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Ernest CZOGAŁA, Witold PEDRYCZ
Politechnika Śląska

Siegfried GOTTWALD
Dept. of Mathematics, University of Leipzig, GDR

ON THE CONCEPTS OF MEASURES OF FUZZINESS AND ITS APPLICATION IN DECISION MAKING

Summary. Since the introduction of the concept of fuzzy set, this notion has proved to be useful in many areas of application and much work was done. Only a small portion of that work deals with the problem to evaluate how fuzzy a fuzzy set is. The works of de Luca and Termini continued by e.g. Knopfmacher, Loo, Gottwald are the most important in this topic. De Luca and Termini have formulated entropy and energy concepts of fuzziness measures and they have given an informational interpretation of the entropy of a fuzzy set with respect to decision processes.

In this paper it is shown that the special kind of energy measure the so called degree of fuzziness is better in many situations in decision making. The application of energy measure as a quality index widely used for ill-defined objects fuzzy controller is discussed in details. It was shown that from the epistemological point of view both these concepts of fuzziness measures are equivalent.

The presented kinds of measures of fuzziness shed new light on them and their connection to the measurement of uncertainty in decision making.

1. Introduction

Since the introduction of the concept of fuzzy set, this notion has proved to be useful in many areas of application and much work was done. Only a small portion of that work is concerned with the problem to evaluate how fuzzy a fuzzy set is. Pioneering is the work of of A. de Luca and S. Termini (of [7] and [8] which was continued e.g. by J. Knopfmacher [5], S. Gottwald [2] and also by S.G. Loo [6]).

In the paper [8] A. de Luca and S. Termini have formulated entropy and energy concepts of fuzziness measures and they have given an informational interpretation of the entropy of a fuzzy set with respect to decision processes. In this paper it is shown that the energy kinds of fuzziness measures are better suited for many situation in decision making.

2. Entropy kinds of measures of fuzziness

In building up models with fuzzy sets it is often advisable to posses means to decide which of the two fuzzy sets is "more fuzzy" than the other, and also to have means to describe how different a fuzzy set is from an usual "crisp" set.

Measures of fuzziness are intended to give this information. There are several proposals of such measures, the most general ones are by Knopfmacher [5], S.G. Loo [6] and A. de Luca and S. Termini [7]. But the main interests of the last-named authors are the connections between probability and fuzziness from the information theory point of view.

With some slight additional assumptions on the functions involved which avoid unnecessary generality (cf. S. Gottwald [2], and for the set $W = [0,1]$ of possible generalized membership values of fuzziness measures of Loo and Knopfmacher are of the kind:

$$h(A) = F\left(\sum_{t \in W} a_t \, f(t)\right) \tag{1}$$

with a_t as the number of points $x \in \mathscr{X}$ of the universe of discourse $\mathscr{X}$ such that $\mu_A(x) = t$, and real functions F and f, of the form

$$h(A) = F\left(\int_W a_t f(t)\,dt\right) \tag{2}$$

The function $F\colon \mathbb{R} \to \mathbb{R}$ should be monotonically increasing. The function $f\colon W \to \mathbb{R}_+$, $\mathbb{R}_+$ the set of reals ≥ 0, is subject to the following restrictions:

$$f(0) = f(1) = 0 \tag{3}$$

$f(t)$ is monotonically increasing for $t \in [0, \frac{1}{2})$ and monotonically decreasing for $t \in [\frac{1}{2}, 1]$ (4)

Therefore, each one of these measures of fuzziness is an entropy measure in the sense of de Luca and Termini [8].

In the case of a finite universe of discourse $\mathscr{X} = \{x_1, x_2, \ldots, x_n\}$ or of a finite support

$$|A| = \{x \in \mathscr{X} \mid \mu_A(x) \neq 0\} \tag{5}$$

of the fuzzy set A or only finitely many values $\mu_A(x)$ formula (1) will do justice - also in each other case where, the sum makes sense (e.g. converges as in some considerations of de Luca and Termini). Otherwise one should use (2).

