Algebra and Discrete Mathematics
Number ?. **(????).** pp. 1–16

RESEARCH ARTICLE

Dense subgroups in the group of interval exchange transformations

Agnieszka Bier and Vitaliy Sushchanskyy

10.03.2014

Abstract. The paper concerns the characterization of the group IET of interval exchange transformations (iet). We investigate a class of rational subgroups of IET. These are subgroups consisting of iet transformations defined by partitions with rational endpoints. We propose a characterization of rational subgroups in terms of infinite supernatural numbers and prove that every such subgroup is dense in IET. We also discuss the properties of these groups.

Introduction

The study of interval exchange transformations (iet) can be considered as a classical topic in modern dynamics. It was initiated in the seventies by works of Keane [14, 15], Rauzy [22], Veech [23], and quite recently it was brought back to attention by Avila, Forni, Gouezel, Viana and Yoccoz [2, 3, 4, 5, 10], who gave solutions to certain open problems. The interest in the set IET of all iets on a given interval is mainly due to its rich dynamical properties, however recently the group structure of IET also draws attention.

In the presented paper we introduce a class of rational subgroups of IET and characterize them by means of supernatural numbers. As a main result we distinguish in this class those subgroups which are dense in IET and discuss their properties.

The paper is organized as follows. Section 2 contains some preliminary notation and basic facts on iets. We summarize dynamical proper-

2001 Mathematics Subject Classification: *37B05, 28D05, 37A05..*
Key words and phrases: *Interval exchange transformations, rational subgroups, dense subgroups, supernatural numbers.*

ties of iets and the group structure of IET. In Section 3, we propose a characterization of a class of rational subgroups of IET using the notion of divisible sequences and supernatural numbers. We prove the necessary and sufficient condition for a subgroup of this class to be dense in IET. The properties of the introduced rational subgroups are investigated in Section 4.

1. Preliminaries

Let $I = [a, b)$ be an interval and let $\pi = \{a_0, a_1, ..., a_n\}$, $a = a_0 \leq a_i < a_{i+1} \leq a_n = b$ be the partition of I into n subintervals $[a_i, a_{i+1})$, $i = 0, ..., n-1$. An interval exchange transformation (iet) of I is a left-continuous bijection, which acts a shuffling of the subintervals $[a_i, a_{i+1})$, i.e. a piecewise translation. We denote the set of all interval exchange transformations of I by IET. Every transformation $f \in IET$ may be represented as a pair $f = (\pi, \sigma)$ for certain n, where π is a partition of I into n subintervals and $\sigma \in S_n$ is a permutation of the set of n elements. Let $f = (\pi_1, \sigma_1)$ and $g = (\pi_2, \sigma_2)$ be the iets defined by $\pi_1 = \{a_0, a_1, ..., a_n\}$, $\pi_2 = \{b_0, b_1, ..., b_m\}$ and $\sigma_1 \in S_n$, $\sigma_2 \in S_m$. It is clear that the composition fg is again an iet (π, σ), where:

$$\begin{array}{rcl} \pi & = & \{c_0, c_1, ..., c_n\}, \qquad a = c_0 \leq c_i < c_{i+1} \leq c_n = b, \\ c_i & \in & \{f(a_0), ..., f(a_n)\} \cup \{b_0, ..., b_n\}, \quad i \in \{0, ..., n+m\}, \end{array}$$

and $\sigma \in S_{n+m}$. The inverse of iet $f = (\pi, \sigma)$ is an iet $f^{-1} = (\pi^\sigma, \sigma^{-1})$, where

$$\pi^\sigma = \{b_0, b_1, ..., b_n\}, \quad b_i \in \{f(a_j) \mid j = 0, 1, ..., n\}, \; b_j < b_{j+1}.$$

Hence the set IET with the operation of composition of transformations forms a group, called *the group of interval exchange transformations*.

Note that the choice of a particular interval I does not decide on the properties of IET, i.e. for any two half-open intervals I_1 and I_2, the respective groups of interval exchange transformations are isomorphic (and the isomorphism is a continuous mapping). Hence in the following we fix $I = [0, 1)$. The interval exchange transformation may be also defined by plotting its graph. For example, the transformation $g = \big(\{0, \frac{1}{2}, 1\}, (1, 2)\big)$ is depicted on the graph Fig. 1.

We note that a particular graph corresponds to many pairs (π, σ). For example, the graph of $h = (\{0, \frac{1}{2}, \frac{3}{4}, 1\}, (132))$ coincides with the one of g and we see that g and h are *equivalent* iets. Hence, to avoid confusion, we introduce the notion of *canonical form* of an iet.

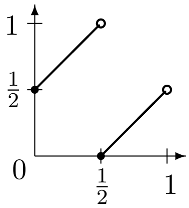

Figure 1: The interval exchange transformation g.

Let f be an iet. By $d(f)$ we denote the number of continuous pieces in the graph of f. This is obviously a finite positive number uniquely determined by f. The pair (π, σ) corresponding to an iet f is called the *canonical form* of f, if π is the partition of interval $[0,1)$ into exactly $d(f)$ subintervals, determined by the discontinuity points of f, and σ is the respective permutation from $S_{d(f)}$.

Directly from the definition we deduce

Lemma 1.1. *Every iet has a unique canonical form.*

Investigations of IET originate from the classical works of Keane [14, 15], Rauzy [22] and Veech [23].

