On lattices of closed subgroups in the group of infinite triangular matrices over a field

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Abstract

We investigate a special type of closed subgroups of the topological group $\mathrm{UT}(\infty, K)$ of infinite-dimensional unitriangular matrices over a field K ($|K| > 2$), considered with the natural inverse limit topology. Namely, we generalize the concept of partition subgroups introduced in [23] and define partition subgroups in $\mathrm{UT}(\infty, K)$. We show that they are all closed and discuss the problem of their invariancy to various group homomorphisms. We prove that a characteristic subgroup of $\mathrm{UT}(\infty, K)$ is necessarily a partition subgroup and characterize the lattices of characteristic and fully characteristic subgroups in $\mathrm{UT}(\infty, K)$. We conclude with some implications of the given characterization on verbal structure of $\mathrm{UT}(\infty, K)$ and $\mathrm{T}(\infty, K)$ and use some topological properties to discuss the problem of the width of verbal subgroups in groups defined over a finite field K.

Keywords: Infinite triangular matrices, Characteristic subgroups, Closed subgroups, Partition subgroups, Verbal subgroups
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1. Introduction

Let K be a field such that $|K| > 2$. By $\mathrm{T}(n, K)$ we denote the group of all invertible upper triangular matrices of size $n \times n$ over the field K. Further, by $\mathrm{UT}(n, K)$ we denote the subgroup of $\mathrm{T}(n, K)$ consisting of all unitriangular matrices (i.e. the triangular matrices having all diagonal entries equal to 1), and by $\mathrm{D}(n, K)$ we denote the the subgroup of $\mathrm{T}(n, K)$ consisting of all diagonal matrices with nonzero diagonal entries. For $i > j$ the group $\mathrm{T}(i, K)$ (and so $\mathrm{UT}(i, K)$, and $\mathrm{D}(i, K)$) may be mapped onto $\mathrm{T}(j, K)$ (respectively $\mathrm{UT}(j, K)$ and $\mathrm{D}(j, K)$) using the projection π_{ij} (or its restrictions $\pi_{ij}|_{\mathrm{UT}(i,K)}$ and $\pi_{ij}|_{\mathrm{D}(i,K)}$), which deletes the last $(i-j)$ rows and the last $(i-j)$ columns of the matrix. The limits of the obtained inverse spectra $(\mathrm{T}(i, K), \pi_{i,i-1})$, $\big(\mathrm{UT}(i, K), \pi_{i,i-1}|_{\mathrm{UT}(i,K)}\big)$ and $\big(\mathrm{D}(i, K), \pi_{i,i-1}|_{\mathrm{D}(i,K)}\big)$ will be denoted by $\mathrm{T}(\infty, K)$, $\mathrm{UT}(\infty, K)$ and $\mathrm{D}(\infty, K)$

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respectively, and called the groups of infinite triangular, infinite unitriangular and infinite diagonal matrices. The elements of $\mathrm{T}(\infty, K)$, $\mathrm{UT}(\infty, K)$ and $\mathrm{D}(\infty, K)$ are the matrices with entries indexed by the set $\mathbb{N} \times \mathbb{N}$. The group $\mathrm{UT}(\infty, K)$ contains as a subgroup the stable group $\mathrm{UT}_f(\infty, K)$ of all finitary infinite matrices, which may be constructed as a direct limit of groups $\mathrm{UT}(n, K)$, $n \in \mathbb{N}$, with natural embeddings. Similarly, the direct limits of triangular and diagonal matrix groups will be denoted by $\mathrm{T}_f(\infty, K)$ and $\mathrm{D}_f(\infty, K)$, respectively.

In the past few years, the groups of infinite matrices have drawn attention of many researchers [5, 6, 7, 16, 17]. Among others one finds results on various aspects of groups $\mathrm{T}(\infty, K)$ and $\mathrm{UT}(\infty, K)$, like those concerning their subgroup structure, their automorphisms, or solvability of special types of equations [3, 4, 18, 19, 20]. Being inverse limits, the groups $\mathrm{T}(\infty, K)$ and $\mathrm{UT}(\infty, K)$ may be considered in a natural way as topological groups, and in particular – profinite groups, if K is finite (for more information on profinite groups see [13] and [14]). In the latter case, the topological properties of $\mathrm{T}(\infty, K)$ and $\mathrm{UT}(\infty, K)$ turn out to be interesting both as a self-contained study and as a tool for investigations of the verbal structure in these groups [15]. This was the motivation of the research presented within this paper.

Throughout the paper all finitely dimensional matrices will be denoted with lowercase letters, while for the infinite matrices we will use the uppercase letters. For every matrix $a \in \mathrm{UT}(n, K)$ (or $A \in \mathrm{UT}(\infty, K)$) and $m \leq n$ by $a[m]$ (and $A[m]$, respectively) we denote the top-left block of size $m \times m$ of matrix a (or A). The identity matrices in the groups $\mathrm{UT}(n, K)$ and $\mathrm{UT}(\infty, K)$ will be denoted by e_n and E. Every finitely dimensional unitriangular matrix $a \in \mathrm{UT}(n, K)$ may be written as a sum:

$$a = e_n + \sum_{1 \leq i < j \leq n} a_{ij} e_{ij},$$

where e_{ij} denotes elementary matrix of size equal to the size of a, which has 1 in the place (i, j) and zeros elsewhere (infinite elementary matrices will be denoted by E_{ij}). Every matrix $A \in \mathrm{UT}_f(\infty, K)$ (or in $\mathrm{T}_f(\infty, K)$ or $(\mathrm{D}_f(\infty, K))$ differs from E only in a finite block $A[n]$ for some n.

In groups $\mathrm{UT}(n, K)$, $\mathrm{UT}(\infty, K)$ and $\mathrm{UT}_f(\infty, K)$ we distinguish the respective subgroups $\mathrm{UT}(n, m, K)$, $\mathrm{UT}(\infty, m, K)$ and $\mathrm{UT}_f(\infty, m, K)$, which consist of all those matrices, whose all entries on the first m superdiagonals are zeros. It is well known (see e.g. [8]) that the series of subgroups

$$\mathrm{UT}(n, K) = \mathrm{UT}(n, 0, K) > \mathrm{UT}(n, 1, K) > ... > \mathrm{UT}(n, n-1, K) = \{e_n\}$$

is the lower central series of $\mathrm{UT}(n, K)$. Analogously, in groups of infinite matrices the two series

$$\mathrm{UT}_f(\infty, K) = \mathrm{UT}_f(\infty, 0, K) > \mathrm{UT}_f(\infty, 1, K) > ...$$

and

$$\mathrm{UT}(\infty, K) = \mathrm{UT}(\infty, 0, K) > \mathrm{UT}(\infty, 1, K) > ...$$

are (infinite) lower central series of $\mathrm{UT}_f(\infty, K)$ and $\mathrm{UT}(\infty, K)$, respectively [1, 19].

In this paper we consider $\mathrm{UT}(\infty, K)$ and $\mathrm{T}(\infty, K)$ as topological groups and investigate their subgroup structure. The inverse limit topology and some basic topological properties of $\mathrm{UT}(\infty, K)$ and $\mathrm{T}(\infty, K)$ are discussed in Section 2. In Section 3 we generalize the concept of a partition subgroup introduced by A. Weir in [23] and define partition subgroups of $\mathrm{UT}(\infty, K)$. We discuss some properties of such subgroups with reference to analogous studies of the partition subgroups of $\mathrm{UT}(n, K)$ in [9]. In particular we show that every partition subgroup is closed and determine the normal closure and normal core of subgroups of this type. Further, in Section 4 we provide the necessary and sufficient condition for a partition subgroup to be characteristic in $\mathrm{UT}(\infty, K)$ and discuss some mutual commutator subgroups of partition subgroups. Namely, we prove that the mutual commutator of a normal partition subgroup and $\mathrm{UT}(\infty, K)$ is also a normal partition subgroup of $\mathrm{UT}(\infty, K)$, and we calculate the coordinates of the defining sequence for that subgroup. Sections 5 and 6 concern the lattices of verbal subgroups of $\mathrm{UT}(\infty, K)$ and $\mathrm{T}(\infty, K)$. In particular, we show that the respective lattice in $\mathrm{UT}(\infty, K)$ is linear and coincides with the lower central series of $\mathrm{UT}(\infty, K)$. In Section 6 we note some implications of our results to the width of verbal subgroups in $\mathrm{UT}(\infty, K)$.

2. $\mathrm{T}(\infty, K)$ and $\mathrm{UT}(\infty, K)$ as topological groups

In this section we describe the inverse limit topology and the respective standard metric on $\mathrm{T}(\infty, K)$, naturally inherited by the group $\mathrm{UT}(\infty, K)$. We introduce a special type of subgroups of $\mathrm{UT}(\infty, K)$ and discuss their topological properties. Then we use it for characterization of characteristic and fully characteristic subgroups of $\mathrm{UT}(\infty, K)$.