These measures of fuzziness have some fundamental properties one likes to have:

A1. In case of a crisp set B with characteristic function $\chi_B : X \to \{0,1\}$ we have:

$$h(B) = 0$$

A2. The "most fuzzy" subset $\xi_{\frac{1}{2}}$ of X with

$$\xi_{\frac{1}{2}} : X \to \left\{\tfrac{1}{2}\right\} \tag{6}$$

has the greatest measure of fuzziness, i.e. for every fuzzy set $A \in \mathcal{F}(X)$ = the set of all fuzzy subsets of X there holds:

$$h(A) \leq h(\xi_{\frac{1}{2}}) \tag{7}$$

A3. As an intuitively satisfactory ordering of the fuzzy sets from $\mathcal{F}(X)$ one can consider the relation:

$$A_1 \leq' A_2 \quad \text{iff} \quad \mu_{A_1}(x) < \mu_{A_2}(x) \quad \text{for } \mu_{A_2}(x) < \tfrac{1}{2}$$

and

$$\mu_{A_1}(x) \geq \mu_{A_2}(x) \tag{8}$$

for

$$\mu_{A_2}(x) \geq \tfrac{1}{2}$$

With this we have the property:

$$\text{-if } A_1 \leq' A_2, \quad \text{then } h(A_1) \leq h(A_2) \tag{9}$$

which can be understood as: "if A_1 is not more fuzzy than A_2, then A_1 has a measure of fuzziness not greater than A_2".

Sometimes ones likes to have as a property of symmetry for the complement $\neg A$ of fuzzy set A with:

$$\mu_{\neg A}(x) = 1 - \mu_A(x) \tag{10}$$

also

A4.
$$h(A) = h(\neg A) \tag{11}$$

To get this, the function f should be subject to the additional restriction

$$f(t) = f(1-t) \tag{12}$$

Let us now consider some examples of such measures of fuzziness for a finite universe of discourse $\mathcal{X} = \{x_1, x_2, \ldots, x_n\}$

1. Take F_1 = id and

$$f_1(t) = \begin{cases} t & \text{for } t \in [0, \frac{1}{2}) \\ 1-t & \text{for } t \in [\frac{1}{2}, 1] \end{cases} \tag{13}$$

We get:

$$h_1(A) = \sum_{t \in W} a_r f_1(t) = \sum_{\substack{i=1,2,\ldots,n \\ \mu_A(x_i) < \frac{1}{2}}} \mu_A(x_i) + \sum_{\substack{i=1,2,\ldots,n \\ \mu_A(x_i) \geq \frac{1}{2}}} (1 - \mu_A(x_i)) \tag{14}$$

which can be written down in a simplified manner if we use the $\frac{1}{2}$ - level set $A^* = \{x \in \mathcal{X} \mid \mu_A(x) \geq \frac{1}{2}\}$ and its characteristic function χ_{A^*}

$$h_1(A) = \sum_{i=1}^{n} \left| \mu_A(x_i) - \chi_{A^*}(x_i) \right| \tag{15}$$

This measure looks like a generalized Hamming distance and was proposed by A. Kaufmann [3].

2. Take $F_2(z) = z^{\frac{1}{2}}$ and

$$f_2(t) = \begin{cases} t^2 & \text{for } t \in [0, \frac{1}{2}) \\ (1-t)^2 & \text{for } t \in [\frac{1}{2}, 1] \end{cases} \tag{16}$$

We get in the same way

$$h_2(A) = \Big(\sum_{\substack{i=1,2,\ldots,n \\ \mu_A(x_i) < \frac{1}{2}}} \mu_A^2(x_i) + \sum_{\substack{i=1,2,\ldots,n \\ \mu_A(x) > \frac{1}{2}}} (1 - \mu_A(x_i))^2 \Big)^{\frac{1}{2}} = \Big(\sum_{i=1}^{n} \left| \mu_A(x_i) - \chi_{A^*}(x_i) \right|^2 \Big)^{\frac{1}{2}} \tag{17}$$

which is the Euclidean norm of the distance of the functions μ_A and χ_{A^*} (cf also [3]).

3. Take F_3 = id and

$$f_3(t) = t \log t - (1-t) \log(1-t) \tag{18}$$

Then we get:

$$h_3(A) = \sum_{t \in W} - a_t(t\log t + (1-t)\log(1-t) =$$
$$= \sum_{i=1}^{n} (- \mu_A(x_i)\log \mu_A(x_i) - \mu_{\neg A}(x_i)\log \mu_{\neg A}(x_i)) \tag{19}$$

whic is already in [7] as entropy of the fuzzy set A.

For some fuzzy sets A let us calculate their measures $h_1(A)$ of fuzziness. Consideration of $h_2(A), h_3(A)$ or other examples would give rise to the same problems.

a) The entropy measure of fuzziness shows no difference between fuzzy sets presented in Fig. 1. Considering equation (13) assuming that F = id,

$$f(t) = \begin{cases} t, & t \in [0,\frac{1}{2}) \\ t-1, & t \in [\frac{1}{2},1] \end{cases}$$

we obtain $h(A_1) = h(A_2) = 0.4$.