The first results attribute to Keane. In his paper [14] he considered ergodic probability measures invariant under interval exchange transformations. It is clear that the Lebesgue measure is invariant to every iet. The question is, whether there exist other measures preserved by iet (that is, whether iet is uniquely ergodic or not). We recall the *minimality condition* for the interval exchange transformation defined in [14]: an iet satisfies the minimality condition, if the orbit $O(x)$ is dense in I for every $x \in I$.

Keane conjectured that every iet satisfying the minimality condition was uniquely ergodic. He proved the minimality condition for transformations of certain classes, such as transformations with *i.o.d.c.* and *irrational interval exchange transformations.* We say that an iet $f = (\pi, \sigma)$, where π and σ are defined as in Section 2, satisfies the *infinite distinct orbit condition (i.d.o.c.)*, if the orbits $O(a_0)$, $O(a_1)$, ..., $O(a_n)$ are infinite and distinct. An irrational interval exchange transformation is an iet $f = (\pi, \sigma)$, where π is the partition such that the relation $\sum_{i=1}^{n}(a_i - a_{i-1}) = 1$ and its multiples are the only rational relations between the lengths of subintervals, and σ is an irreducible permutation, i.e. does not fix any subset $\{1, 2, ..., j\}$ for $1 \le j \le n-1$.

There had been already known examples by Keynes and Newton [16], Keane [15] and Veech [24] of non uniquely-ergodic transformations, which satisfy the minimality condition. The restricted conjecture, that almost all minimal transformations are uniquely ergodic was proved later, in 1982, independently by Veech [25] and Masur [18]. We also mention the work of Boshernitzan [7], who proved the unique ergodicity for transformations satisfying the diophantine condition. More recent results include for example the weak-mixing property of iets [2], or the investigation of subsets of IET, for which the Keane's conjecture holds [1].

The dynamics of interval exchange transformations can be also considered in terms of group actions. We showed that the set IET considered with the operation of composing transformations forms a group. The major question is whether a given group G may be realized as a subgroup of IET. For example, it is a well known open problem, raised by Katok [13], whether IET contains a free subgroup of rank 2. Another one is due to Grigorchuk and concerns the problem of existence of subgroups of intermediate growth in IET.

By now these problems in general remain open, however there are some partial results. Katok's problem is discussed by Dahmani, Fujiwara and Giurardel in [9]. For instance it is proved that the non-abelian free subgroups of rank 2 in IET are quite rare in the following sense. The authors consider subgroups of IET generated by two iets $f \in IET$ and $g \in IET_\sigma$, where IET_σ is the set of all iets with canonical form defined for a given permutation $\sigma \in S_n$. They prove the following

Theorem 1.2. *There is a dense open subset A of $IET \times IET_\sigma$ such that the subgroup generated by the pair $(f, g) \in A$ is not free.*

In the above statement the notion "dense subset" is used in terms of the *natural topology* on IET. The topology arises from the mutual correspondence between the subsets IET_σ, $\sigma \in S_{n+1}$, and the standard n-dimensional open simplex

$$\Delta_n = \{(x_1, x_2, ..., x_{n+1}) \in \mathbb{R}_+{}^{n+1} \mid \sum_{i=1}^{n+1} x_i = 1\},$$

which is given by:

$$\begin{aligned}
&\psi : IET_\sigma \longrightarrow \Delta_n \\
&\psi(\{a_0, a_1, ..., a_{n+1}\}, \sigma)) = (a_1 - a_0, a_2 - a_1, ..., a_{n+1} - a_n).
\end{aligned}$$

The assumptions imply that IET is a disjoint union of all IET_σ, and hence if all of IET_σ are defined to be open, we obtain the topology on IET. We note that IET is not a topological group with this topology, as the operation of composition of iets is not continuous. For group topologies on IET see [20].

In [9] the authors provide also an example of a 2-generated subgroup G of IET, which contains a free subsemigroup and is universal in the way that it contains an isomorphic copy of any finite group. They also prove that every finitely presented subgroup of IET is residually finite and that IET does not contain infinite Kazhdan groups.

The Katok's problem is also discussed in [21]. The author proves that a free subgroup of IET (if it exists) contains no disjoint rotations, that is iets which act as rotations on finite number of invariant subintervals.

Another scope of research concerns the group theoretical properties of IET. In [19] it is proved that the group IET contains no distortion elements. The result yields examples of groups that do not act faithfully by iets. Additionally, the author provides the classification of centralizers in IET and the characterization of the automorphism group of IET:

$$Aut(IET) \cong IET \rtimes \mathbb{Z}/2\mathbb{Z}$$

The derived subgroup IET' of IET was first characterized by Sah, who also proved that IET' is simple and characterized the quotient group $IET \,/\, IET'$. However these results have not been published until a paper of Veech [26]. In [28] the author formulates some of Sah previous results and proves even more:

Theorem 1.3. *The derived subgroup IET' of the group IET coincides with each of the following groups:*

(i) *subgroup of iets with zero SAF invariant*

(ii) *subgroup generated with iets of order 2*

(iii) *subgroup generated with iets of finite order*

The subset of interval exchange transformations which are defined by the rational partitions (i.e. partitions $\pi = \{a_0, a_1, ..., a_n\}$ with all endpoints a_i being rational numbers) is denoted by $RIET$. The elements of $RIET$ are called rational interval exchange transformations. It is clear that $RIET$ is a subgroup of IET. This subgroup is an object of interest of [11, 12]. The paper [11] concerns the characterization of the K_0-functor and investigations of characters in $RIET$. The latter topic is developed in [12]. It is proved that every indecomposable character of $RIET$ is a power of the so-called *natural* character, i.e. the function that maps given iet f to the measure of the set of the fixed points of f.

