On identities defining t-groups

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Abstract

We discuss the situation leading to the conjecture that *a variety of groups consists of t-groups if and only if it is pseudoabelian.*

A variety of groups is the class of groups satisfying a given set of identities. For example the variety of all abelian groups satisfies the identity $[x, y] = 1$, where $[x, y] = x^{-1}y^{-1}xy$. The variety of all metabelian groups is defined by the identity $[[x, y], [z, t]] = 1$. The variety of 2-nilpotent groups satisfies the identity $[[x, y], z] = 1$. By Birkhoff, a variety of groups is also the class of groups closed with respect to taking subgroups, quotient groups and cartesian products.

A group G is called a *t-group* (or is said to have a t-property) if every subnormal subgroup in G is normal. By another words, G is a t-group if for any two subgroups A and B the fact that A is normal in B, and B is normal in G implies that A is normal in G. So the "t" stands for transitivity of normality. The t-groups with additional conditions were studied in [1], [8]. As examples of t-groups we have abelian groups, direct products of simple groups, a quaternion group. A nilpotent group G is a t-group if and only if every subgroup is normal, i.e. either G is abelian or it is a Dedekind group (a direct product of a quaternion group, an elementary abelian 2-group, and an abelian group with all elements of odd order) [8].

The smallest group, which is not a t-group is the dihedral group D_4 of order eight (the symmetry group of a square). It is known ([6], p. 166) that the quaternion group Q_8 and the dihedral group D_4 generate the same variety which is defined by the identities $x^4 = 1,\ [x, y]^2 = 1,\ [[x, y], z] = 1$. Since D_4 is not a t-group, so these identities do not imply the t-property, while the identity $[x, y] = 1$ does imply the t-property.
A natural question arises:

Question 1 *What identities imply the t-property?*

For conveniens we introduce

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Definition 1 *A variety is called a t-variety if it is not abelian and every group in the variety is a t-group.*

The question is answered in [4]. We formulate the result below, where $\langle x \rangle$ denotes an infinit cyclic group geneated by x.

Theorem 1 *An identity implies the t-property if and only if it is of the form*

$$[x, y] = u(x, y), \tag{1}$$

where $u(x, y)$ *belongs to* $[[\langle x \rangle, \langle y \rangle], \langle y \rangle]$.

It is interesting to notice that simple identities of the form (1), as for example $[x, y] = [[x, y], y]$, imply the abelian identity [2], and hence do not define t-varieties. A natural question arises:

Question 2 *Does there exist a t-variety?*

This question has a positive answer. To discuss it we need to define a special type of variety.

Definition 2 *A variety is called pseudoabelian if it is not abelian, but all its finite groups are abelian.*

The pseudoabelian varieties can be defined similarly by mean of metabelian groups instead of finite groups [6].

Theorem 2 *A variety is pseudoabelian if and only if it is not abelian, but all its metabelian groups are abelian.*

Proof Let $\mathcal{M}$ be a pseudoabelian variety and G be a metabelian group in it. If G is not abelian, then we can find a two-generator non-abelian subgroup H in G. By Lemma 7.2 of B. Neumann [5], H has a non-abelian finite factor group, which gives a contradiction.

Conversely, let $\mathcal{M}$ be a non-abelian variety such that all its metabelian groups are abelian. If G is a finite non-abelian group of minimal order in $\mathcal{M}$, then all subgroups in G are abelian, and by a theorem of O. Schmidt, G is a metabelian (non-abelian) group in $\mathcal{M}$ which gives a contradiction.

The question of existence of pseudoabelian varieties was formulated in 1967 [6] and is known as "The Fifth Problem of Hanna Neumann". The Problem was solved positively in 1985 by A. Yu. Ol'shanskii [7], who gave example of a pseudoabelian variety. This example gives a positive answer for the Question 2 because of the following theorem [4].

Theorem 3 *The Ol'shanskii pseudoabelian variety is a t-variety.*

The proof is based on checking that the Ol'shanskii variety satisfies an identity of the type (1).

The following natural question arises:

Question 3 *Is every t-variety pseudoabelian ?*

We can show that the answer is positive.

Theorem 4 *Every variety of t-groups is pseudoabelian.*

Proof. In view of Theorems 1, 2, we have to show that if G is metabelian and satisfies an identity (1), then G is abelian.

By substituting $[y, z]$ for y in the equation $[x, y] = u(x, y)$, we obtain that for $x, y, z \in G$: $[x, [y, z]] \in G'' = 1$ and hence G is nilpotent of class at most 2. Now $[x, y] = u(x, y) \in [[G, G], G] = 1$ and hence G is abelian as required.

We consider now converse to the above statement.

Question 4 *Is every pseudoabelian variety a t-variety?*

This Question is open. To give a positive answer it is enough to show that if an element $g \in G$ fails to normalize a subnormal subgroup of G, then G has a metabelian factor and so cannot belong to any pseudoabelian variety.
There exists an unpublished partial result obtained independently by L. Kovács and Peter M. Neumann [3]: If an element of *squarefree order* fails to normalize a subnormal subgroup of a group G, then G has a metabelian factor. It is shown in [4] that the condition of squarefree order can be weakened to only odd squares.

So, we can only conjecture that *a variety consists of t-groups if and only if it is pseudoabelian.*

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