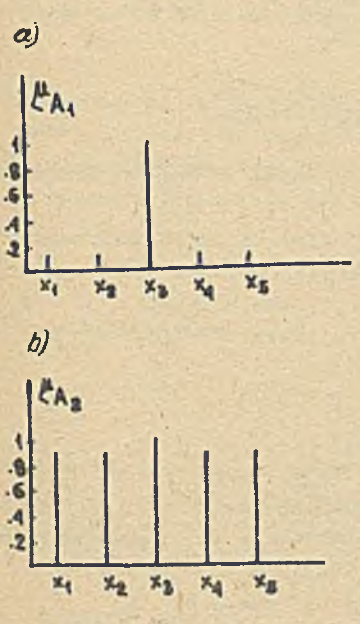

Fig. 1a,b. Membership functions of fuzzy sets A_1 and A_2

From the decision making point of view there is a difference because it is easy to see that the alternative x_3 in A_1 is a dominant one and there is no doubt that a decision maker should choose x_3 in A_1. There is another situation with the fuzzy set A_2 from the decision making point of view. The decision maker could hesitate which x_i to choose. We can say intuitively that fuzziness of the fuzzy set A_2 is greater than fuzziness of A_1.

Let us consider now two crisp sets (sharp situation) presented in Fig.2. In set A_1 there is no doubt that the decision maker will choose x_1, for set A_2 each x_i could be good. The measure of fuzziness is also the same for both sets because $h(A_1) = h(A_2) = 0$ although one can easily feel that the "fuzziness" of set A_2 is greater than the "fuzziness" of the set A_1. Taking into account the above presented examples and situations from decision making point of view we try to use energy measure of fuzziness which gives different results.

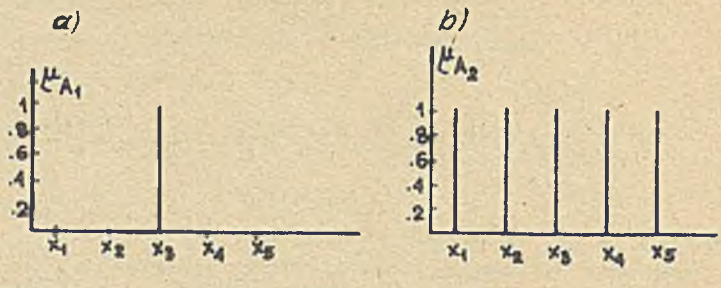

Fig. 2a,b. Membership functions of fuzzy sets A_1 and A_2

3. Energy kinds of measures of fuzziness

All the mentioned measures of fuzziness describe a kind of difference between fuzzy sets and give the crisp sets the measure zero.

Our discussion has convinced us that another kind of measure of fuzziness is also desirable: one which is strongly connected with the incertainty of a decision maker, he has to choose one alternative out of a fuzzy set of alternatives.

Let us consider only normalized fuzzy sets. Then the very last situation for the decision maker is to have such a fuzzy set A of given alternatives which has a 1-level set

$$A^{(1)} = \left\{ x \in \mathscr{X} \mid \mu_A(x) = 1 \right\} \tag{20}$$

which is a singleton. Furthermore, always a fuzzy set as in Fig 1a should be "more fuzzy" than in Fig. 1b.

We will try to get formulas (1) and (2) also to cover this new situation. The essential change we will do with the function f; F again is supposed to be monotonically increasing. For f, the property f(1) = 0 is without value now. Hence, let us place the following restriction on functions f:

(i) f(0) = 0

(ii) f(1) = 1

(iii) f is monotonically increasing.

Property (ii) is only subject to some normalization $f(1) \neq 0$ would be equally good. Let us write $h^*(A)$ instead of $h(A)$ to indicate that f should have these properties (i), (ii), (iii).

The main properties of the new measures h^* now are:

B1. In case of a crisp singleton $B = \left\{x_o\right\}$ we have:

$$h^*(B) = 1 \tag{21}$$

B2. Let A be a normalized fuzzy set, which therefore has nonempty support. Then

$$h^*(A) \geq 1 \tag{22}$$

and $h^*(A) = 1$ if and only if A is a singleton.