2. Supernatural numbers and subgroups of IET.

In this section we characterize a class of subgroups of IET, using a construction which is consistent with the general construction of the group $RIET$. We also investigate certain properties of the constructed subgroups.

2.1. Subgroups of $\mathbb{Q}$ containing $\mathbb{Z}$.

For the construction we use the notion of divisible sequences of integer numbers, which arise naturally in the study of subgroups of $\mathbb{Q}$ – the additive group of rational numbers. If $\frac{p}{q} \in \mathbb{Q}$ is an irreducible fraction, then the smallest subgroup of $\mathbb{Q}$ containing $\mathbb{Z}$ and $\frac{p}{q}$ will be denoted by $\mathbb{Z}\left(\frac{p}{q}\right)$. We shall start here with a few simple observations (in the following $GCD(a,b)$ and $LCM(a,b)$ denote the greatest common divisor and the least common multiple of numbers a and b, respectively).

Lemma 2.1. *(Properties of subgroups $\mathbb{Z}\left(\frac{p}{q}\right)$)*

(i) *If $\frac{p}{q}$ is an irreducible fraction, then $\mathbb{Z}\left(\frac{p}{q}\right) = \mathbb{Z}\left(\frac{1}{q}\right)$.*

(ii) $\mathbb{Z}\left(\frac{1}{q_1}\right) \subseteq \mathbb{Z}\left(\frac{1}{q_2}\right) \quad \Leftrightarrow \quad q_1$ *divides* q_2.

(iii) *For every natural numbers $q_1, q_2 \in \mathbb{N}$ we have*

$$\langle \mathbb{Z}\left(\frac{1}{q_1}\right) \cup \mathbb{Z}\left(\frac{1}{q_2}\right) \rangle = \mathbb{Z}\left(\frac{1}{LCM(q_1,q_2)}\right) \subseteq \mathbb{Z}\left(\frac{1}{q_1 q_2}\right).$$

Proof.

(i) The inclusion $\mathbb{Z}\left(\frac{p}{q}\right) \subseteq \mathbb{Z}\left(\frac{1}{q}\right)$ is obvious.

Now, let $\frac{p}{q}$ be an irreducible fraction, i.e. $GCD(p,q) = 1$. Then there exist integers $m,n \in \mathbb{Z}$ such that $mp + nq = 1$, and since $\frac{p}{q}$ is an element of $G = \mathbb{Z}\left(\frac{p}{q}\right)$, then $m \cdot \frac{p}{q} + n \in G$. Thus $\frac{mp+nq}{q} = \frac{1}{q}$ lies in G and the reverse inclusion holds.

(ii) Assume first the inclusion $\mathbb{Z}\left(\frac{1}{q_1}\right) \subseteq \mathbb{Z}\left(\frac{1}{q_2}\right)$ holds. It follows that $\frac{1}{q_1} \in \mathbb{Z}\left(\frac{1}{q_2}\right)$ and hence there exist $p \in \mathbb{Z}$ such that $\frac{1}{q_1} = \frac{p}{q_2}$. Thus $q_2 = p \cdot q_1$, i.e. q_1 divides q_2.

Conversely, assume q_2 to be divisible by q_1, i.e. there exists $s \in \mathbb{Z}$ such that $q_2 = q_1 \cdot s$. Let x be an arbitrary element of $\mathbb{Z}\left(\frac{1}{q_1}\right)$. Then we have

$$x = \frac{p}{q_1} = \frac{p \cdot s}{q_2} \in \mathbb{Z}\left(\frac{1}{q_2}\right),$$

thus $\mathbb{Z}\left(\frac{1}{q_1}\right) \subseteq \mathbb{Z}\left(\frac{1}{q_2}\right)$.

(iii) The inclusion $\mathbb{Z}\left(\frac{1}{q_i}\right) \subseteq \mathbb{Z}\left(\frac{1}{LCM(q_1,q_2)}\right)$, $i = 1, 2$, follows by (ii). Hence we have

$$\langle \mathbb{Z}\left(\frac{1}{q_1}\right) \cup \mathbb{Z}\left(\frac{1}{q_2}\right) \rangle \subseteq \mathbb{Z}\left(\frac{1}{LCM(q_1,q_2)}\right).$$

Now let $q = LCM(q_1, q_2)$ and $d = GCD(q_1, q_2)$. The extended Euclidean algorithm provides the existence of integers m and n, such that

$$d = mq_1 + nq_2.$$

Hence we obtain:

$$\frac{1}{q} = \frac{d}{q_1 q_2} = \frac{mq_1 + nq_2}{q_1 q_2} = \frac{m}{q_2} + \frac{n}{q_1} \in \mathbb{Z}\left(\frac{1}{q_1}\right) \cup \mathbb{Z}\left(\frac{1}{q_2}\right).$$

It follows that $\mathbb{Z}\left(\frac{1}{LCM(q_1,q_2)}\right) \subseteq \langle \mathbb{Z}\left(\frac{1}{q_1}\right) \cup \mathbb{Z}\left(\frac{1}{q_2}\right) \rangle$.