2.1. Ultrametric on $\mathrm{T}(\infty, K)$

Let A and B be two matrices in $\mathrm{T}(\infty, K)$. We define the distance $d(A, B)$ between A and B to be equal to $\frac{1}{2^k}$, where k is the largest natural number such that $A[k] = B[k]$, and 1 if $A_{1,1} \neq B_{1,1}$. If there is no such number, that is $A = B$, we fix $d(A, B) = 0$. It is clear that the defined function $d : \mathrm{T}(\infty, K) \times \mathrm{T}(\infty, K) \longrightarrow [0, 1]$ is an ultrametric on $\mathrm{T}(\infty, K)$ and the group operations: multiplication $\cdot$ and inversion $^{-1}$ are continuous.

The open ball $\mathcal{B}(A, r)$ centered at $A \in \mathrm{T}(\infty, K)$ and with radius r consists of all matrices $T \in \mathrm{T}(\infty, K)$ such that T and A have the same top left block of size $\lfloor \log_{\frac{1}{2}} r \rfloor$. For $s > 1$ we define $\mathcal{T}_s = \mathcal{B}(E, \frac{1}{2^{s-1}})$. We say that a matrix $A \in \mathrm{T}(\infty, K)$ has depth s, if it is contained in $\mathcal{T}_s$ but not in $\mathcal{T}_{s+1}$, and we denote this fact by $dp(A) = s$. For every $s > 1$, $\mathcal{T}_s$ is the kernel of the natural projection $\pi_s : \mathrm{T}(\infty, K) \longrightarrow \mathrm{T}(s, K)$, where $\pi_s(A) = A[s]$. Hence $\mathcal{T}_s$ is normal in $\mathrm{T}(\infty, K)$

and

$$\mathrm{T}(\infty, K)/\mathcal{T}_s \cong \mathrm{T}(s, K). \tag{1}$$

The group $\mathrm{T}(\infty, K)$ together with the ultrametric d defined above is a topological group, with the inverse limit topology defined by the open balls $\mathcal{T}_s$, $s \in \mathbb{N}$ as the basis of neighborhoods of the identity. Then $\mathrm{UT}(\infty, K)$ considered with the subspace topology, is also a topological group. Moreover, the family of open balls $\mathcal{U}_s = \mathcal{T}_s \cap \mathrm{UT}(\infty, K)$, $s \in \mathbb{N}$, is a basis of neighborhoods of the identity for this topology and we have

$$\mathrm{UT}(\infty, K)/\mathcal{U}_s \cong \mathrm{UT}(s, K). \tag{2}$$

Further we discuss some properties of this topology.

2.2. Continuous maps in $\mathrm{UT}(\infty, K)$

To start our discussion we describe certain basic types of epimorphisms of the group $\mathrm{UT}(\infty, K)$. The four of them are listed below (by A^U we denote the conjugation, i.e. $A^U = U^{-1}AU$):

1. inner automorphisms $Inn_U : A \longmapsto A^U$, $U \in \mathrm{UT}(\infty, K)$;
2. diagonal automorphisms $Diag_D : A \longmapsto A^D$, $D \in \mathrm{D}(\infty, K)$;
3. epimorphisms induced by field automorphisms $\bar{\varsigma} : A \longmapsto A'$, where $(A')_{i,j} = \varsigma(A_{i,j})$ and ς is an automorphism of the field K;
4. shifts up: $Sh_n(A) := A|_n$, $n \in \mathbb{N}$.

In the above definitions A denotes an arbitrary unitriangular matrix in $\mathrm{UT}(\infty, K)$, and $A|_n$ denotes the matrix A after deleting the first n rows and n columns of it.

It has been shown recently (see [20]) that there are no other epimorphisms of $\mathrm{UT}(\infty, K)$, but the epimorphisms of the four types defined above and their compositions. We recall this result in the following

Lemma 1. *Let K be a field such that $|K| > 2$. Every epimorphism f of* $\mathrm{UT}(\infty, K)$ *onto itself is a composition of epimorphisms of types 1-4.*

We are interested only in group automorphisms. The epimorphisms of type 4 clearly are not injective, thus we have the following consequence of Lemma 1:

Corollary 1. *Every automorphism of* $\mathrm{UT}(\infty, K)$*, where* $|K| > 2$*, is a composition of automorphisms of types 1, 2 and 3.*

Since $\mathrm{UT}(\infty, K)$ is a topological group, it is natural to discuss the continuity of group epimorphisms. We recall that an endomorphism $f : A \to A$ of a topological group A is continuous, if the preimage of every open subset in $f(A)$ is open in A. We now prove

Proposition 1. *Every epimorphism of* $\mathrm{UT}(\infty, K)$*, where* $|K| > 2$*, is continuous.*

Proof. It is enough to show that epimorphisms of each of the four types 1, 2, 3 and 4 are continuous. The inner automorphism are continuous, because they involve only group operations, which are continuous by definition. Similarly, the diagonal automorphisms, when considered as inner automorphisms of the group $\mathrm{T}(\infty, K)$, are continuous by definition. Automorphisms of type 3 are induced by field automorphisms. Let ς be an automorphism of the field K, and $\bar{\varsigma}$ - the respective induced automorphism of $\mathrm{UT}(\infty, K)$. Consider an open ball $\mathcal{U}_s$ and its preimage under $\bar{\varsigma}$. It is clear that $\varsigma(a) \neq 0$ for all $a \in K^* = K \setminus \{0\}$, hence $\bar{\varsigma}(A)_{ij} = 0$ if and only if $A_{ij} = 0$. It follows that $\bar{\varsigma}^{-1}(\mathcal{U}_s) \subseteq \mathcal{U}_s$. Also it is clear that the bijectivity of σ implies the bijectivity of $\bar{\varsigma}$, and hence $\bar{\varsigma}^{-1}(\mathcal{U}_s) = \mathcal{U}_s$ is open.

Finally, let $Sh_n(A) := A|_n$ be the shift up by n, $n \geq 1$ and let $\mathcal{U}_s$ be an open ball in $UT(\infty, K)$, $s > 1$. Then the preimage $Sh_n^{-1}(\mathcal{U}_s)$ contains all matrices $A \in \mathrm{UT}(\infty, K)$, such that $A|_n \in \mathcal{U}_s$, i.e. $d(A|_n, E) \leq \frac{1}{2^{s-1}}$. Then we have:

$$Sh_n^{-1}(\mathcal{U}_s) = \bigcup_{A \in \mathrm{UT}(\infty, K),\ d(A|_n, E) \leq \frac{1}{2^{s-1}}} \mathcal{B}(A, \frac{1}{2^{s-1+n}}),$$

i.e. $Sh_n^{-1}(\mathcal{U}_s)$ is open in $\mathrm{UT}(\infty, K)$. Similarly, if $\mathcal{U}$ is an arbitrary open set in $UT(\infty, K)$, then it is a sum of basic open sets and their cosets. The arguments shown above that the preimage of every such open set is open, and so is the preimage of $\mathcal{U}$. The statement follows. □

Having all automorphisms of $\mathrm{UT}(\infty, K)$ characterized we discuss characteristic subgroups of $\mathrm{UT}(\infty, K)$. We begin with the following observation

Lemma 2. *Let* K *be a field such that* $|K| > 2$*. For every* $s \in \mathbb{N}$ *the basic set* $\mathcal{U}_s$ *is a characteristic subgroup of* $\mathrm{UT}(\infty, K)$

Proof. Observe first that $\mathcal{U}_s$ is a kernel of the natural projection π_s restricted to $\mathrm{UT}(\infty, K)$), which maps $\mathrm{UT}(\infty, K)$ onto $\mathrm{UT}(s, K)$ and thus $\mathcal{U}_s$ is a normal subgroup of $\mathrm{UT}(\infty, K)$. Now it is enough to check that $\mathcal{U}_s$ is invariant to the diagonal and field induced automorphisms of $\mathrm{UT}(\infty, K)$. Indeed, every field induced automorphism $\bar{\varsigma}$ preserves all entries of the matrix that are equal to zero or one and thus $\bar{\varsigma}(\mathcal{U}_s) \subseteq \mathcal{U}_s$. Similarly for every matrix $A \in \mathcal{U}_s$ and $D \in D(\infty, K)$ we have $D^{-1}AD \in \mathcal{U}_s$. Therefore $\mathcal{U}_s$ is characteristic, as stated. □

In next paragraphs we provide detailed description of all characteristic subgroups of $\mathrm{UT}(\infty, K)$.