B3. With the usual inclusion of fuzzy set A_1, A_2

$$A_1 \subseteq A_2 \quad \text{if } (\forall x \in \mathscr{X}) \; (\mu_{A_1}(x) \leq \mu_{A_2}(x)) \tag{23}$$

we have:

$$\text{if} \quad A_1 \subseteq A_2, \quad \text{then} \quad h^*(A_1) \leq h^*(A_2) \tag{24}$$

B4. The "most fuzzy" set is now the fuzzy subset of $\mathscr{X}$ with:

$$\xi_1 : \mathscr{X} \to \{1\}$$

Regarding the examples of the previous section one can obtain that for 1. we get:

$$h^*(A_1) = 1 + 0.4 = 1.4$$

$$h^*(A_2) = 1 + 3.6 = 4.6$$

Similarly for 2. $h^*(A_1) = 1$, $h^*(A_2) = 4$, so the set A_2 has greater value of proposed measure of fuzziness than A_1.

It could be shown that the measure of fuzziness proposed by E. Czogała and W. Pedrycz [1] called degree of fuzziness given in the form:

$$\varphi(A) = \sum_{x \in \mathscr{X}} \mu_A(x) \tag{25}$$

or in normalized form:

$$\psi(A) = \frac{1}{\max\limits_{x \in \mathscr{X}} \mu_A(x)} \sum_{x \in \mathscr{X}} \mu_A(x) \tag{26}$$

is a special case of the measure proposed above because the following

$$\varphi(A) = \sum_{x \in \mathscr{X}} \mu_A(x) = \sum_{t \in W} (\text{number of } x \in \mathscr{X} \text{ with } \mu_A(x) = t), \; t = h^*(A) \tag{27}$$

holds true.

We can prove that the degree of fuzziness does not recognize such a situation as shown in Fig. 3. We get $\varphi(A_1) = 2$ and $\varphi(A_2) = 2$, but if we put down:

$$f(t) = t^2 \quad \text{for} \quad t \in [0,1] \tag{28}$$

or generally

$$f(t) = t^n \quad \text{for} \quad t \in [0,1] \tag{29}$$

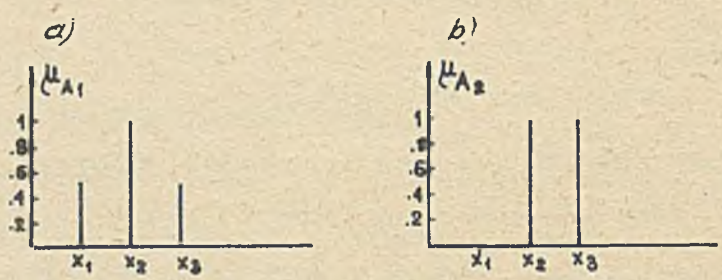

Fig. 3a,b. Membership functions of fuzzy sets A_1 and A_2

we will get different results for such sets. Taking n=2 we calculat $h^*(A_1) = 1.5$, $h^*(A_2) = 2$, what is a good result from intuition and decision making point of view, because in the A_1 there is no doubt about choosing the element x_2 and in A_2 there is uncertainty if to choose x_2 or x_3.

4. Application of energy measure of fuzziness in decision making processes

Now we present some ideas of the use of energy measure of fuzziness as a quality index in decision making processes such as fuzzy controller in the form presented by Mamdani [4] or Tsukamoto [10]. Fuzzy controller usually used in control of ill-defined objects consists of a collection of control decision making rules:

- if the situation state of the object is X_i then control or decision variable is equal to U_i.

where X_i, U_i, $i = 1,2,\ldots,N$ are fuzzy sets defined in the spaces $\mathcal{X}$ and $\mathcal{U}$ respectively. Thus having a set of rules Eq. for each $X' \in \mathcal{F}(\mathcal{X})$ U' could be computed using fuzzy compositional rule of inference:

$$U' = X' \circ R \tag{30}$$

where assuming a noncompetitivity of rules, R is calculated as

$$R = \bigcup_{i=1}^{N} (X_i \times U_i) \tag{31}$$

i.e.

$$\mu_R(x,u) = \max_{1 \leq i \leq n} \left[\min(\mu_{X_i}(x), \mu_{U_i}(u))\right] \tag{32}$$

Similarly:

$$\mu_{U'}(u) \quad \max_{x \in \mathcal{X}} \left[\min(\mu_{X'}(x), \mu_R(x,u))\right] \tag{33}$$

Now we sould discuss the quality of fuzzy controller taking into account a degree of fuzziness given by Eq. 25.