Obviously, by (ii) we have $\mathbb{Z}\left(\frac{1}{LCM(q_1,q_2)}\right) \subseteq \mathbb{Z}\left(\frac{1}{q_1 q_2}\right)$.

□

A sequence of natural numbers $\bar{n} = (n_1, n_2, ...)$ is called the *divisible sequence*, if $n_i | n_{i+1}$ (n_i divides n_{i+1}) for every $i \in \mathbb{N}$. In the divisible sequence $\bar{n}$ the set of prime divisors of n_i is contained in the set of prime divisors of n_{i+1} and this includes also the multiplicities of these divisors. Thus with every divisible sequence $\bar{n}$ one may associate the *supernatural number* $\hat{n}$, defined as the formal product

$$\hat{n} = \prod_{p_i \in P} p_i^{\varepsilon_i},$$

where P denotes the (naturally ordered) set of all primes, and $\varepsilon_i \in \mathbb{N} \cup \{0, \infty\}$ for every $i \in \mathbb{N}$. The supernatural number $\hat{n}$ associated to the divisible sequence $\bar{n}$ is called the *characteristic* of this sequence. Obviously, two different divisible sequences may have the same characteristic.

The set of all supernatural numbers will be denoted by $s\mathbb{N}$. Note that $s\mathbb{N}$ contains $\mathbb{N}$ and every natural number n is the characteristic of a constant divisible sequence $\bar{n} = (n, n, n, ...)$. Also, for convenience, in the set $s\mathbb{N}$ we introduce the following equivalence relation. We say that the supernatural numbers

$$\hat{m} = \prod_{p_i \in P} p_i^{\alpha_i}, \qquad \hat{n} = \prod_{p_i \in P} p_i^{\beta_i}, \qquad \alpha_i, \beta_i \in \mathbb{N} \cup \{0, \infty\},$$

are *equivalent*, which we denote $\hat{m} \equiv \hat{n}$, if $\alpha_i = \beta_i$ for almost all $i \in \mathbb{N}$, and if $\alpha_i \neq \beta_i$, then both of these exponents are finite.

Let $\bar{n} = (n_1, n_2, ...)$ be a divisible sequence. For every n_i we construct the subgroup $\mathbb{Z}\left(\frac{1}{n_i}\right)$ of $\mathbb{Q}$. By Lemma 2.1 (iii) we obtain an ascending series of proper subgroups in $\mathbb{Q}$:

$$\mathbb{Z}\left(\frac{1}{n_1}\right) \subseteq \mathbb{Z}\left(\frac{1}{n_2}\right) \subseteq \mathbb{Z}\left(\frac{1}{n_3}\right) \subseteq ...,$$

and we define

$$\mathbb{Z}(\bar{n}) = \bigcup_{i=1}^{\infty} \mathbb{Z}\left(\frac{1}{n_i}\right),$$

which is proper subgroup of $\mathbb{Q}$.

A complete characterization of subgroups of the additive group $\mathbb{Q}$ was given by Beaumont and Zuckerman in [6]. We take our interest on these subgroups of $\mathbb{Q}$, which contain the group $\mathbb{Z}$.

Theorem 2.2.

(i) *Every proper subgroup of $\mathbb{Q}$ containing the group $\mathbb{Z}$ is of the form $\mathbb{Z}(\bar{n})$ for some divisible sequence $\bar{n}$.*

(ii) *The subgroups $\mathbb{Z}(\bar{n})$ and $\mathbb{Z}(\bar{m})$ of $\mathbb{Q}$ are isomorphic if and only if $\hat{n} \equiv \hat{m}$.*

Proof.

(i) Let G be a group such that $\mathbb{Z} \leq G \leq \mathbb{Q}$. Then G is generated by $\mathbb{Z}$ and a subset (possibly infinite) $\{\frac{p_i}{q_i} \mid i \in I\}$ of rational numbers, where $I = \{1, 2, ...\}$. Then, by Lemma 2.1 we have:

$$G = \mathbb{Z} \cup \bigcup_{i \in I} \mathbb{Z}\left(\frac{p_i}{q_i}\right) = \bigcup_{i \in I} \mathbb{Z}\left(\frac{1}{q_i}\right).$$

By taking $n_i = LCM\{q_1, q_2, ..., q_i\}$ we obtain a finite or infinite sequence of integers $\{n_i\}$ in which every n_i divides n_{i+1}. If I is finite nad M is the maximal element in I, then we additionally put $n_i = n_M$ for all $i > M$. The obtained sequence $\bar{n} = \{n_i\}$ is infinite and divisible. Moreover, by Lemma 2.1 we have

$$\bigcup_{j=1}^{i} \mathbb{Z}\left(\frac{1}{q_j}\right) = \mathbb{Z}\left(\frac{1}{n_i}\right), \quad i \in \mathbb{N},$$

and hence

$$G = \bigcup_{i=1}^{\infty} \mathbb{Z}\left(\frac{1}{q_i}\right) = \lim_{i\to\infty} \bigcup_{j=1}^{i} \mathbb{Z}\left(\frac{1}{q_j}\right) = \lim_{i\to\infty} \mathbb{Z}\left(\frac{1}{n_i}\right) = \mathbb{Z}(\bar{n}).$$

(ii) Let A and B be two subgroups of $\mathbb{Q}$, both containing $\mathbb{Z}$. Then, by (i), there exist divisible sequences $\bar{m}$ and $\bar{n}$ such that $A = \mathbb{Z}(\bar{m})$ and $B = \mathbb{Z}(\bar{n})$. In particular, A and B contain rational numbers with arbitrary numerators. In [6] the authors proved, that such A and B are isomorphic if and only if the respective two supernatural numbers $\hat{m}$ and $\hat{n}$ are equivalent.