3. Partition subgroups of $\mathbf{UT}(\infty, \boldsymbol{K})$

In the paper [23] A. Weir described characteristic subgroups of $\mathrm{UT}(n, K)$ for a finite field K using a concept of so-called *partition subgroups*. The characterization was completed to all fields by V. Levchuk in [9, 10]. We recall briefly these results and propose a natural generalization of the concept of normal partition subgroups to the case of infinite dimensional matrices.

Let $\hat{u} = \langle u_2, u_3, ..., u_n\rangle$ be a sequence of nonnegative integers. By $H(\hat{u})$ we denote the set of all matrices $a \in \mathrm{UT}(n, K)$ such that $a_{i,j} = 0$ for all indices with $j - u_j \leq i < j$. Then $H(\hat{u})$ is a subgroup of $\mathrm{UT}(n, K)$, which admits a nice graphical presentation using a diagram (we explain these diagrams and give some examples further in this section). To some extent the diagrams resemble Young diagrams of partitions (with possible permutations of components) and for this reason Weir called subgroups of this type partition subgroups [1]. We summarize the results of [23] and [9] in the following

Lemma 3. *Let K be a field such that $|K| > 2$.*

1. *If a subgroup H of $\mathrm{UT}(n, K)$ is normal, then whenever there exists $a \in H$ with $a_{i,j} \neq 0$ and $i < j$, the partition subgroup $Q_{i,j} = Q_{i,j}(\hat{u})$ with $\hat{u} = \langle u_2, u_3, ..., u_n\rangle$ and*

   $$u_k = \begin{cases} k-1, & k = 2, 3, ..., j-1, \\ j-i+1 & k = j, \\ j-i+t & k = j+t, t = 1, 2, ..., n-j+i, \end{cases} \qquad k = 2, 3, ..., n,$$

   is contained in H.

2. *A partition subgroup $H = H(\langle u_2, u_3, ..., u_n\rangle)$ of $\mathrm{UT}(n, K)$ is normal in $\mathrm{UT}(n, K)$ if and only if $u_{i+1} \leq u_i + 1$ for all $i = 2..., n-1$.*

3. *A subgroup H of $\mathrm{UT}(n, K)$ is invariant to all inner, diagonal and field induced automorphisms of $\mathrm{UT}(n, K)$ if and only if H is a normal partition subgroup.*

4. *A normal partition subgroup H of $\mathrm{UT}(n, K)$ is characteristic in $\mathrm{UT}(n, K)$ if and only if with every matrix $a \in H$ it contains the matrix H', which is symmetric to H with respect to the auxiliary diagonal.*

The latter condition results from the the fact that every group $\mathrm{UT}(n, K)$ admits an automorphism, that maps every matrix $A \in \mathrm{UT}(n, K)$ to the inverse

[1]We note that the notion of a partition subgroup introduced in [23] is more general, but for the purpose of our study it is enough to discuss only a special type of them, which can be defined by a sequence as shown above.

of A', which is symmetric to A with respect to the auxiliary diagonal. We note also that statement (3) of the above lemma concerns all subgroups $H \leq \mathrm{UT}(n, K)$ which are normal in $T(n, K)$, i.e. invariant to any triangular automorphism that conjugates a matrix by a triangular matrix from $\mathrm{T}(n, K)$.

Remark 1. *In the case of $|K| = 2$ one the above lemma does not hold.*

A simple counterexample was given in [9] for the group $\mathrm{UT}(3, \mathbb{F}_2)$, where one finds a cyclic subgroup $S = \langle e_3 + e_{1,2} + e_{2,3} \rangle$ of order 4, which is a characteristic non-partition subgroup. One can observe the exceptionality of the case of $K = \mathbb{F}_2$ looking into the group automorphisms of $\mathrm{UT}(n, \mathbb{F}_2)$. It is clear tht the diagonal automorphisms fall into the class of inner automorphisms in this case. Consequently, the description of characteristic subgroups of $\mathrm{UT}(n, \mathbb{F}_2)$ is substantially different.

Extending the definition to infinite dimensional matrix groups and following the terminology, the subgroup $H = H(\hat{u})$ of $\mathrm{UT}(\infty, K)$ will be called a partition subgroup, if it may be characterized by an infinite sequence of nonnegative integers $\hat{u} = \langle u_2, u_3, ... \rangle$ in the following way: $A \in H(\hat{u})$ if and only if $A_{i,j} = 0$ for all indices with $j - u_j \leq i < j$, $j = 2, 3, ...$. Direct calculations show that $H(\hat{u})$ is a subgroup of $\mathrm{UT}(\infty, K)$. For instance, if $\hat{u} = \langle u_2, u_3, ... \rangle$ is a sequence such that

$$u_k = \begin{cases} k-1, & k = 2, 3, ..., j-1, \\ j-i+1 & k = j, \\ j-i+t & k = j+t, t \geq 1, \end{cases} \qquad k = 2, 3, ...,$$

then $\bar{Q}_{i,j} = \bar{Q}_{i,j}(\hat{u})$ is a partition subgroup of $\mathrm{UT}(\infty, K)$, a generalization of the concept of $Q_{i,j} \leq \mathrm{UT}(n, K)$. If $\hat{v} = \langle v_2, v_3, ... \rangle$ is obtained from $\hat{u}$ by modifying the j-th coordinate of $\hat{u}$, i.e.:

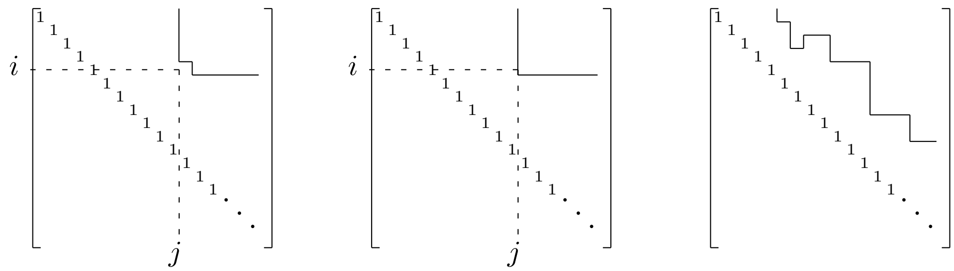
$$v_k = \begin{cases} k-1, & k = 2, 3, ..., j-1, \\ j-i & k = j, \\ j-i+t & k = j+t, t \geq 1, \end{cases} \qquad k = 2, 3, ...,$$

then the obtained partition subgroup $H(\hat{v})$ is called *rectangular* and denoted by $R_{i,j}$.

Following the ideas of Weir, we introduce a more illustrative way of characterizing partition subgroups – the diagrams of matrix groups. The diagrams of exemplary partition subgroups $\bar{Q}_{i,j}$, $R_{i,j}$ and $H(\hat{u})$, where $\hat{u} = \langle u_2, u_3, ... \rangle$ are presented in Fig.1.

The polyline denotes the border between the zero over-diagonal entries and arbitrary entries. It's shape is related to the sequence $\hat{u} = \langle u_2, u_3, ... \rangle$.

In the following theorem we summarize some general observations on partition subgroups.



(a) Diagram of $\bar{Q}_{i,j}$. (b) Diagram of $R_{i,j}$. (c) Diagram of $H(\hat{u})$.

Figure 1: Diagrams of partition subgroups.

Theorem 1.

1. *Let $H = H(\hat{u})$ be a partition subgroup defined by the sequence $\hat{u} = \langle u_2, u_3, ...\rangle$.*
   (a) *If $\hat{u} = \langle u_2, u_3, ...\rangle$ is an almost zero sequence, i.e. if there exists N such that $u_i = 0$ for all $i > N$, then H is open. Otherwise H has an empty interior.*
   (b) *H is closed in $\mathrm{UT}(\infty, K)$.*
   (c) *H is invariant to any automorphism induced by a field automorphism.*
2. *All partition subgroups of $\mathrm{UT}(\infty, K)$ constitute a distributive lattice $\mathcal{L}_{part}$.*

Proof.

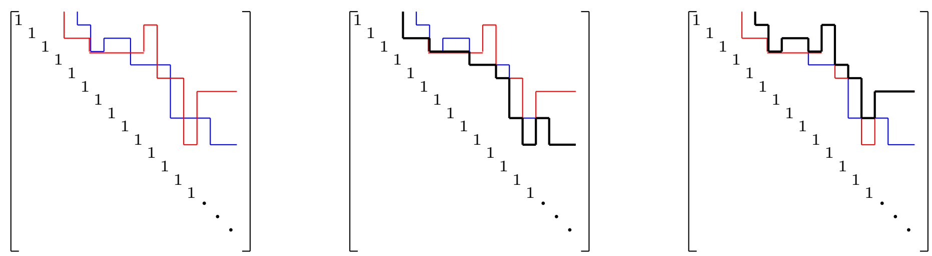
(1a) Let $\hat{u} = \langle u_2, u_3, ...\rangle$ be an infinite sequence of integers. Let us assume that it is an almost zero sequence and let N be such that $u_i = 0$ for all $i > N$, and let $r = \lfloor \log_{\frac{1}{2}} N \rfloor$. Then we have

$$H = H(\hat{u}) = \bigcup_{a \in \mathrm{UT}(N,K), A|_n = a} K(A, r),$$

hence H is open.

