On verbal subgroups of finitely generated nilpotent groups

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Abstract

The paper concerns the problem of characterization of verbal subgroups in finitely generated nilpotent groups. We introduce the notion of verbal poverty and show that every verbally poor finitely generated nilpotent group is a finite p-group with the lower p-central series for certain prime p. We conclude with few examples of verbally poor groups.

1 Introduction and main results.

The characterization of verbal subgroups in a group is an interesting and difficult problem. The full description of the verbal structure has been found only for few specific kinds of groups. The examples are given in the last section of this paper. All the groups presented there admit rather poor verbal structure. We consider then an inverse problem and provide conditions which a finitely generated nilpotent group should meet to have such poor verbal structure.

Let G be a group and F be a set of words from the free group of countably infinite rank, freely generated by an alphabet $X = \{x_1, x_2, ..., x_n\}$. The *verbal subgroup* $V_F(G)$ of group G is the subgroup generated by all values of the words from F in group G. If $F = \{f\}$ then the verbal subgroup generated by F will be denoted by $V_f(G)$. In a nilpotent group every verbal subgroup is generated by a finite set of words, hence we restrict here our considerations to the case of F being finite [3]. Let us denote by c_i the following words:

$$c_1 = x_1, \quad c_{i+1} = [x_{i+1}, c_i(x_1, ..., x_i)]$$

For any group G the verbal subgroups $V_{c_i}(G)$ constitute the lower central series

$$G = \gamma_1(G) \geq \gamma_2(G) \geq ...,$$

in which $\gamma_i(G) = V_{c_i}(G)$. In the case of a nilpotent group of class l we obviously have $\gamma_l(G) = V_{c_l}(G) = \{\mathbf{1}\}$.

The group is said to be *verbally simple* if it has no proper verbal subgroups. In the class of residually nilpotent groups we introduce yet another notion concerning verbal subgroups in the group. We will say that the group G is *verbally poor* if it has no verbal subgroups but the terms of its lower central series. In other words group G is verbally poor if every verbal subgroup $V_F(G)$ coincides with $\gamma_i(G)$ for certain $i \in \mathbf{N}$. Of course, every verbally simple group is verbally poor. One can also easily check that a cyclic group C_{p^n} of order p^n, where p is a prime is verbally poor if and only if $n = 1$.

It is an interesting question, whether a subgroup of a verbally poor group is also verbally poor. A partial answer to that is given in the following statements.

Proposition 1 *Let G be a group and H be a normal subgroup of G. If G is verbally poor then so is the quotient group G/H.*

Proof. Let $\varphi : G \longrightarrow G/H$ be the natural homomorphism. Since G is verbally poor, then there exists $i \in \mathbf{N}$ such that

$$V_F(G/H) = \varphi(V_F(G)) = \varphi(V_{c_i}(G)) = V_{c_i}(G/H).$$

Hence G/H is verbally poor. $\square$

Proposition 2 *Let $\{G_1, G_2, ..., G_n\}$ be a finite family of groups and G be the direct product of G_i, i.e. $G = \prod_{i=1}^{n} G_i$. If G is verbally poor then so is G_i for every $i \in \{1, 2, ..., n\}$.*

Proof. Consider $G = A \times B$. Then obviously $B \cong G/A$ and $A \cong G/B$. Since G is verbally poor, then following Proposition 1, both A and B are verbally poor too. The rest of the proof is simple induction on n. $\square$

The following part of the paper contains considerations on verbal subgroups in a finitely generated nilpotent group G. For the purpose of this work we recall the notion of p-central series. In a nilpotent group G the central series

$$G = \zeta_1(G) \geq \zeta_2(G) \geq ... \geq \zeta_k(G) = \{1\}$$

is called a *p-central series*, if all the quotients $\zeta_i(G)/\zeta_{i+1}(G)$ for all $i = 1, 2, ..., k-1$ are elementary abelian of exponent p. An example of such series is the lower p-central series $G = \lambda_1(G) \geq \lambda_2(G) \geq ... \geq \lambda_k(G) = \{1\}$, which is defined recursively as:

$$\lambda_1(G) = G, \qquad \lambda_i(G) = [\lambda_{i-1}(G), G]\lambda_{i-1}(G)^p \quad for \quad i > 1.$$

Here, G^p denotes the verbal subgroup $V_{x^p}(G)$. In the next section we prove

Main Theorem *Every verbally poor finitely generated nilpotent group is a (finite) p-group with the lower central series being a p-central series for certain prime p.*

In Section 3 we give some examples of finite as well as infinite verbally poor groups. These are gathered from literature and the author's own work.