□

2.2. Subgroups of IET defined by supernatural numbers

An interval exchange transformation $f = (\pi, \sigma)$, such that

$$\pi = \{0, a_1, ..., a_{n-1}, 1\}, \quad \text{where} \quad a_i = \frac{p_i}{q_i} \in \mathbb{Q}.$$

is called a *rational interval exchange transformation* (rational iet).

Take $q = LCM(q_1, ..., q_{n-1})$ and define π_q to be the partition of I into q subintervals of equal length. Since f shuffles the n subintervals defined by π, then in a natural way it shuffles the q subintervals defined by π_q. Hence there exists a permutation $\sigma' \in S_q$ such that the action of $f' = (\pi_q, \sigma')$ on I is equivalent to the action of f. The arguments above allow us to interpret every rational interval exchange transformation as a transformation defined on the partition π_n of interval I into n subintervals of equal length. Moreover, if f and g are two rational iets, then one may easily find a partition π_m of the interval I into m equally sized subintervals, such that f and g act on I by shuffling the subintervals defined by π_m (it is enough to take m as the least common multiplier of all endpoints of partitions defining f and g). Then the composition fg is clearly a rational iet and hence the set $RIET$ of all rational iets is a proper subgroup of IET.

We begin our discussion with the following observation:

Remark 2.3. *$RIET$ is a dense subgroup of IET.*

Proof. Let $f = (\pi, \sigma)$ be an iet in its canonical form, where $\sigma \in S_n$. Then obviously $f \in IET_\sigma$, while IET_σ belongs to the basis of the topological space IET. Moreover, it is clear that IET_σ contains rational iets, i.e.

$$IET_\sigma \cap RIET \neq \emptyset.$$

We see that every neighbourhood of any element f of IET contains elements from $RIET$, thus $RIET$ is dense subset of IET. $\square$

Now, let $RIET(n) = \{f \in RIET \quad | \quad f = (\pi_n, \sigma), \sigma \in S_n\}$, $n \in \mathbb{N}$. It is clear that $RIET(n)$ is the subgroup of $RIET$, isomorphic to S_n. Thus, if $\bar{n} = (n_1, n_2, ...)$ is a divisible sequence, we may define the *diagonal embeddings* $\varphi_i : RIET(n_i) \hookrightarrow RIET(n_{i+1})$, where $n_i | n_{i+1}$, which correspond to the diagonal embeddings of S_{n_i} into $S_{n_{i+1}}$ (see [17] for details). In particular, if $f = (\pi_{n_i}, \sigma) \in RIET(n_i)$ and $n_{i+1} = k \cdot n_i$ then the diagonal embedding φ_i is defined by the following rule:

$$f^{\varphi_i}\left(\left[\frac{l \cdot n_i + j}{n_{i+1}}, \frac{l \cdot n_i + j + 1}{n_{i+1}}\right)\right) = \left[\frac{l \cdot n_i + \sigma(j)}{n_{i+1}}, \frac{l \cdot n_i + \sigma(j) + 1}{n_{i+1}}\right),$$

for all $j = 0, 1, ..., n_i - 1$ and $l = 0, 1, ..., k - 1$. The groups $RIET(n)$ together with the diagonal embeddings form a direct system of groups. The corresponding direct limit:

$$RIET(\bar{n}) = \lim_{i} RIET(n_i)$$

is a subgroup of $RIET$. In particular, if $\hat{M} = \prod_{p_i \in P} p_i^\infty$ is the characteristic of the divisible sequence $\bar{m}$, then $RIET(\bar{m}) = RIET$. However, we emphasize that the presented construction is substantially different than the construction of $RIET$ proposed in [11]. The difference lies in the direct systems of groups and the respective embedding schemes – the so called *periodic* embeddings discussed in [11] and the diagonal embeddings defined above are not the same. Nevertheless, for any divisible sequence $\bar{n}$ the subgroup $RIET(\bar{n})$ may be also naturally constructed via periodic embeddings. We note that both embedding schemes refer to the embedding schemes of symmetric groups which are diagonal in the sense of definition in [17]. However, the embeddings φ_i are strictly diagonal, while the periodic embeddings are not.

We begin with a characterization of groups $RIET(\bar{n})$.