We say that a fuzzy controller has a $\mathcal{E}$-degree of fuzziness if the following condition is hold:

$$\forall_{X' \in \mathcal{F}(\mathcal{X})} \quad \varphi(X') \leq \delta \Longrightarrow \varphi(U') \leq \mathcal{E} \qquad \delta, \mathcal{E} > 0 \tag{34}$$

so for each fuzzy information X' with degree of fuzziness less than δ, fuzzy controller should give a fuzzy set U' (decision) with degree of fuzziness which is smaller than $\mathcal{E}$, so a decision maker may not doubt which x_i to choose $(i = 1,2,\ldots,n)$.

Similarly using the notion of fuzzy Łukasiewicz logic in fuzzy controller, the degree of fuzziness may be used in creation of the following criterion of quality of controller [9]:

$$\forall_{X' \in \mathcal{F}(\mathcal{X})} \quad \exists_{1 \leq j \leq n} \quad \varphi(U'_j) \leq \mathcal{E} \qquad \mathcal{E} > 0$$

where j stands for the index of decision rule.

Let us illustrate our considerations presenting numerical examples, discussing two collections of control rules:

controller no. 1

if X is <u>big</u> then U is <u>big</u>

if X is <u>small</u> then U is <u>small</u>

where <u>big</u>, <u>small</u> are fuzzy sets defined on $\mathcal{X}$ and $\mathcal{U}$ as follows:

$\mathcal{X}$	x_1	x_2	x_3	x_4	x_5
μ_{big}	0	0	0	.9	1
μ_{small}	1	.9	0	0	0

U	u_1	u_2	u_3	u_4
$\mu_{\underline{big}}$	.1	.2	.6	1
$\mu_{\underline{small}}$	1	.3	.1	0

Controller no. 2. It consists of the same rules presented in controller no. 1 but fuzzy sets have different membership functions:

X	x_1	x_2	x_3	x_4	x_5
$\mu_{\underline{big}}$	0	.3	.6	.9	1
$\mu_{\underline{small}}$	1	.8	.5	.2	0
U		u_1	u_2	u_3	u_4
$\mu_{\underline{big}}$		.1	.4	.9	1
$\mu_{\underline{small}}$		1	.8	.5	.2

Thus the matrices relations R_1, R_2 of fuzzy controllers are equal to:

$$R_1 = \begin{bmatrix} 1 & .3 & .1 & 0 \\ .9 & .3 & .1 & 0 \\ 0 & 0 & 0 & 0 \\ .1 & .2 & .6 & .9 \\ .1 & .2 & .6 & 1 \end{bmatrix}$$

$$R_2 = \begin{bmatrix} 1 & .8 & .5 & .2 \\ .8 & .8 & .5 & .3 \\ .5 & .5 & .6 & .6 \\ .2 & .4 & .9 & .9 \\ .1 & .4 & .9 & .1 \end{bmatrix}$$

Considering a collection of testing sets (normal sets with degree of fuzziness equal to 1) i.e.

$$\begin{gathered} [1 \;\; 0 \ldots 0] \\ [0 \;\; 1 \ldots 0] \\ \vdots \\ [0 \ldots 0 \;\; 1] \end{gathered}$$

after calculations we get the following:

controller no. 1. $\varphi(X') = 1 \Rightarrow \varphi(U') \leq 2$

controller no. 2. $\varphi(X') = 1 \Rightarrow \varphi(U') \leq 3.7$

what is in agreement with the intuition viz. fuzzy controller no. 2. gives more "fuzzy" information about control variable.

5. Concluding remarks

A. de Luca and S. Termini provided us a general setting of entropy and energy measures of a fuzzy set. Let us remind essential definitions of the concepts of energy and entropy of the fuzziness. Regarding a sextuple $\langle P, \leq, \leq', i, o, o \rangle$ where P is a set partially ordered by $\leq$ and $\leq'$ i, o are the maximum and minimum of P relative to $\leq$ and both are minimal elements with respect to $\leq'$ o is the maximal element of P relative to $\leq'$. Denoting by M the set of minimal elements of P with respect to $\leq'$ there are introduced the following general definitions:

1. An E-function is any map $e: P \to R$ such that e is isotone with the order $\leq$, i.e. for all $p,q \in P$ $p \leq q \Rightarrow h(p) \leq h(q)$.

