-d}]^d, [x^d, y^{-d}]^d],$$

we can see that $v(x,y)$ belongs to $[[X,Y],Y]$ by the Lemma. Therefore $u(x,y)$ belongs to $[[X,Y]Y,]$ in view of the Lemma and the equation $\varepsilon_1 + \cdots + \varepsilon_{h-1} = 0$. Hence $\mathfrak{M}$ is a variety of t-groups by the Theorem.

The following theorem will help us to find conditions under which a pseudoabelian variety consists of t-groups only.

THEOREM 2. *Let a group G contain an element g of finite order n such that the exponents of the odd prime divisors in the prime decomposition of n are at most 1. Then if g fails to normalize a subnormal subgroup of a group G, then G has a metabelian factor.*

PROOF. Let H be a normal subgroup of G and F a normal subgroup of H such that $g^{-1}Fg \neq F$. Without loss of generality we can assume that G is generated by g and H. Let $F_0 = F$, $F_1 = g^{-1}F_0 g$, $F_2 = g^{-1}F_1 g$, $\ldots$, $F_{n-1} = g^{-1}F_{n-2}g$. We shall call a group P an intersection of weight k if P is the intersection of k distinct subgroups from the list $F_0, F_1, \ldots, F_{n-1}$. It is clear that $F_0 \cap F_1 \cap \ldots \cap F_{n-1}$ is a normal subgroup of G. Hence there exists a subgroup Q such that Q is an intersection of certain weight which is not normal in G but any intersection of bigger weight is normal in G. Therefore considering Q instead of H and taking the quotient of G by the product of all the intersections of weight bigger than the weight of Q, we can assume without loss of generality that in the set $F_0, F_1, \ldots, F_{n-1}$ any two groups either coincide or intersect trivially.

If n is a prime, then all $F_0, F_1, \ldots, F_{n-1}$ are distinct and therefore $F_0 F_1 \ldots F_{n-1} = F_0 \times F_1 \times \ldots \times F_{n-1}$. Hence if a is a non-trivial element of F_0, then the group generated by g and a is a nonabelian metabelian group.

Now let $n = 2^m$ where $m > 1$. Consider a non-trivial element a from F_0 such that $g^{-2^i} a g^{2^i} = a$ and i is the smallest positive integer with the property that

g^{2^i} stabilizes at least one non-trivial element from F_0 when conjugating by g^{2^i}. Let $h = g^{2^{i-1}}$, then $h^{-1}(h^{-1}ah)h = a$. Hence $h^{-1}bh = b^{-1}$ where $b = ah^{-1}a^{-1}h$. If $h^{-1}ah$ is not from F_0, then the group generated by h and a is a nonabelian metabelian group. If $h^{-1}ah$ is from F_0, then it follows from the definition of i that b cannot be equal to b^{-1}. Therefore the group generated by h and b is a nonabelian metabelian group.

It remains to observe that the last two paragraphs prove the theorem.

Corollary 3. *Let $\mathfrak{M}$ be a pseudoabelian variety of exponent n such that the exponents of the odd prime divisors in the prime decomposition of n are at most 1. Then $\mathfrak{M}$ is a variety of $t-$ groups.*

Proof. It follows immediately from Theorem 2.

In conclusion, we would like to mention here the fact that according to [3] and [9], there are plenty of examples of pseudoabelian varieties of finite exponent that satisfy the conditions of Corollary 3.

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