Theorem 2.4. *For an arbitrary divisible sequence $\bar{n}$, the respective subgroup $RIET(\bar{n})$ in $RIET$ coincides with the group of all rational iets defined by partitions with endpoints lying in $\mathbb{Z}(\bar{n})$.*

Proof. Let $\bar{n}$ be a divisible sequence and let $f = (\pi, \sigma)$ be an iet such that partition $\pi = (a_0, a_1, ..., a_k)$ has endpoints $a_i = \frac{p_i}{q_i} \in \mathbb{Z}(\bar{n})$, where q_i is a term of the sequence $\bar{n}$, i.e. $q_i = n_{k_i}$ for $i = 0, ..., k$. Let $m = \max_i n_{k_i}$. Then π_m is a subpartition of pi and there exists $\sigma' \in S_m$ such that $f = (\pi_m, \sigma')$. Thus f is contained in $RIET(m)$, which is one of the groups in the direct system defined by $\bar{n}$. It follows that the direct limit $RIET(\bar{n})$ contains f.

Conversely, if f is an iet contained in $RIET(\bar{n})$, where $\bar{n} = (n_1, n_2, ...)$ is a divisible sequence, then it is contained in $RIET(n_i)$ for some i. Hence, f is defined be a partition π_{n_i} with endpoints in $\mathbb{Z}(\bar{n})$. □

In [17] the authors investigate the so called homogeneous symmetric groups $S_{\bar{n}}$, defined for an infinite divisible sequence $\bar{n} = (n_1, n_2, ...)$. These are exactly the direct limits of symmetric groups with diagonal embeddings $S_{n_i} \hookrightarrow S_{n_{i+1}}$ corresponding to the sequence $\bar{n}$.

Theorem 2.5. *Let $\hat{n}$ be an infinite supernatural number with characteristic $\bar{n}$. Then*

(i) *The subgroup $RIET(\bar{n})$ is isomorphic to the homogeneous symmetric group $S_{\bar{n}}$.*

(ii) *The subgroup $RIET(\bar{n})$ is dense in $RIET$.*

Proof.

(i) Let $\bar{n} = (n_1, n_2, ...)$ be an infinite divisible sequence. By definition, there is a mutual correspondence between the symmetric group S_{n_i} and the subset of $RIET(n_i)$ consisting of all iets defined for a given partition π_{n_i} of I into n_i subintervals of equal length. We note that every iet f form $RIET(n_i)$ has a unique representation $f = (\pi_{n_i}, \sigma)$, where $\sigma \in S_{n_i}$, although this is not necessarily the canonical form of f. It is clear that the discussed correspondence

$$\iota : \sigma \mapsto (\pi_{n_i}, \sigma)$$

is a group isomorphism of $RIET(n_i)$ and S_{n_i}. Since $RIET(n_i) \cong S_{n_i}$ for every $i \in \mathbb{N}$ and the diagonal embeddings φ_i of $RIET(n_i)$ correspond to the diagonal embeddings of the symmetric groups S_{n_i}, then the isomorphism of both direct limits $RIET(\bar{n})$ and $S_{\bar{n}}$ is clear.

(ii) Let $RIET(\bar{n})$ be a subgroup of IET defined for the divisible sequence $\bar{n} = (n_1, n_2, ...)$ and let f be an iet in its canonical form, where $\sigma \in S_{\mathbb{N}}$.

Since $\bar{n}$ is an infinite divisible sequence, then there exists $i \in \mathbb{N}$, such that $N \leq n_i$. We construct the rational partition of interval $I = [0, 1)$ in the following way. First, divide the interval I into n_i subintervals of equal length. If $n_i = N$ then we are done. Otherwise, glue together the first two subintervals in this partition into one subinterval, obtaining another partition with the number of subintervals less by 1 then the initial partition had. If the obtained partition has exactly N subintervals, then stop; otherwise apply the second step repeatedly. The construction will ultimately stop providing a rational partition into exactly N subintervals, determined by endpoints in $\mathbb{Z}\left(\frac{1}{n_i}\right)$. Hence, there exists $g \in RIET(\bar{n})$ being an element from the neighbourhood IET_σ of f. It follows that $RIET(\bar{n})$ is dense in IET.

□

3. Properties of groups of rational iets

In the previous section we introduced groups of rational iets which are characterized in terms of supernatural numbers. We also distinguished those of them which are dense in IET and gave a simple isomorphism condition. In this section we give certain properties of the discussed groups.

3.1. The lattice of subgroups of $RIET$

The set $s\mathbb{N}$ of all supernatural numbers is partially ordered with respect to the (extended) divisibility relation $\preceq$ defined as follows. Let $\hat{m}$ and $\hat{n}$ be supernatural numbers, such that

$$\hat{m} = \prod_{p_i \in P} p_i^{\alpha_i}, \quad \hat{n} = \prod_{p_i \in P} p_i^{\beta_i}, \qquad \alpha_i, \beta_i \in \mathbb{N} \cup \{0, \infty\}, \quad i \in \mathbb{N}.$$

We say that $\hat{m}$ *divides* $\hat{n}$ and write $\hat{m} \preceq \hat{n}$, if for every $i \in \mathbb{N}$ we have $\alpha_i \leq \beta_i$. In the above definition we assume $\infty \leq \infty$ and $n \leq \infty$ for every $n \in \mathbb{N}$.

Observe that the ordering $\preceq$ of $s\mathbb{N}$ corresponds to the ordering of subgroups $\mathbb{Z}(\bar{n})$ of $\mathbb{Q}$. For convenience, we use the alternative notation $RIET(\hat{n})$ for the subgroup $RIET(\bar{n})$. The following observation is clear.