2. An H-function is any map $h: P \to R$ such that h is isotone with the order $\leq'$ i.e. for all $p,q \in P$ $p \leq' q \Rightarrow h(p) \leq h(q)$ and $h(p) = 0$ iff $p \in M$.

Energy and entropy of a fuzzy set are meant an E-function and H-function defined on $\mathcal{L}(\mathcal{X})$ respectively, where $\mathcal{L}(\mathcal{X})$ is a complete lattice with respect to the operations $(\vee)$ and $(\wedge)$.

Each of these measures of fuzziness given by the formulas 1 and 2 given in this paper under assumptions 1. and 2. is an entropy measure in the sense of de Luca and Termini.

From the epistemological point of view both conceptions of fuzziness measures are equivalent. But from the decision making point of view the energy kind of measure seems to be more useful and intuitively clear. It is shown that the presented kinds of measures of fuzziness shed new light on these functions and their connection to the measurement of uncertainty in decision making.

REFERENCES

[1] Czogala E., Pedrycz W.: An approach to the identification of fuzzy systems, Archiwum Automatyki i Telemechaniki, 1, 1980, 43-50.

[2] Gottwald S.: A note on measures of fuzziness, EIK, 15, 1979, 221-223.

[3] Kaufmann A.: Theorie des sous-ensembles flous, vol. 1, Masson et Cie., Paris, 1973.

[4] King P.J., Mamdani E.: The application of fuzzy control systems to industrial processes, IFAC 6th Triennal World Congress, Boston Cambridge 1975.

[5] Knopfmacher J.: On measures of fuzziness, J. Math. Anal.Appl.49,1975. 529-534.

[6] Lee S.G.: Measures of fuzziness, Cybernetica, 20, 1977, 201-210.

[7] de Luca A, Termini S.: A definition of a nonprobabilistic entropy in the setting of fuzzy sets theory, Information Control, 20, 1972,301-312.

[8] Entropy and energy measures of a fuzzy set. Advances in Fuzzy Set Theory and Appl., North Holland, Amsterdam, 1979, 321-338.

[9] Pedrycz W.: On fuzzy Łukasiewicz logic in control problem in fuzzy systems, Archiwum Automatyki i Telemechaniki (to appear).

[10] Tsukamoto Y., Takagi T., Sugeno M.: Fuzzification of $L_{\text{Aleph-1}}$ and its application to control. Proc. Int. Conf. on Cybernetics and Society, 2, 1217-1221, Tokyo 1978.

Recenzent: Prof. dr hab. inż. Andrzej Tylikowski

МЕРЫ РАСПЛЫВЧАТОСТИ И ИХ ПРИМЕНЕНИЕ В ПРОЦЕССЕ ПРИНИМАНИЯ РЕШЕНИЙ

Р е з ю м е

Концепция расплывчатых множеств оказалась плодотворной во многих применениях. Однако только небольшое количество работ связано с проблемой оценивания (характеризации) расплывчатого множества. В этом отношении основными является работы де Люка, Термини, Ли, Кнопфмахера и Готтвадьда.

Де Люка и Термини сформулировали понятия о энтропийных и энергетических мерах расплывчатого множества, представляя одновременно применение энтропии к вопросам принятия решений.

В данной работе представлено особый вид энергетической меры, так называемый уровень расплывчатости множества, которая во многих случаях применимая в процессе принятия решений (например для проектирования расплывчатого регулятора). С эпистомологической точки зрения обе концепции мер тождественные.

KONCEPCJA MIAR ROZMYTOŚCI I ICH ZASTOSOWANIE W PODEJMOWANIU DECYZJI

S t r e s z c z e n i e

Od czasu wprowadzenia koncepcji zbiorów rozmytych pojęcie to okazało się użyteczne w wielu dziedzinach zastosowań, co potwierdza szereg prac. Tylko niewielka ich liczba związana jest z problemem oceny (charakteryza-

cji zbioru rozmytego. W tym zakresie podstawowymi pracami są prace de Luca, Terminiego, Loc, Knopfmachera, Gottwalda. De Luca i Termini sformułowali pojęcia miar entropowych i energetycznych zbioru rozmytego, prezentując jednocześnie wykorzystanie entropii w problemach podejmowania decyzji.

W niniejszej pracy zaprezentowano specjalny rodzaj miary energetycznej, tzw. stopień rozmycia zbioru, w wielu sytuacjach przydatny w podejmowaniu decyzji (np. dla projektowania regulatora rozmytego). Z epistemologicznego punktu widzenia obie koncepcje miar są równoważne.