If $\hat{u}$ is not an almost zero sequence, then it has infinitely many nonzero terms. It is now sufficient to show that H does not contain any open ball $\mathcal{B}(A, r)$ centered at an arbitrary matrix $A \in H$. Indeed, if $\mathcal{B}(A, r) \subseteq H$, then H contains every matrix B, such that $B|_n = A|_n$ for $n = \lfloor -\log_2 r \rfloor$. In particular, one finds in H a matrix B such that $B_{i,i+1} \neq 0$ for all $i > n$, and we get a contradiction. Therefore H must be defined by $\hat{u} = \langle u_2, u_3, ...\rangle$ such that $u_i = 0$ for all $i > n$. This completes the proof.

(1b) Let $H = H(\hat{u})$ be a partition subgroup defined by a sequence $\hat{u} = \langle u_2, u_3, ...\rangle$ and let $G = \mathrm{UT}(\infty, K) \setminus H$. If $H = \mathrm{UT}(\infty, K)$, i.e. $k_j = 0$ for all



(a) Diagrams of groups $H(\hat{w})$ (red) and $H(\hat{u})$ (blue). (b) Diagram of group $H(\hat{w}) \cup H(\hat{u})$ (bold). (c) Diagram of group $H(\hat{w}) \cap H(\hat{u})$ (bold).

Figure 2: The diagrams of a sum and an intersection of partition subgroups.

$j \in \mathbb{N}$, then H is closed. So assume $G \neq \emptyset$. Then G contains all matrices from $\mathrm{UT}(\infty, K)$ that have a nonzero entry in the place (i,j), where $i \in \{j-u_j, ..., j-1\}$, $k_j > 0$. In particular, if $A \in G$ is such that $A_{i,j} \neq 0$ for certain (i,j) with $i \in \{j-u_j, ..., j-1\}$, $k_j > 0$, then the open ball $\mathcal{B}(A, \frac{1}{2^{j-1}})$ is contained in G. Thus G is open and hence H is closed.

(1c) Let us consider an automorphism $\bar{\varsigma}$ induced by a field automorphism ς and observe that $\bar{\varsigma}(A_{i,j}) = 0$ if and only if $A_{i,j} = 0$ and $\bar{\varsigma}(A_{i,j}) = 1$ if and only if $A_{i,j} = 1$. It follows that $\bar{\varsigma}(H(\hat{u})) = H(\hat{u})$.

(2) In the set of all infinite sequences of nonnegative integers we introduce a partial order $\preceq$. Namely, if $\hat{u} = \langle u_2, u_3, ...\rangle$ and $\hat{w} = \langle w_2, w_3, ...\rangle$, then we set $\hat{u} \preceq \hat{w}$ if and only if $u_i \geq w_i$ for every $i \in \mathbb{N}$. Observe that by definition, every infinite sequence of nonnegative integers $\hat{u}$ defines a unique partition subgroup $H(\hat{u})$, and two distinct sequences define distinct partition subgroups. The partial order $\preceq$ on sequences agrees with the relation of being a subgroup, i.e. $H(\hat{u}) \leq H(\hat{w})$ if and only if $\hat{u} \preceq \hat{w}$. Indeed, if $H(\hat{u}) \leq H(\hat{w})$, then for every $i \geq 2$ we have $u_i \geq w_i$, that is $\hat{u} \preceq \hat{w}$. Direct calculations show that the subgroup $H(\hat{w}) \cup H(\hat{u})$ generated by $H(\hat{w})$ and $H(\hat{u})$, and $H(\hat{w}) \cap H(\hat{u})$ are also partition subgroups $H(\hat{m})$ and $H(\hat{M})$ defined by the sequences (see Fig. 2):

$$\hat{m} = \langle \min\{u_1, w_1\}, \min\{u_2, w_2\}, ...\rangle = \inf_{\preceq}\{\hat{u}, \hat{w}\},$$

and

$$\hat{M} = \langle \max\{u_1, w_1\}, \max\{u_2, w_2\}, ...\rangle = \sup_{\preceq}\{\hat{u}, \hat{w}\}.$$

Moreover, as a lattice of sets, $\mathcal{L}_{part}$ is distributive. $\square$

Our next result concerns those partition subgroups, which are normal in $\mathrm{UT}(\infty, K)$ (some general remarks on normal subgroups of $\mathrm{UT}(\infty, K)$ may be found in [19]). Given a subgroup H of G, by H^G we denote the normal closure of H in G, i.e. the smallest normal subgroup of G containing H, and by H_G we denote the normal core of H in G, i.e. the largest normal subgroup of G contained in H. We have:

Theorem 2. *Let $H = H(\hat{u})$ be a partition subgroup of $\mathrm{UT}(\infty, K)$, $|K| > 2$, defined by a sequence $\hat{u} = \langle u_2, u_3, ...\rangle$.*

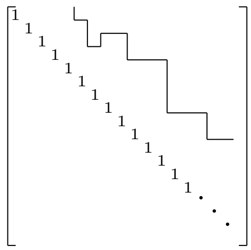
1. *H is normal in $\mathrm{UT}(\infty, K)$ if and only if for all $i = 2, 3...$ we have $u_{i+1} \leq u_i + 1$.*
2. *The normal closure $H^{\mathrm{UT}(\infty,K)}$ of H in $\mathrm{UT}(\infty, K)$ is the partition subgroup $H(\hat{w})$ defined by a sequence $\hat{w} = \langle w_2, w_3, ...\rangle$, where $w_2 = u_2$ and $w_i = \min\{u_i, w_{i-1} + 1\}$ for all $i > 2$.*
3. *The normal core $H_{\mathrm{UT}(\infty,K)}$ of H in $\mathrm{UT}(\infty, K)$ is the partition subgroup $H(\hat{v})$ defined by a sequence $\hat{v} = \langle v_2, v_3, ...\rangle$, where $v_i = \max\limits_{j \geq i} u_j$ for all $i \geq 2$.*

Proof.

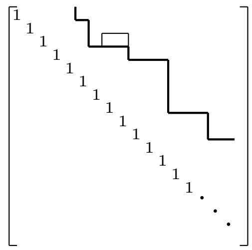
1. Let $H = H(\hat{u})$ defined by a sequence $\hat{u} = \langle u_2, u_3, ...\rangle$ be a normal subgroup of $\mathrm{UT}(\infty, K)$. Let us consider the subgroup $H(\hat{u}) \cdot \mathcal{U}_s$ of $\mathrm{UT}(\infty, K)$, $s \geq 1$. Being the kernel of the natural projection of $\mathrm{UT}(\infty, K)$ onto $\mathrm{UT}(s, K)$, the subgroup $\mathcal{U}_s$ is normal in $\mathrm{UT}(\infty, K)$. Therefore
$$H(\hat{u}) \cdot \mathcal{U}_s/\mathcal{U}_s \cong H(\langle u_2, ..., u_s\rangle)$$
is a normal partition subgroup of $\mathrm{UT}(\infty, K)/\mathcal{U}_s \cong \mathrm{UT}(s, K)$. By Lemma 3 we obtain that $u_{i+1} \leq u_i + 1$ for all $i = 2, ...s - 1$. As s was chosen arbitrarily, the statement follows.

   Now let $H = H(\hat{u})$ be the partition subgroup defined by an infinite sequence $\hat{u} = \langle u_2, u_3, ...\rangle$, where $u_{i+1} \leq u_i + 1$ for all $i \geq 2$. By Lemma 3 we have that for every $s > 1$ the partition subgroup
$$H(\langle u_2, ..., u_s\rangle) \cong H(\hat{u}) \cdot \mathcal{U}_s/\mathcal{U}_s$$
is normal in $UT(s, K) \cong \mathrm{UT}(\infty, K)/\mathcal{U}_s$. Hence $H(\hat{u}) \cdot \mathcal{U}_s$ is a normal subgroup of $\mathrm{UT}(\infty, K)$ and so is
$$H(\hat{u}) = \bigcap_{s \geq 2} H(\hat{u}) \cdot \mathcal{U}_s.$$

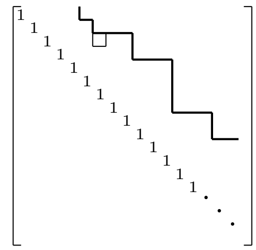
2. Let $H = H(\hat{u})$ be a partition subgroup of $\mathrm{UT}(\infty, K)$. If $H \trianglelefteq \mathrm{UT}(\infty, K)$ then the statement holds. Assume the contrary. By Lemma 3 it follows that there exists $i \geq 2$ such that $u_{i+1} > u_i + 1$. It is clear that $H(\hat{w})$ defined by a sequence $\hat{w} = \langle w_2, w_3, ...\rangle$, where $w_2 = u_2$ and $w_i = \min\{u_i, w_{i-1} + 1\}$



(a) Diagram of $H(\hat{u})$.



(b) Diagram of $H(\hat{u})^{\mathrm{UT}(\infty,K)}$.



(c) Diagram of $H(\hat{u})_{\mathrm{UT}(\infty,K)}$.

Figure 3: The diagrams of the normal closure and the normal core of a given partition subgroup $H(\hat{u})$.

for all $i > 2$ is the smallest (in the lattice of partition subgroups) partition normal subgroup of $\mathrm{UT}(\infty, K)$ containing H (see Fig. 3(b) for the diagram of $H(\bar{w})$). Let $N \trianglelefteq \mathrm{UT}(\infty, K)$ containing H. Then by the same arguments as above,

$$N_s \cong N \cdot \mathcal{U}_s/\mathcal{U}_s \trianglelefteq \mathrm{UT}(\infty, K)/\mathcal{U}_s$$

implies $N_s \trianglelefteq \mathrm{UT}(s, K)$, and moreover $H(\langle u_2, u_3, ..., u_s\rangle) \subseteq N_s$. Therefore, by statement (1) of Lemma 3, N_s contains all partition subgroups $Q_{i,j}$ for which $i < j$ and $a_{i,j} \neq 0$ for at least one matrix $a \in N_s$. In particular,

$$H(\langle u_2, u_3, ..., u_s\rangle) \cup \bigcup_{i=j-u_j-1} Q_{i,j} \subseteq N_s,$$

and hence $H(\langle w_2, w_3, ..., w_s\rangle) \subseteq N_s$ where $w_2 = u_2$ and $w_i = \min\{u_i, w_{i-1} + 1\}$ for all $i = 2, 3, ..., s$. It follows that N contains $H(\hat{w})$ such that $w_2 = u_2$ and $w_i = \min\{u_i, w_{i-1} + 1\}$ for all $i > 2$. Thus $H^{\mathrm{UT}(\infty,K)} = H(\hat{w})$.

3. Let $H = H(\hat{u})$ be a partition subgroup of $\mathrm{UT}(\infty, K)$, which is not normal. It is clear that H contains $H(\hat{v})$ in the statement (see Fig. 3(c) for the diagram of $H(\hat{v})$). Now let N be a normal subgroup contained in H and let $N_s \cong N \cdot \mathcal{U}_s/\mathcal{U}_s$ be the respective normal subgroups of $\mathrm{UT}(s, K)$ for $s \geq 2$. Obviously $N_s \subseteq H(\langle u_2, u_3, ..., u_s\rangle)$. Moreover, the statement (1) in lemma 3 implies that N_s does not contain matrices a for which $a_{i,j} \neq 0$ where $i = j - \max\limits_{i \leq k \leq n} u_k$ (otherwise $Q_{i,j} \subseteq N_s$, a contradiction). Therefore $N_s \subseteq H(\langle v_2, v_3, ..., v_s\rangle)$, where $v_i = \max\limits_{i \leq j \leq s} u_j$ for all $2 \leq s \leq n$. Then statement (3) follows easily. □

Remark. From the above theorem it follows that every rectangular partition subgroup is normal in $\mathrm{UT}(\infty, K)$.

4. Characteristic subgroups of $\mathrm{UT}(\infty, K)$

In this section we distinguish characteristic subgroups among all partition subgroups and prove that there are no other characteristic subgroups in $\mathrm{UT}(\infty, K)$ but those distinguished partition subgroups.

In the following we use the abbreviation NP-subgroup for the normal partition subgroup of $\mathrm{UT}(\infty, K)$.

Theorem 3. *Let K be a field such that $|K| > 2$.*

1. *A partition subgroup $H = H(\hat{u})$ defined by a sequence $\hat{u} = \langle u_2, u_3, ...\rangle$ is characteristic in $\mathrm{UT}(\infty, K)$ if and only if it is normal in $\mathrm{UT}(\infty, K)$.*

2. *Every characteristic subgroup of $\mathrm{UT}(\infty, K)$ is a NP-subgroup.*

Proof. 1. As the other is obvious, we need to deal only with one implication. Let $H = H(\hat{u})$ be a normal partition subgroup, i.e. a subgroup defined by a sequence $\hat{u} = \langle u_2, u_3, ...\rangle$, where $u_{i+1} \leq u_i + 1$ for all $i \geq 2$. By theorem 1, H is invariant to all field-induced automorphisms, so we have to show the invariancy of H to the diagonal automorphisms. By Lemma 3 for every $s > 1$ we have that

$$H(\langle u_2, ..., u_s\rangle) \cong H(\hat{u}) \cdot \mathcal{U}_s/\mathcal{U}_s$$

is invariant to all diagonal automorphisms of $UT(s, K) \cong \mathrm{UT}(\infty, K)/\mathcal{U}_s$. If $a \in \mathrm{UT}(s, K) \cup D(s, K)$ and A is an arbitrary representative of the coset of infinite matrices in $\mathrm{UT}(\infty, K)/\mathcal{U}_s$ or in $\mathrm{D}(\infty, K)/\mathcal{U}_s$ respective to a (e.g. $A[s] = a$), then $H(\langle u_2, ..., u_s\rangle)^a = H(\langle u_2, ..., u_s\rangle)$ implies $(H(\hat{u}) \cdot \mathcal{U}_s)^A = H(\hat{u}) \cdot \mathcal{U}_s$. Therefore $H(\hat{u}) \cdot \mathcal{U}_s$ is invariant to all diagonal automorphisms and hence

$$H(\hat{u}) = \bigcap_{s\geq 2} H(\hat{u}) \cdot \mathcal{U}_s$$

is a characteristic subgroup of $\mathrm{UT}(\infty, K)$.

2. Let H be a subgroup of $\mathrm{UT}(\infty, K)$, which is invariant to all inner, diagonal and field-induced automorphisms. Then $H \cdot \mathcal{U}_s$ is also a characteristic subgroup of $\mathrm{UT}(\infty, K)$ and hence $H \cdot \mathcal{U}_s/\mathcal{U}_s$ is invariant to all inner, diagonal and field-induced automorphisms of $\mathrm{UT}(s, K)$. By Lemma 3 the group H has to be a partition subgroup and there exists a sequence $\hat{u} = \langle u_2, ..., u_s\rangle$ such that $H \cdot \mathcal{U}_s/\mathcal{U}_s \cong H(\hat{u})$ and $u_{i+1} \leq u_i + 1$ for all $i \in \mathbb{N}$. Moreover, if $H' = H(\hat{u'}) \cong H \cdot \mathcal{U}_{s'}/\mathcal{U}_{s'}$ where $s < s'$ then $u' = \langle u_2, ..., u_{s-1}, u_s, ..., u_{s'}\rangle$. Therefore

$$H = \bigcap_{s\geq 2} H(\langle u_2, ..., u_s\rangle) \cdot \mathcal{U}_s = H(\hat{w}),$$

where $\hat{w} = (u_2, u_3, ...)$ is an infinite sequence such that $w_i = u_i$ for $i > 1$. $\square$

The above theorem is an analogue to statement (1) of Lemma 3.

Due to the exceptionality of the case $K = \mathbb{F}_2$ shown in Remark 1, the question whether or not the characteristic subgroups of the group $UT(\infty, \mathbb{F}_2)$ follow the statements of Theorem 3, remains open and requires a separate investigation.

Given two subgroups G_1 and G_2 of a group G by $[G_1, G_2]$ of G_1 and G_2 we denote the *mutual commutator* of G_1 and G_2, i.e. the subgroup generated by all commutators $[g_1, g_2]$, where $g_1 \in G_1$ and $g_2 \in G_2$. We note that if both G_1 and G_2 are characteristic subgroups of G, then so is $[G_1, G_2]$. In the following we discuss the mutual commutators of NP-subgroups with the whole group $\mathrm{UT}(\infty)$. We begin our investigations with rectangular partition subgroups.

Proposition 2. *Let $R_{i,j}$, $i < j$, be a rectangular partition subgroup of* $\mathrm{UT}(\infty, K)$. *Then* $[R_{i,j}, \mathrm{UT}(\infty, K)] = \bar{Q}_{i,j}$.

Proof. Let $i < j$ and $R_{i,j} = H(\hat{v})$ be a rectangular partition subgroup of $\mathrm{UT}(\infty, K)$ with $\hat{v} = (v_2, v_3, ...)$, where

$$v_k = \begin{cases} k-1, & k = 2, 3, ..., j-1, \\ n+t, & k = j+t, \ t \geq 0, \end{cases} \qquad k = 2, 3, ...,$$

and $n = j - i$. It is clear that $R_{i,j} \leq \mathrm{UT}(\infty, n, K)$. Thus it follows that

$$[R_{i,j}, \mathrm{UT}(\infty, K)] \leq [\mathrm{UT}(\infty, n, K), \mathrm{UT}(\infty, K)] = \mathrm{UT}(\infty, n+1, K).$$

On the other hand, if $A \in \mathrm{UT}(\infty, K)$, $B \in R_{i,j}$ then $B_{s,t}$ and $B^{-1}_{s,t} = 0$ for every (s,t) such that $t - v_t \leq s < t$. Then we have

$$\begin{aligned} [A, B]_{s,t} &= (A^{-1}B^{-1}AB)_{s,t} = \\ &= (A^{-1}B^{-1}A)_{s,t} + \sum_{k=s}^{t-v_t-1} \left(\sum_{l=s}^{k} \left(\sum_{m=s}^{l-v_l-1} A^{-1}_{s,m} B^{-1}_{m,l} + A^{-1}_{s,l} \right) A_{l,k} \right) B_{k,t} \end{aligned}$$

and for all (s,t) such that $t - v_t \leq s < t$, since $A^{-1}B^{-1}A \in R_{i,j}$ we have $[A, B]_{s,t} = (A^{-1}B^{-1}A)_{s,t} = 0$. Thus $[A, B] \in R_{i,j}$ and hence

$$[R_{i,j}, \mathrm{UT}(\infty, K)] \subseteq R_{i,j} \cap \mathrm{UT}(\infty, n+1, K) = \bar{Q}_{i,j}.$$

Now, we prove the reverse inclusion. The mutual commutator $[R_{i,j}, \mathrm{UT}(\infty, K)]$ is a characteristic subgroup of $\mathrm{UT}(\infty, K)$ and hence it is a NP-subgroup. Thus for our proof it suffices to find a matrix $C \in [R_{i,j}, \mathrm{UT}(\infty, K)]$ such that $C_{i,i+n+2} \neq 0$ and (in the case $i > 1$) additionally $C_{i-1,i+n+1} \neq 0$. Assume first $i > 1$ and take

$$\begin{aligned} A &= E + a \cdot E_{i,i+n+1} \in R_{i,j}, \\ B &= E + \sum_{k=1}^{\infty} E_{k,k+1} \in \mathrm{UT}(\infty, K). \end{aligned}$$

Then

$$[A,B]_{i-1,i+n+1} = \sum_{k=i-1}^{i+n+1} \left(\sum_{l=i-1}^{k} \left(\sum_{m=i-1}^{l} A^{-1}_{i-1,m} B^{-1}_{m,l} + A^{-1}_{i-1,l} \right) A_{l,k} \right) B_{k,i+n+1} = -a,$$

$$[A,B]_{i,i+n+2} = \sum_{k=i}^{i+n+2} \left(\sum_{l=i}^{k} \left(\sum_{m=i}^{l} A^{-1}_{i,m} B^{-1}_{m,l} + A^{-1}_{i,l} \right) A_{l,k} \right) B_{k,i+n+2} = a,$$

thus whenever $a \neq 0$ we have $C = [A,B]$ satisfying our requirements.

In the case $i = 1$ we also have

$$[A,B]_{1,n+3} = \sum_{k=1}^{n+3} \left(\sum_{l=1}^{k} \left(\sum_{m=1}^{l} A^{-1}_{1,m} B^{-1}_{m,l} + A^{-1}_{1,l} \right) A_{l,k} \right) B_{k,n+3} = a,$$

and again, if $a \neq 0$ then $C = [A,B]$ is the required matrix. The proposition follows. $\square$

Proposition 3. *Let H_1, H_2 be two characteristic subgroups of $\mathrm{UT}(\infty, K)$, $|K| > 2$. Then*

$$[H_1 H_2, \mathrm{UT}(\infty, K)] = [H_1, \mathrm{UT}(\infty, K)] \cdot [H_2, \mathrm{UT}(\infty, K)].$$

Proof. The inclusion $[H_1 H_2, \mathrm{UT}(\infty, K)] \supseteq [H_1, \mathrm{UT}(\infty, K)] \cdot [H_2, \mathrm{UT}(\infty, K)]$ is clear. To prove the reverse, take $h_1 \in H_1$, $h_2 \in H_2$ and $g \in \mathrm{UT}(\infty, K)$. Then

$$[h_1 h_2, g] = [h_1, g]^{h_2} [h_2, g],$$

and since $[H_1, \mathrm{UT}(\infty, K)]$ is a characteristic subgroup of $\mathrm{UT}(\infty, K)$, we have $[h_1, g]^{h_2} \in [H_1, \mathrm{UT}(\infty, K)]$. It follows that

$$[H_1 H_2, \mathrm{UT}(\infty, K)] \subseteq [H_1, \mathrm{UT}(\infty, K)] \cdot [H_2, \mathrm{UT}(\infty, K)],$$

and the proof is complete.

In particular, Proposition 2 applies to rectangular subgroups. The result can be extended easily by induction to a product of arbitrary finite number of rectangular partition subgroups:

Corollary 2. *If $H = H(\hat{w})$ is an NP-subgroup, which is a product of a finite number of rectangular partition subgroups we have:*

$$[H(\hat{w}), \mathrm{UT}(\infty, K)] = H(\hat{v}),$$

where

$$v_i = \begin{cases} \min\{i-1, w_i + 1\}, & \text{if} \quad w_i < w_{i-1} + 1, \\ w_i, & \text{if} \quad w_i = w_{i-1} + 1. \end{cases}$$

Propositions 1 and 2 have also direct implications on respective properties of characteristic partition subgroups of $\mathrm{UT}(s,K)$, $s \in \mathbb{N}$:

Corollary 3. *Let $H(\hat{w}|_s)$ be a normal partition subgroup of $\mathrm{UT}(s,K)$, isomorphic to $H(\hat{w}|_s)\cdot\mathcal{U}_s/\mathcal{U}_s$. Then $H(\hat{w}|_s)$ is a product of finite number of rectangular subgroups of $\mathrm{UT}(s,K)$ and*

$$[H(\hat{w}|_s), \mathrm{UT}(s,K)] = H(\hat{v}|_s),$$

where

$$v_i = \begin{cases} \min\{i-1, w_i+1\}, & \text{if} \quad w_i < w_{i-1}+1, \\ w_i, & \text{if} \quad w_i = w_{i-1}+1, \end{cases} \qquad 2 \le i \le s.$$

So far we have investigated only mutual commutators of rectangular partition subgroups and their finite products. This does not cover the whole class of normal partition subgroups, e.g. the basic subgroups $\mathcal{U}_s$ are not finite products of rectangular partition subgroups. Thus the next step in our discussion is to determine the mutual commutator of the basic subgroups $\mathcal{U}_s$.

Proposition 4. *For $s \in \mathbb{N}$ it holds:*

$$[\mathcal{U}_s, \mathrm{UT}(\infty,K)] = \mathcal{U}_s \cap \mathrm{UT}(\infty,1,K).$$

Proof. Observe that every matrix $A \in \mathcal{U}_s$ can be represented as a product $B\cdot U$

$$\left(\begin{array}{c|c} e_s & B \\ \hline 0 & U \end{array}\right),$$

where $B \in R_{s,s+1}$ and $U \in \mathcal{G} \cong \mathrm{UT}(\infty,K)$. Hence, as $R_{s,s+1}$ is normal, the basic group $\mathcal{U}_s$ is a semidirect product:

$$\mathcal{U}_s = R_{s,s+1} \rtimes \mathcal{G} \quad \sim \quad \left(\begin{array}{c|c} e_s & R_{s,s+1} \\ \hline 0 & \mathcal{G} \end{array}\right).$$

For $B_i := E + \sum\limits_{k=i}^{\infty} E_{k,k+1}$ we have

$$[E - E_{i+1,i+2}, B_i] = E + E_{i,i+2}.$$

Note that $E - E_{i+1,i+2} \in \mathcal{U}_s$ and $B_i \in \mathcal{G}$ for all $i \geq s-1$. Hence $E + E_{i,i+2} \in [\mathcal{U}_s, \mathrm{UT}(\infty,K)]$, and since $[\mathcal{U}_s, \mathrm{UT}(\infty,K)]$ is a characteristic subgroup of $\mathrm{UT}(\infty,K)$ then it is a normal partition subgroup. It follows that $R_{i,i+2} \subseteq [\mathcal{U}_s, \mathrm{UT}(\infty,K)]$ for every $i \geq s-1$. Thus

$$\mathcal{U}_s \cap \mathrm{UT}(\infty,1,K) \subseteq [\mathcal{U}_s, \mathrm{UT}(\infty,K)].$$

To see the reverse inclusion we first note that

$$[\mathcal{U}_s, \mathrm{UT}(\infty, K)] \subseteq [\mathrm{UT}(\infty, K), \mathrm{UT}(\infty, K)] = \mathrm{UT}(\infty, 1, K).$$

Moreover, for every pair of matrices $C \in \mathcal{U}_s$, $U \in \mathrm{UT}(\infty, K)$ we have

$$[C, U] = C^{-1}C^U \in \mathcal{U}_s,$$

thus $[\mathcal{U}_s, \mathrm{UT}(\infty, K)] \subseteq \mathcal{U}_s \cap \mathrm{UT}(\infty, 1, K)$. □

We are now ready for the final step in our discussion of mutual commutators.

Theorem 4. *Let $H = H(\hat{w})$ be a NP-subgroup defined by the sequence $\hat{w} = \langle w_2, w_3, ...\rangle$. Then $[H, \mathrm{UT}(\infty, K)]$ coincides with the partition subgroup $H(\hat{v})$ defined by the sequence $\hat{v} = \langle v_2, v_3, ...\rangle$, where*

$$v_i = \begin{cases} \min\{i-1, w_i+1\}, & \text{if} \quad w_i < w_{i-1}+1, \\ w_i, & \text{if} \quad w_i = w_{i-1}+1. \end{cases}$$

Proof. $H = H(\hat{w})$ be a NP-subgroup defined by the sequence $\hat{w} = \langle w_2, w_3, ...\rangle$. Consider a descending sequence of subgroups

$$H_s := H(\hat{w}) \cdot \mathcal{U}_s, \qquad s \in \mathbb{N}.$$

Obviously we have $H(\hat{w}) = \bigcap_{s=1}^{\infty} H_s$. For every $s \in \mathbb{N}$, H_s is a NP-subgroup and it can be represented as a product of a finite number of rectangular subgroups and the basic subgroup $\mathcal{U}_s$:

$$H_s = H(\hat{w}|_s) \cdot \mathcal{U}_s,$$

where $H(\hat{w}|_s) = \prod_{(i,j)\in\mathcal{J}} R_{i,j}$ and $\mathcal{J} = \{(i,j) \mid w_{j-1} \geq w_j \ \wedge \ i = j - w_j - 1 \leq s\}$.

As both $H(\hat{w}|_s)$ and $\mathcal{U}_s$ are characteristic subgroups of $\mathrm{UT}(\infty, K)$, thus by Proposition 3 we have

$$\begin{aligned} [H_s, \mathrm{UT}(\infty, K)] &= [H(\hat{w}|_s) \cdot \mathcal{U}_s, \mathrm{UT}(\infty, K)] = \\ &= [H(\hat{w}|_s), \mathrm{UT}(\infty, K)] \cdot [\mathcal{U}_s, \mathrm{UT}(\infty, K)] = \\ &= H(\hat{v}|_s) \cdot (\mathcal{U}_s \cap \mathrm{UT}(\infty, 1, K)), \end{aligned}$$

where

$$v_i = \begin{cases} \min\{i-1, w_i+1\}, & \text{if} \quad w_i < w_{i-1}+1, \\ w_i, & \text{if} \quad w_i = w_{i-1}+1, \end{cases} \qquad 2 \leq i \leq s.$$

It follows that

$$\begin{aligned} [H(\hat{w}), \mathrm{UT}(\infty, K)] &= [\bigcap_{s=1}^{\infty} H_s, \mathrm{UT}(\infty, K)] \subseteq \bigcap_{s=1}^{\infty} [H_s, \mathrm{UT}(\infty, K)] = \\ &= \bigcap_{s=1}^{\infty} H(\hat{v}|_s) \cdot (\mathcal{U}_s \cap \mathrm{UT}(\infty, 1, K)) = H(\hat{v}). \end{aligned}$$

where $\hat{v}$ is the infinite sequence form the statement of the theorem.

For the reverse inclusion observe from Proposition 2 that $Q_{i,j} \subseteq [H(\hat{w}), \mathrm{UT}(\infty, K)]$ whenever $R_{i,j} \subseteq H(\hat{w})$. The latter condition holds for $i = j - w_j - 1$ and $i \geq 2$. It follows that $H(\hat{v}) \subseteq [H(\hat{w}), \mathrm{UT}(\infty, K)]$, and this completes the proof. □

We conclude this section with the following remark:

Remark 2. Since $\mathrm{T}(\infty, K) = \mathrm{D}(\infty, K) \ltimes \mathrm{UT}(\infty, K)$, it follows that every subgroup H of $\mathrm{UT}(\infty, K)$ invariant to all inner and diagonal automorphisms of $\mathrm{UT}(\infty, K)$, is invariant to all inner automorphisms of $\mathrm{T}(\infty, K)$, as:

$$H^T = H^{D \cdot U} = (H^D)^U = H^U = H,$$

for every $T \in \mathrm{T}(\infty, K)$, $T = D \cdot U$, $U \in \mathrm{UT}(\infty, K)$, $D \in \mathrm{D}(\infty, K)$. Thus every NP-subgroup is a normal subgroup of $\mathrm{T}(\infty, K)$.

5. Verbal subgroups in $\mathbf{UT}(\infty, \boldsymbol{K})$ and $\mathbf{T}(\infty, \boldsymbol{K})$

Let $X = \{x_1, x_2, ...\}$ be the set of free generators of the free group F_∞. Given a set of words $W = \{f_i\}_{i \in I} \subseteq F_\infty$ and a group G, by $W(G)$ we denote the *verbal subgroup* of G, that is the subgroup generated by all values of the words w_i, $i \in I$, in group G. If $W = \{w\}$, we will simply write $w(G)$ instead of $W(G)$. For instance, let c_i denote the basic commutator words:

$$c_1 = x_1, \quad c_{i+1} = [x_{i+1}, c_i(x_1, ..., x_i)]$$

For any group G the verbal subgroups $c_i(G)$ constitute the lower central series $G = \gamma_1(G) > \gamma_2(G) > ...$, with $\gamma_i(G) = c_i(G)$. If every verbal subgroup of G coincides with one of $\gamma_i(G)$, $i \in \mathbb{N}$, then G is called *verbally poor*.

In particular, for $G = \mathrm{UT}(n, K)$, $n \in \mathbb{N}$, we have $\gamma_m(G) = \mathrm{UT}(n, m - 1, K)$ and in [1] group G was proved to be verbally poor. One straightforward consequence of the above facts is that the stable unitriangular group $\mathrm{UT}_f(\infty, K)$ is also verbally poor, that is every verbal subgroup $W(\mathrm{UT}_f(\infty, K))$ coincides with $\mathrm{UT}_f(\infty, m, K)$ for some $m \in \mathbb{N}$. This implication relies on the properties of direct limits of verbal subgroups and it fails when considering the inverse limit instead. Although an analogous result for the group $\mathrm{UT}(\infty, K)$ seems naturally expected, one needs another technique to prove this fact. Some partial results on certain specific verbal subgroups of $\mathrm{UT}(\infty, K)$ may be found in [3] and [4]. For the group of triangular matrices, only the finitely-dimensional case of $T(n, K)$ has been investigated [22].

It is known (see [12] for the reference) that every verbal subgroup is fully characteristic. We use this fact for the characterization of all verbal subgroups in $\mathrm{UT}(\infty, K)$. By now we have determined all characteristic subgroups of $\mathrm{UT}(\infty, K)$ – they are proved to be exactly the NP - subgroups. When considering the action of endomorphisms on these subgroups, one gets the following characterization of fully characteristic (and also verbal) subgroups in $\mathrm{UT}(\infty, K)$.

Theorem 5. *Let K be a field, $|K| > 2$.*

1. *Every fully characteristic subgroup of* $\mathrm{UT}(\infty, K)$ *coincides with* $\mathrm{UT}(\infty, m, K)$ *for some* $m \in \mathbb{N} \cup \{0\}$.

2. *The group* $\mathrm{UT}(\infty, K)$ *is verbally poor.*

3. *Every verbal subgroup* $W(\mathrm{T}(\infty, K))$ *coincides either with* $\mathrm{UT}(\infty, m, K)$ *for some* $m \in \mathbb{N} \cup \{0\}$ *or with the product* $W(\mathrm{D}(\infty, K)) \cdot \mathrm{UT}(\infty, K)$.

Proof.

1. Let H be a fully characteristic subgroup of $\mathrm{UT}(\infty, K)$. Since it is a characteristic subgroup, then $H = H(\hat{u})$ and $\hat{u} = (u_2, u_3, ...)$, where $u_{i+1} \leq u_i + 1$. Now let $n = \min_{i>1} u_i$ and consider the action of the shift Sh_1 on H. Since H is invariant to Sh_1, then $u_{i+1} = u_i = n$ for all $i > 2$. Hence H is a NP-subgroup defined by a constant sequence $\hat{u} = (n, n, n, ...)$. It follows that $H = \mathrm{UT}(\infty, n, K)$.

2. As every verbal subgroup H is necessarily fully characteristic, then clearly $H = \mathrm{UT}(\infty, n, K) = \gamma_n(\mathrm{UT}(\infty, K))$ for some $m \in \mathbb{N}$, i.e. every verbal subgroup of $\mathrm{UT}(\infty, K)$ coincides with a term of the lower central series of $\mathrm{UT}(\infty, K)$.

3. Let $W(\mathrm{T}(\infty, K)$ be a verbal subgroup of $\mathrm{T}(\infty, K)$, generated by a set of words W. Assume first that $W(\mathrm{T}(\infty, K)) \subseteq \mathrm{UT}(\infty, K)$. As a normal subgroup of $\mathrm{T}(\infty, K)$ and invariant to shifts, $W(\mathrm{T}(\infty, K))$ is a fully characteristic subgroup of $\mathrm{UT}(\infty, K)$. Hence by statement (1) we have $W(\mathrm{T}(\infty, K)) = \mathrm{UT}(\infty, m, K)$ for some $m \in \mathbb{N} \cup \{0\}$. Using the concepts of the proof in [2] in the case of finite matrices, it is not hard to see that $w(\mathrm{T}(\infty, K)) \subseteq \mathrm{UT}(\infty, K)$ whenever w is a commutator word, or - in the case of a finite field $K = \mathbb{F}_q$ - a word equivalent to a power x^m, where $(q-1)|m$ and $char K = p$. Here by a word equivalent to x^m we mean every word w in which the sum of exponents of one of the letters is equal to m.

   If $W(\mathrm{D}(\infty, K))$ is a nontrivial subgroup of $W(\mathrm{T}(\infty, K))$, then by the semiproduct decomposition of $\mathrm{T}(\infty, K)$:

   $$\mathrm{T}(\infty, K) = \mathrm{D}(\infty, K) \ltimes \mathrm{UT}(\infty, K)$$

   we have that

   $$W(\mathrm{T}(\infty, K)) = W(\mathrm{D}(\infty, K)) \cdot \mathcal{H},$$

   where $\mathcal{H}$ is a subgroup of $\mathrm{UT}(\infty, K)$ consisting of all unitriangular matrices $B \in \mathrm{UT}(\infty, K)$ such that $DB \in W(\mathrm{T}(\infty, K))$ for some $D \in W(\mathrm{D}(\infty, K))$. Moreover, as $W(\mathrm{T}(\infty, K))$ is fully characteristic subgroup of $\mathrm{T}(\infty, K)$, it is invariant to all inner automorphisms of $\mathrm{T}(\infty, K)$ (i.e. all inner and diagonal automorphisms of $\mathrm{UT}(\infty, K)$), all field-induced automorphisms and all shifts.

Let α be an endomorphism of $\mathrm{T}(\infty, K)$ of one of the above listed types. Then

$$\alpha(W(\mathrm{T}(\infty, K))) = W(\mathrm{T}(\infty, K)),$$

and in particular:

$$\alpha(W(\mathrm{D}(\infty, K)) \cdot \mathcal{H}) = W(\mathrm{D}(\infty, K)) \cdot \alpha(\mathcal{H}),$$

and since $\alpha(\mathrm{UT}(\infty, K)) \subseteq \mathrm{UT}(\infty, K)$ we have that $\alpha(\mathcal{H}) \subseteq \mathcal{H}$. Thus $\mathcal{H}$ is a fully characteristic subgroup of $\mathrm{UT}(\infty, K)$, and hence by statement (1) we have $\mathcal{H} = \mathrm{UT}(\infty, m, K)$ for some $m \in \mathbb{N} \cup \{0\}$. Moreover in [2] it was proved that if $W(\mathrm{T}_f(\infty, K)) \nsubseteq \mathrm{UT}_f(\infty, K)$ then $UT_f(\infty, K) \subseteq W(\mathrm{T}_f(\infty, K))$. Hence $UT_f(\infty, K) \subseteq W(\mathrm{T}(\infty, K))$ and thus $\mathcal{H} = \mathrm{UT}(\infty, K)$, as stated.

□

Remark 3. It is easy to see, that the verbal subgroup $W(\mathrm{D}(\infty, K))$ is isomorphic to the cartesian product of countably many isomorphic copies of $W(K^*)$, the respective verbal subgroup of the multiplicative group of the field. Some examples and detailed verbal structure of K^* for certain types of fields are discussed in [2].

We mention that statements (1) and (2) in the above theorem can be deduced alternatively from the results in [18] and [21]. Statement (3) is an analogue to results obtained in [1] and [2] for groups $\mathrm{T}(n, K)$ of finite dimensional matrices.

6. Verbal width in UT(∞, K) over a finite field

Another interesting consequence of the results obtained in Section 3, are the implications concerning the verbal width in the considered groups. We recall that the width $wid_W(G)$ of verbal subgroup $W(G)$ is defined to be the smallest (if such exists) number $l \in \mathbf{N}$ such that every element $A \in W(G)$ can be represented as a product of l values of words from W in group G (if such number does not exist, we say that the width is infinite). A well known result of Merzlyakov [11] states that every verbal subgroup in an algebraic group of matrices over a field K has finite width. Groups admitting this property are called verbally elliptic. In particular, Merzlyakov's result applies to groups $\mathrm{UT}(n, K)$. It is not known in general, whether the group $\mathrm{UT}(\infty, K)$ is verbally elliptic. In the following we prove verbal ellipticity of groups $\mathrm{UT}(\infty, K)$ over finite fields K, $|K| > 2$.

Let K be a finite field. Then every group $\mathrm{UT}(n, K)$ is finite and the limit $\mathrm{UT}(\infty, K)$ of the respective inverse spectrum $(\mathrm{UT}(n, K), \pi_{ij})$ is clearly a profinite group. The profinite topology agrees with the topology we introduced in Section 2; the cosets of all normal subgroups $\mathcal{U}_s$, $s \in \mathbb{N}$, constitute the basis of the profinite topology. We will use the equivalency (see [15]):

Lemma 4. *For every word $w \in F_\infty$ and a profinite group G the following conditions are equivalent:*

1. *$w(G)$ is closed,*
2. *$w(G)$ has finite width in G.*

We mention here that in general case for an arbitrary group G the above conditions are not equivalent and only the implication (2)$\Rightarrow$ (1) holds.

From Theorems 1 and 5, and the above lemma we deduce

Theorem 6. *If K is a finite field, then $\mathrm{UT}(\infty, K)$ is verbally elliptic.*

In other words, every verbal subgroup $W(\mathrm{UT}(\infty, K))$ has finite width. This result has also an application to the groups of finitely dimensional matrices. Namely, let us assume that $wid_W(\mathrm{UT}(\infty, K)) = n$ for some set of words W, i.e. every matrix $A \in W(\mathrm{UT}(\infty, K))$ is a product of at most n W-values in $\mathrm{UT}(\infty, K)$:

$$A = W_1 W_2 ... W_n,$$

each W_i being a W-value in $\mathrm{UT}(\infty, K)$. Consider the image of A under the projection π_s of $\mathrm{UT}(\infty, K)$ to $\mathrm{UT}(s, K)$:

$$\pi_s(A) = \pi_s(W_1 W_2 ... W_n) = \pi_s(W_1)\pi_s(W_2)...\pi_s(W_n).$$

Every W-value in $\mathrm{UT}(\infty, K)$ is mapped to a W-value in $UT(s, K)$. Moreover, as

$$\pi_s(W(\mathrm{UT}(\infty, K))) = W(\pi_s(\mathrm{UT}(\infty, K))) = W(\mathrm{UT}(s, K))$$

it follows that the width of $W(\mathrm{UT}(s, K))$ is at most n, i.e. the verbal subgroups of all groups of unitriangular matrices of finite dimension over a finite field K, which are generated by the same set of words W have the width bounded by $wid_W(\mathrm{UT}(\infty, K))$.

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