Remark 3.1. Let $\hat{m}$ and $\hat{n}$ be supernatural numbers. The group $RIET(\hat{m})$ is a subgroup of $RIET(\hat{n})$ if and only if $\hat{m} \preceq \hat{n}$.

It follows that the subgroups $RIET(\hat{n})$ with respect to inclusion constitute a lattice $\mathcal{L}$, which corresponds to the lattice $(s\mathbb{N}, \preceq)$. This lattice contains infinitely many minimal subgroups, each of them of the form $RIET(p)$, where p is a prime. There is a unique maximal subgroup $RIET(\hat{M})$, where $\hat{M} = \prod_{p_i \in P} p_i^{\infty}$.

Below we list some basic properties of elements of $\mathcal{L}$.

Theorem 3.2.

(i) *$RIET(\hat{n})$ is either finite or locally finite. In particular $RIET(\hat{n})$ is finitely generated if and only if it is finite.*

(ii) *If $\bar{n} = (n_1, n_2, ...)$ and $\bar{m} = (n_i, n_{i+1}, ...)$ $i > 1$, then $RIET(\bar{n}) = RIET(\bar{m})$.*

(iii) *For every prime p the subgroup $RIET(p^{\infty})$ is the minimal dense subgroup of IET in the lattice $\mathcal{L}$.*

Proof.

(i) If $\hat{n}$ is finite, then there are only finitely many partitions of I defined by endpoints characterized by $\hat{n}$, and for each partition there are only finitely many permutations, defining a particular iet. Thus in that case, $RIET(\hat{n})$ is finite.

On the contrary, if $\hat{n}$ is infinite, then obviously $RIET(\hat{n})$. However, every finite subset A of $RIET(\hat{n})$ generates a finite subgroup of $RIET(m)$, where m is the least common multiplier of all denominators of fractions defining the endpoints in partitions of iets in A. Thus, $RIET(\hat{n})$ is locally finite. It also follows, that every infinite $RIET(\hat{n})$ is not finitely generated.

(ii) Both sequences $\bar{n}$ and $\bar{m}$ have the same characteristic, i.e. $\hat{n} = \hat{m}$. Thus the statement holds.

(iii) Since p^{∞} is an infinite supernatural number, then by Theorem 2.5, $RIET(p^{\infty})$ is dense in IET. It is enough to see that all proper subgroups of $RIET(p^{\infty})$ in $\mathcal{L}$ are finite.

Let $H \in \mathcal{L}$ be a proper subgroup of $RIET(p^{\infty})$. It consists of iets defined by partitions with endpoints of the type $\frac{q}{p^k}$, where $q \in \mathbb{Z}$ and $k \in \mathbb{N}$. If there exists a maximal exponent k, such that every element in H is defined by a partition of I into at most p^k subintervals, then $H < RIET(p^k)$ is finite and therefore not dense in IET. Otherwise H coincides with $RIET(p^{\infty})$ and the statement follows.

□

3.2. Derived subgroups of $RIET(\hat{n})$

The presented construction of $RIET(\hat{n})$ corresponds to the construction of the homogeneous symmetric group $S_{\mathbb{N}}$ in terms of direct limit of groups. Using this correspondence and the results of [17], we describe the derived subgroups of $RIET(\hat{n})$.

Theorem 3.3. *Let $\hat{n}$ be a supernatural number.*

(i) *If $2^\infty|\hat{n}$ then the group $RIET(\hat{n})$ is perfect, i.e. $RIET(\hat{n})' = RIET(\hat{n})$.*

(ii) *If $2^\infty \nmid \hat{n}$ then the derived subgroup $RIET(\hat{n})'$ is a proper subgroup of $RIET(\hat{n})$ and consists of all the iets from $RIET(\hat{n})$, which are defined by even permutations.*

Proof. Consider first the derived subgroup of $RIET(n_i)$, $n_i \in \mathbb{N}$. It was shown that $RIET(n_i)$ and S_{n_i} are isomorphic, and we denoted this isomorphism by

$$\iota : \sigma \mapsto (\pi_{n_i}, \sigma).$$

Thus it is clear that

$$RIET(n_i)' = \iota(S'_{n_i}) = \iota(A_{n_i}) = \{(\pi_{n_i}, \sigma) \mid \sigma \in A_{n_i}\},$$

where A_n denotes the alternating group on the set of n elements.

Now, let $\bar{n} = (n_1, n_2, ...)$ be a divisible sequence with characteristic $\hat{n}$. Since $RIET(\hat{n}) = \lim_i RIET(n_i)$, then by direct limit properties we have

$$RIET(\hat{n})' = \lim_i RIET(n_i)' = \{(\pi_k, \sigma) \mid k|\hat{n}, \sigma \in A_k\}.$$

Now, if $2^\infty \nmid \hat{n}$ it is clear, that the set $RIET(\hat{n})'$ is a proper subgroup of $RIET(\hat{n})$.