2 Proof of the Main Theorem.

First, we recall from [2] a useful characterization of torsion-free finitely generated nilpotent groups. It is well known that a finitely generated torsion-free nilpotent group G has the lower central series with infinite cyclic quotients. This fact allows us to introduce in G integer coordinates. If $G = G_1 > G_2 > ... > G_{s+1} = \{\mathbf{1}\}$ is the lower central series of G, then we can choose the Malcev basis $a_1, a_2, ..., a_s$ of group G such that $G_i = \langle a_i, G_{i+1}\rangle$. Every element $x \in G$ can be uniquely represented as $x = a_1^{t_1(x)} a_2^{t_2(x)} ... a_s^{t_s(x)}$ where $t_i(x) \in \mathbf{Z}$ are the Malcev coordinates. The notion of Malcev basis and coordinates in group G allows to represent elements of G as unitriangular matrices with integer entries. The latter is stated in the following

Lemma 1 *Every torsion-free finitely generated nilpotent group is isomorphic to a subgroup of $\mathbf{UT}_n(\mathbf{Z})$ for a certain $n \in \mathbf{N}$.*

The group $\mathbf{UT}_n(\mathbf{Z})$ is the group of upper triangular reversible matrices with ones on the main diagonal, zeros below and integer entries above. A detailed proof of Lemma 1 can be found in [2]. From the characterization it follows that for description of verbal subgroups of any torsion-free finitely generated nilpotent group, it is sufficient to investigate the latter in the subgroups of the group $\mathbf{UT}_n(\mathbf{Z})$, $n \in \mathbf{N}$.

For further considerations we introduce some necessary notation first. Let $A = [a_{ij}]$ be an arbitrary matrix from $\mathbf{UT}_n(\mathbf{Z})$ such that for all indices $i < j < i+t_i+1$ there is

$$a_{ij} = 0 \quad \textit{and} \quad i + t_i + 2 > n$$

or

$$a_{i,i+t_i+2} \neq 0.$$

Let $w(A) = (t_1, t_2, ..., t_{n-1})$ be a vector of size $n-1$, in which each coordinate t_i is the number of zeroes placed between the main diagonal of the matrix A and the first nonzero element in i-th row of the matrix. The vector $w(A)$ will be called *the type of matrix A*. For example, the type of the unit matrix I_n is equal to $w(I_n) = (n-1, n-2, ..., 2, 1)$. Directly from the given definition one can observe that

Lemma 2 *If $H \leq \mathbf{UT}(n, \mathbf{Z})$ contains matrices A and B of the types*

$$w(A) = (t_1, t_2, ..., t_{n-1}) \quad \textit{and} \quad w(B) = (s_1, s_2, ..., s_{n-1})$$

respectively, then H contains a matrix C of the type

$$w(C) = \min\{w(A), w(B)\} \stackrel{def}{=} (\min\{t_1, s_1\}, \min\{t_2, s_2\}, ..., \min\{t_{n-1}, s_{n-1}\}).$$

Proof. It is enough to take $C = A^i B$ for adequate $i \in \mathbf{N}$. Obviously, for every $i \in \mathbf{Z}$ we have $C \in H$. Now, if for every $k \in \{1, 2, ..., n-1\}$ there is $t_k \neq s_k$ or $t_k = s_k \ \wedge \ A_{k,k+t_k+2} \neq (B_{k,k+t_k+2})^{-1}$, then for $i = 1$ the assumed matrix C will be of the desired type. Otherwise, we take

$$i = 1 + \max_{k \in \{1,2,\ldots,n-1\}} |B_{k,k+t_k+2}|,$$

and this completes the proof. □

As a consequence of Lemma 2 we are able to define *the type W_H of the subgroup H in the group* $\mathbf{UT}_n(\mathbf{Z})$ as

$$W_H = \min_{A \in H} w(A).$$

In the set of the types W_H of subgroups H of the group $\mathbf{UT}_n(\mathbf{Z})$ we define the order:

$$(a_1, ..., a_n) = W_{H_1} \leq W_{H_2} = (b_1, ..., b_n) \quad \Leftrightarrow \quad \forall i = 1, ..., n \quad a_i \leq b_i.$$

If $W_{H_1} \leq W_{H_2}$ and there exists $i \in \{1, 2, ..., n\}$ such that $a_i \neq b_i$, then we write

$$W_{H_1} < W_{H_2}.$$

Lemma 3 *Let $\gamma_i(H)$, $i = 2, 3, ..., c$ be the i-th term of the lower central series in the group H being the subgroup of $UT_n(Z)$ of nilpotency class c. Then*

$$W_{\gamma_c(H)} > W_{\gamma_{c-1}(H)} > ... > W_{\gamma_2(H)} > W_H.$$

Proof. If $W_{\gamma_i(H)} = (t_1, t_2, ..., t_{n-1})$, then $W_{\gamma_{i+1}(H)} = (t_1 + 1, t_2 + 1, ..., t_{n-1} + 1)$.□

Lemma 4 *Let $w(A) = (t_1, t_2, ..., t_{n-1})$ be the type of the matrix A. Then $w(A^k) = w(A)$ for every $k \in \mathbf{Z} \setminus \{0\}$.*

Proof. Indeed, it can be easily observed that

$$(A^k)_{i,i+t_i+2} = k \cdot A_{i,i+t_i+2} \neq 0$$

for $k \neq 0$. Moreover, if $i < j < i+t_i+1$ then $(A^k)_{ij} = 0$, and hence $w(A^k) = w(A)$.□

Lemma 5 *The group* $\mathbf{UT}(n, \mathbf{Z})$ *does not contain divisible subgroups different from* $\{I_n\}$.

Proof. Let us assume that H is a nontrivial divisible subgroup of $\mathbf{UT}(n, \mathbf{Z})$. Then there exists $A \in H$ such that $A \neq I_n$, ie. $w(A) = (t_1, t_2, ..., t_{n-1})$ and there exists $i_0 \in \{1, 2, ..., n-1\}$ such that $t_{i_0} < n - i_0$. Since $A_{i_0,i_0+t_{i_0}+2} \in \mathbf{Z} \setminus \{0\}$, then there exists $m \in \mathbf{Z} \setminus \{0\}$ such that $m \nmid A_{i_0,i_0+t_{i_0}+2}$. Indeed, it is enough to take $m > |A_{i_0,i_0+t_{i_0}+2}|$. If there exists a matrix B such that $B^m = A$, then following Lemma 4 we have $w(B) = w(A)$ and the equality

$$A_{i_0,i_0+t_{i_0}+2} = m \cdot B_{i_0,i_0+t_{i_0}+2}$$

holds. Then we get a contradiction and hence there exists no such matrix B. Therefore H is not a divisible group. □

Proposition 3 *Every verbally poor finitely generated nilpotent group is a (finite) torsion group.*

Proof. The proof will be carried out in a few steps. Successively, we will show the following:

A If H is a non-trivial subgroup of group $\mathbf{UT}_n(\mathbf{Z})$, then for every $k \neq 0, \pm 1$ the verbal subgroup $V_{x^k}(H)$ is a proper subgroup of H.

B A non trivial subgroup of $\mathbf{UT}_n(\mathbf{Z})$ is not a verbally poor group.

C Correctness of the thesis in the Proposition.

A. The first part of the proof consists of two steps:

1. At first we will show that if $A \in V_{x^k}(H)$ and $w(A) = (t_1, t_2, ..., t_{n-1})$, then the elements $A_{i,i+t_i+2}$ for $i = 1, 2, ..., n-1$ are divisible by k.

   Indeed. Let A be a matrix from $V_{x^k}(H)$. Then there exist matrices $A_1, A_2, ..., A_s \in H$ such that $A = A_1^k A_2^k ... A_s^k$. If $w(A_i) = (u_1^{(i)}, u_2^{(i)}, ..., u_{n-1}^{(i)})$ for $i = 1, 2, ..., s$, then
   $$w(A) = \min_i w(A_i)$$
   and
   $$A_{j,j+t_j+2} = \sum_{i=1}^{s} k \cdot A_{j,j+t_j+2} = k \cdot \sum_{i=1}^{s} A_{j,j+t_j+2} \quad \Box$$

2. Now, assume $M_i \subset \mathbf{Z}$ to be defined as follows: $M_i \overset{def}{=} \{A_{i,i+t_i+2} | A \in H\}$. Since $H \neq \{1_n\}$, then there exists $i_0 \in \{1, 2, ..., n-1\}$ such that $M_{i_0} \neq \{0\}$. Let us denote by m the element of the smallest nonzero absolute value in M_{i_0} (if there are more than one such elements we choose one of those that are positive numbers). The following two cases may occur:

   a) $k \nmid m$. Then there exists matrix $A \in H$ such that $A_{i_0,i_0+t_{i_0}+2} = m$ and following the first step of the proof $A \notin V_{x^k}(H)$.

b) $m = k \cdot l$ for certain $l \in \mathbf{Z} \setminus \{\mathbf{0}\}$ such that $|l| < |m|$. Let us assume that matrix $A \in H$ satisfying condition: $A_{i_0,i_0+t_{i_0}+2} = m$ is contained in $V_{x^k}(H)$. Then there exist matrices $A_1, A_2, ..., A_s \in H$ such that

$$A = A^{(1)k} A_2^{(2)k} ... A_s^{(s)k}, \quad s > 1.$$

Hence $m = k \cdot l = k \cdot \sum_{i=1}^{s} A^{(i)}_{i_0,i_0+t_{i_0}+2}$, and therefore

$$l = \sum_{i=1}^{s} A^{(i)}_{i_0,i_0+t_{i_0}+2} \in M_{i_0}.$$

By the fact that $|l| < |m|$ and the assumptions involving m we get a contradiction. Hence $A \notin V_{x^k}(H)$.

From a) and b) we directly conclude that $V_{x^k}(H) \neq H$ for $k \neq 0, \pm 1$. □

B. Now we can prove the second part. If $A \in H$ is a matrix of the type W_H, then after Lemma 4 the matrix A^k, $k \neq 0$ is a matrix of the type W_H, hence the type of the verbal subgroup $V_{x^k}(H)$ of the group H generated by the word x^k is equal W_H. Simultaneously if $k \neq \pm 1$, then $V_{x^k}(H) \neq H$.

We obtained that while $k \neq 0$ the inequalities hold: $W_{V_{x^k}(H)} = W_H < W_{\gamma_i(H)}$ for $i = 2, 3, ..., c$. Hence

$$V_{x^k}(H) \neq \gamma_i(H) \ \ dla \ \ i = 2, 3, ..., c,$$

i.e. group H has verbal subgroups $V_{x^k}(H)$, that do not coincide with any term of its lower central series. □

C. Let G be a finitely generated nilpotent group with non-trivial torsion-free part. We denote by T the torsion part of G. Of course, $T \lhd G$ and the quotient group G/T is torsion-free. Then G/T is isomorphic to a subgroup of $\mathbf{UT}_n(\mathbf{Z})$, which - as proved above - is not a verbally poor group. Hence by Proposition 1, neither is G. Therefore, a finitely generated verbally poor nilpotent group G has a trivial torsion-free part, i.e. it is a finite torsion group. □

Proof of the Main Theorem. As a consequence of Proposition 3, the research of verbally poor finitely generated nilpotent groups can be limited to finite torsion groups. The structure of finite nilpotent groups has been thoroughly investigated and we recall here a famous result of Burnside and Wielandt, that a finite nilpotent group is a direct product of its maximal p-subgroups. Then from Proposition 2 we obtain a simple observation, that if a finite nilpotent group G has a maximal p-subgroup which is not verbally poor, then neither is G.

Let us consider the case of G being the direct product of groups A and B such that

$$exp(A) = p^k, \quad exp(B) = q^l, \quad p \neq q,$$

where p and q are two different primes. We will show that G is not verbally poor. Obviously

$$V_{x^{p^k}}(A) = \{1_A\} \quad and \quad V_{x^{q^l}}(B) = \{1_B\}.$$

Also, since $LCD(p^k, q^l) = 1$ then $V_{x^{q^l}}(A) = A$ and $V_{x^{p^k}}(B) = B$, hence

$$V_{x^{p^k}}(G) = \{1_A\} \times B \quad and \quad V_{x^{q^l}}(G) = A \times \{1_B\}.$$

However, the subgroups $\{1_A\} \times B$ and $A \times \{1_B\}$ do not coincide with any of the terms of the lower central series in G. As A and B are nilpotent, then $\gamma_i(A) \neq A$ and $\gamma_i(B) \neq B$ for $i > 1$ and therefore

$$A \times \{1_B\} \neq \gamma_i(G) \neq \{1_A\} \times B.$$

For $i = 1$ we have $\gamma_i(G) = G$ and this term also does not coincide with the verbal subgroups determined above. Hence G is not verbally poor, if it is not a p-group.

Now, consider a verbally poor finite p-group G of nilpotency class c. As a p-group, G has the lower central series with abelian sections being p-subgroups. We will show that the lower central series of G is a p-central series, that is $\gamma_i(G) = \lambda_i(G)$ for all $i = 1, 2, ..., c + 1$.

The proof is inductive. We start with $\lambda_2(G) = G'G^p$ and note that the second term of the lower p-central series of group G is the Frattini subgroup $\Phi(G)$ of G. It is well known that $\gamma_2(G) = G' \subseteq \Phi(G)$, hence we need only to prove the reverse inclusion. It is enough to show that $G^p \subseteq G'$. Since G is verbally poor, then G^p coincides with one of the terms of the lower central series of G, say $G^p = \gamma_l(G)$ for certain l. Moreover, since G is a finite p-group, G^p is a proper subgroup of G and $l \geq 2$. Then $G^p = \gamma_l(G) \subseteq \gamma_2(G)$.

Now, assume that $\gamma_i(G) = \lambda_i(G)$ for all $i \leq k$. We take

$$\lambda_{k+1}(G) = [\lambda_k(G), G]\lambda_k(G)^p.$$

By induction it is equal to $[\gamma_k(G), G]\gamma_k(G)^p = \gamma_{k+1}(G)\gamma_k(G)^p$. We need only to show that $\gamma_k(G)^p \subseteq \gamma_{k+1}(G)$. Indeed, $\gamma_k(G)^p$ is a proper verbal subgroup of $\gamma_k(G)$ and hence it is a verbal subgroup in G. Since G is verbally poor, $\gamma_k(G)^p = \gamma_m(G)$ for certain $m \geq k + 1$ and in particular

$$\gamma_k(G)^p \subseteq \gamma_{k+1}(G).$$

This completes the proof of Main Theorem. □

3 Examples of verbally poor groups

We shall recall a few examples of finite verbally poor nilpotent groups that can be found in the bibliography. We will also introduce some interesting new results.

Example 1 The group $UT_n(K)$ of unitriangular matrices of size n over a field K of characteristic $p \neq 2$, where p is a prime, is a verbally poor group. Every verbal subgroup of $UT_n(K)$ coincides with one of its subgroups of the type $UT_n^l(K)$, $0 \leq l \leq n - 1$.

Example 2 The Sylow p-subgroup $Syl_p(S_n)$ of a finite symmetric group is verbally poor. The proof and details can be found in [5].

Example 3 The group W_n defined as a wreath product $W_n = C_p^n \wr C_p^n \wr ... \wr C_p^n$ is verbally poor group.

Example 4 The group of automorphisms $AutT_2$ of the homogeneous 2-adic rooted tree is verbally poor.

Example 5 The Sylow p-subgroup of the group of automorphisms $AutT_p$ of a p-adic rooted tree for $p > 2$.

Please note, that Examples 4 and 5 are infinite groups and are given just to illustrate that the notion of verbal poverty refers also to such groups. As another example of infinite verbally poor group the groups constructed as in Example 1 can serve, whenever they are defined over a field of characteristic 0 (for details see [1]). An observation can be made for the case of finite abelian group G, namely

Example 6 A finite abelian group G is verbally poor if and only if G is elementary abelian.

Proof. Let G be an elementary abelian group, i.e. $G = (C_p)^n$. Then $V_F(G) = (V_F(C_p))^n$ and since C_p is verbally simple then so is G.

Now, if G is a finite abelian group then it is a direct product of cyclic subgroups of the form $G = \prod_{i=1}^{n} C_{p_i^{k_i}}$. If there exist two factors $C_{p_1^{k_1}}$ and $C_{p_2^{k_2}}$ such that $p_1 \neq p_2$, then - as shown in point B of the proof of Proposition 3 - group G is not verbally poor and we get a contradiction. Otherwise, if G is a product of $C_{p^{k_i}}$, $i \in \{1, 2, ..., n\}$, and there is at least one $k_i > 1$ then, again, following the fact shown if the proof of the Main Theorem G is a verbally poor group neither. Hence, $G = (C_p)^n$, i.e. G is elementary abelian. □

Example 7 If a finite metabelian p-group G has lower central series which is a p-central series and G' is cyclic of order p, then G is verbally poor.

Proof. We assume that G has the lower central series

$$G \geq G' \geq \{1\}$$

such that $G/G' \cong (C_p)^k$ and $G'/\{1\} \cong G' \cong C_p$. We recall here a useful fact

Lemma 6 *[2] If G is nilpotent group and A is its subgroup such that $A[G, G] = G$, then $A = G$.*

We denote by φ the natural homomorphism of G onto G/G'. Let $V_F(G)$ be an arbitrary verbal subgroup of G. Then $\varphi(V_F(G)) = V_F(G/G')$ and since $G/G' \cong C_p^k$ is verbally simple, then there are two options:

1. $V_F(G/G') = G/G'$. Then $V_F(G)$ contains representatives of all cosets of G/G' and we have $G = V_F(G) \cdot G'$. Following Lemma 6, $V_F(G) = G$.
2. $V_F(G/G') = 1 \cdot G'$. Then $V_F(G) \leq G'$, and since $G' \cong C_p$ then $V_F(G) = \{1\}$ or $V_F(G) = G'$.

Overall, the only verbal subgroups of G are G, G' and $\{1\}$, i.e. G is verbally poor.□

References

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