Otherwise, if $2^\infty|\hat{n}$, then $\hat{n}$ is the characteristic of the divisible sequence $(n_1, n_2, ...)$, such that for every index k n_{k+1}/n_k is even. Hence, the image of the diagonal embedding $\varphi_k : \iota(S_{n_k}) \hookrightarrow \iota(S_{n_{k+1}})$ lies in the group $\iota(A_{n_{k+1}})$, as the embedding multiplies the number of transpositions in the respective permutation by an even number. Hence, every element $f = (\pi_{n_k}, \sigma)$ lies in the set $RIET(n_{k+1})' = \{(\pi_{n_{k+1}}, \sigma) \mid \sigma \in A_{n_{k+1}}\}$. This implies

$$RIET(\hat{n}) = \lim_{k\to\infty} RIET(n_{k+1})' = RIET(\hat{n})'.$$

$\square$

References

[1] J. Athreya, M. Boshernitzan, *Ergodic properties of compositions of interval exchange maps and rotations*, Nonlinearity N.**26**, 2013, pp.417-421.

[2] A. Avila, G. Forni, *Weak mixing for interval exchange transformations and translation flows*, Ann. of Math., to appear.

[3] A. Avila, S. Gouezel, J.-C. Yoccoz, *The Teichmuller flow is exponentially mixing*, preprint.

[4] A. Avila, M. Viana, *Symplicity of Lyapunov spectra: Proof of the Zorich-Kontsevich conjecture*, 2005, preprint, available at www.preprint.impa.br.

[5] A. Avila, M. Viana, *Dynamics in the moduli space of Abelian differentials*, Port. Math. (N.S.), N.**62**, 2005, pp.531-547.

[6] R. Beaumont, H. Zuckerman, *Characterization of the subgroups of the additive rationals*, Pacific J. Math, N.**1**, 1951, pp.169-177.

[7] M. Boshernitzan, *A condition for minimal interval exchange transformations to be uniquely ergodic*, Duke J. of Math., N.**52**(3), 1985, pp.723-752.

[8] M. Boshernitzan, *Subgroup of interval exchanges generated by torsion elements and rotations*, arXiv:1208.1023 [math.DS].

[9] F. Dahmani, K. Fujiwara, V. Giurardel, *Free groups of interval exchange transformations are rare*, Groups, Geometry, and Dynamics, N.**7**(4), 2013, pp.883-910.

[10] G. Forni, *Deviation of ergodic averages for area-preserving flows on surfaces of higher genus*, Ann. of Math., N.**155**, 2002, pp.1-103.

[11] E. Goryachko, *The K_0-functor and characters of the group of rational rearrangements of the segment* (English. Russian original) J. Math. Sci., N.**158**(6), 2009, pp.838-844.

[12] E. Goryachko, F. Petrov, *Irreducible characters of the group of rational interval exchange transformations*, Zap. Nau. Seminars POMI, N.**378**, 2010, pp.17-31.

[13] A. Katok, *Interval exchange transformations and some special flows are not mixing*, Israel J. Math., N.**35**(4), 1980, pp.301-310.

[14] M. Keane, *Interval exchange transformations*, Math. Z. N.**141**, 1975, pp.25-31.

[15] M. Keane, *Non-ergodic interval exchange transformations*, Israel J. Math., N. **26**, 1977, pp.188-196.

[16] H. Keynes, D. Newton, *A minimal, non-uniquely ergodic interval exchange transformation*, Math. Z. N.**148**, 1976, pp.101-105.

[17] N. Kroshko, V. Sushchanskyy, *Direct limits of symmetric and alternating groups with strictly diagonal embeddings*, Arch. Math., N.**71**, 1998, pp.173-182.

[18] H. Masur, *Interval exchange transformations and measured foliations*, Annals of Mathematics, N.**115**, 1982, pp.169-200.

[19] C. Novak, *Discontinuity growth of interval exchange maps*, J. Mod. Dyn., N.**3**, 2009, pp.379-405.

[20] C. Novak, *Continuous Interval Exchange Actions*, Algebraic and Geometric Topology, N.**10**, 2010, pp.1609-1625.

[21] C. Novak, *Interval exchanges that do not embedd in free groups*, arXiv:1007.3940v1 [math.DS]

[22] G. Rauzy, *Echanges d'intervalles et transformations induites*, Acta Arith., N.**34**, 1979, pp.315-328.

[23] W.A. Veech, *Interval exchange transformations*, J. Analyse Math., N.**33**, 1978, pp.222-272.

[24] W.A. Veech, *Strict ergidicity in zero dimensional dynamical systems and the Kronecker - Weyl theorem mod 2*, Trans. Amer, Math. Soc., N.**140**, 1969, pp.1-33.

[25] W.A. Veech, *Gauss measures for transformations on the space of interval exchange maps*, Annals of Mathematics, N.**115**, 1982, pp.201-242.

[26] W.A. Veech, *The metric theory of interval exchange transformations III. The Sah-Arnoux-Fathi invariant.*, Amer. J. Math., N.**106**, 1984, no. 6, pp.1389-1422.

[27] M. Viana, *Ergodic theory of interval exchange maps*, Rev. Mat. Complut., N.**19**, 2006, no. 1, pp.7-100.

[28] Y. Vorobets, *Notes on the commutator group of the group of interval exchange transformations*, arXiv:1109.1352v1 [math.GR]

Contact information

A. Bier
Institute of Mathematics
Silesian University of Technology
ul. Kaszubska 23, 44-100 Gliwice, Poland
E-Mail: agnieszka.bier@polsl.pl
URL:

V. Sushchanskyy
Institute of Mathematics
Silesian University of Technology
ul. Kaszubska 23, 44-100 Gliwice, Poland
E-Mail: vitaliy.sushchanskyy@polsl.pl
URL: