On Solvability of Engel Equations In the Group of Triangular Matrices Over a Field

Agnieszka Bier

Institute of Mathematics, Silesian University of Technology, ul. Kaszubska 23, 44-100 Gliwice, POLAND tel/fax: +48 32 237 28 64

Abstract

In the following paper we investigate commutator-type matrix equations and discuss the existence of their solutions. In particular, we derive the solutions of the Engel equations $A = e_k(X, Y)$, where $e_k(x, y)$ denotes the k-th Engel word, in the groups of unitriangular and triangular matrices over field K of arbitrary characteristic. The results directly apply to the discussion on the width of k-Engel subgroups in the considered groups. We conclude with a few observations on that matter.

Keywords: triangular matrices, commutator equalities, Engel subgroup, Engel width

2000 MSC: 20G15, 20F12, 20F45

1. Introduction

Recall that the commutator $[x, y]$ is the word in the free group F defined as $x^{-1}y^{-1}xy$. Then we define the k-Engel words recursively as follows:

$$e_1(x, y) = [x, y], \quad e_k(x, y) = [e_{k-1}(x, y), y] = [x, \underbrace{y, y, ..., y}_{k}].$$

Further, for brevity, we will use the common notation $e_k = [x,_k , y]$. An *Engel equation* is the equation of the type

$$e_k(x, y) = c,$$

where k is a fixed natural number, x and y are variables and c is an arbitrary constant. Given group G and a word $w(x, y) \in F$ we may substitute the letters x and y in w by elements of G and compute the resulting element. This element is called *the value* of the word w in group G or simply w-value. Hence the problem

Email address: agnieszka.bier@polsl.pl (Agnieszka Bier)

of finding solutions to Engel equations in a group is closely related to the problem of determining the Engel e_k - values in this group.

Throughout the paper we assume K to be a field of arbitrary characteristic. By $UT_n(K)$, $D_n(K)$ and $T_n(K)$ we denote the subgroups of the general linear group $GL_n(K)$, which are respectively the groups of upper unitriangular, diagonal and triangular matrices of size $n \times n$ with all entries from the field K. In the group $UT_n(K)$ we distinguish subgroups

$$UT_n^m(K) = \{\mathbf{1_n} + \sum_{i<j-m\leq n} a_{i,j}e_{i,j}, \quad a_{i,j} \in K\}, \quad 0 \leq m \leq n-1,$$

where $\mathbf{1_n}$ denotes the unity matrix of size $n \times n$ and $e_{i,j}$ denotes the matrix with unity in the place (i,j) and zeros elsewhere.

Groups which admit the identity $e_k(x,y) = 1$ for some $k \in \mathbb{N}$ are called k-Engel groups. These are exactly the groups in which all values of the given word e_k are trivial (unity). It is an interesting question whether a k-Engel group must be nilpotent, and if so - what is the dependence of the nilpotency class on k. It was proved in [4] that every Engel subgroup of the linear group is nilpotent. In [3] the authors characterized the maximal Engel subgroups of the group $T_n(R)$, R being a local ring, and calculated their class of nilpotency. The main result of that paper is that every maximal Engel subgroup of $T_n(K)$ up to conjugacy in $GL_n(K)$ is a direct product of the form

$$N_{n_1,n_2,\ldots n_s}(K) = N_{n_1}(K) \times N_{n_2}(K) \times \ldots \times N_{n_s}(K),$$

where $n = n_1 + n_2 + \ldots + n_s$ and $N_{n_i}(K)$ for every $i = 1,2,...,s$ denotes the direct product $S_{n_i}(K^*) \times UT_{n_i}(K)$. Here $S_{n_i}(K^*)$ stands for the group of all scalar matrices of size $n_i \times n_i$, i.e. the group $S_{n_i}(K^*) = \{\alpha \cdot \mathbf{1}_{n_i} \mid \alpha \in K^*\}$. It is clear from this description, that $N_{n_1,n_2,\ldots n_s}(K)$ is nilpotent of nilpotency class equal to

$$c = \max\{n_i \mid i = 1,2,...,s\} - 1.$$

Other examples of Engel subgroups in $T_n(K)$ include the group $UT_n(K)$ and its subgroups $UT_n^m(K)$, $m = 1,2,...,n-1$. In the presented paper we establish the dependence between the Engel property and the class of nilpotency for these groups.

The main part of this work covers the problem of characterization of Engel values in the considered groups by deriving the solutions of the respective Engel equations. In particular, for $UT_n(K)$ we prove:

Theorem 1. *Let K be an arbitrary field. Then for every matrix C contained in $UT_n^m(K)$ the Engel equation*

$$e_m(x,y) = C$$

has a solution in $UT_n(K)$.

For the group $T_n(K)$ we obtain a similar result with an additional constraint on the size of the field K.

Theorem 2. *Let K be a field containing at least $n+1$ elements. Then for every matrix C contained in $UT_n(K)$ the Engel equation*

$$e_k(x,y) = C$$

has a solution in $T_n(K)$, $k \in \mathbb{N}$.

We note that the results of [7] imply that for $k = 1$ the statement of Theorem 2 is valid regardless of the size of field K, since $e_1 = [x_1, x_2]$ is an outer commutator word. However, the methods used for the proof of this result can not be applied to the k-Engel values for $k > 2$.

Observe that the above results can be easily generalized to the groups of finitary matrices. Let $\varphi_n : T_n(K) \hookrightarrow T_{n+1}(K)$ denote the natural embedding, i.e.

$$\varphi_n(A) = \begin{pmatrix} A & \underline{0}^T \\ \underline{0} & \mathbf{1} \end{pmatrix} \qquad \text{for all } A \in T_n(K),$$

where $\underline{0}$ denotes a zero vector from K^n. Then obviously $\varphi_n(UT_n(K)) \subseteq UT_{n+1}(K)$ and $\varphi_n(D_n(K)) \subseteq D_{n+1}(K)$.

As a direct consequence of results of Theorems 1 and 2 we may describe the Engel values in the groups $UT_\infty(K) = \varinjlim_n (UT_n(K), \varphi_n)$ and $T_\infty(K) = \varinjlim_n (T_n(K), \varphi_n)$ of finitary unitriangular and triangular matrices over field K, respectively. Namely, we have:

Theorem 3. *Let K be an arbitrary field.*

1. *For every matrix A contained in $UT^m_\infty(K)$ the Engel equality*

   $$e_m(x,y) = A$$

   has a solution in $UT_\infty(K)$.

2. *If K is infinite, then for every matrix A contained in $UT_\infty(K)$ and any natural number k the Engel equation*

   $$e_k(x,y) = A$$

   has a solution in $T_\infty(K)$.

It is worth mentioning here that the characterization of word values in other groups of infinite matrices (such as row-finite or column -finite infinite matrices) can not be derived from the characterization of the finite case so easily. There are however some results concerning the values of commutators in the Vershik-Kerov group and some of its subgroups (see [5], [6])

Another straightforward generalization of Theorems 1 and 2 is the following. We introduce the *generalized Engel word* $e_{(k_1,k_2,...,k_s)}(x, y_1, y_2, ..., y_s)$ as the word of the form:

$$e_{(k_1,k_2,...,k_s)}(x, y_1, y_2, ..., y_s) = [x, \underbrace{y_1, y_1, ..., y_1}_{k_1}, \underbrace{y_2, y_2, ..., y_2}_{k_2}, ..., \underbrace{y_s, y_s, ..., y_s}_{k_s}],$$

where $y_i \neq y_{i+1}$ for $i \in \{1, 2, ..., s-1\}$ and they all are different from x. In the following we will use the notation:

$$e_{(k_1,k_2,...,k_s)}(x, y_1, y_2, ..., y_s) = [x, k_1, y_1, k_2, y_2, ..., k_s, y_s].$$

Then we define the *generalized Engel equation* to be the equation of the form:

$$e_{(k_1,k_2,...,k_s)}(x, y_1, y_2, ..., y_s) = c,$$

where $k_1, k_2, ..., k_s$ are fixed natural numbers, $x, y_1, y_2, ..., y_s$ are the unknown variables and c is a constant. Then we have:

Corollary 1. *Let K be an arbitrary field.*

1. *For every matrix C contained in $UT_n^m(K)$ the generalized Engel equation*

   $$e_{(k_1,k_2,...,k_s)}(x, y_1, y_2, ..., y_s) = C, \quad \textit{where} \;\; m = k_1 + k_2 + ... + k_s,$$

   is solvable in $UT_n(K)$;

2. *If K has at least $n+1$ elements, then for every matrix C contained in $UT_n(K)$ and for arbitrary coice of natural numbers $k_1, k_2, ..., k_s$ the generalized Engel equation*

   $$e_{(k_1,k_2,...,k_s)}(x, y_1, y_2, ..., y_s) = C$$

   is solvable in $T_n(K)$;

3. *For every matrix C contained in $UT_\infty^m(K)$ the generalized Engel equation*

   $$e_{(k_1,k_2,...,k_s)}(x, y_1, y_2, ..., y_s) = C, \quad \textit{where} \;\; m = k_1 + k_2 + ... + k_s,$$

   is solvable in $UT_\infty(K)$;

4. *If K is infinite, then for every matrix C contained in $UT_\infty(K)$ and for arbitrary choice of natural numbers $k_1, k_2, ..., k_s$ the generalized Engel equation*

   $$e_{(k_1,k_2,...,k_s)}(x, y_1, y_2, ..., y_s) = C$$

   is solvable in $T_\infty(K)$.

We summarize the paper with a discussion on the presented results in terms of verbal subgroups and verbal width.

Given group G and a word $w(x,y) \in F$ we define the *verbal subgroup* $V_w(G)$ of G to be the subgroup generated by all values of w in G. By definition, every element $x \in V_w(G)$ is a product of a finite number of w-values. For a given verbal subgroup an interesting question is the existence of the least number n such that every element of $V_w(G)$ can be presented as a product of at most n w-values. Such number n is then called the *width of verbal subgroup* (or verbal width) and denoted by $wid_w(G)$. There are also known examples of verbal subgroups having infinite verbal width.

The lattices of verbal subgroups in $UT_n(K)$ and $T_n(K)$ have been characterized in [1, 2]. In particular, it was proved that every verbal subgroup in $UT_n(K)$ coincides with one of the terms of the lower central series of $UT_n(K)$, i.e. with one of the subgroups $\gamma_{m+1}(UT_n(K)) = UT_n^m(K)$. In the case of the group $T_n(K)$, the following characterization of verbal subgroups was introduced: every verbal subgroup $V_w(T_n(K))$ is either a verbal subgroup of $UT_n(K)$ or can be represented as a product $V_w(D_n(K)) \cdot UT_n(K)$.

Knowing the lattice of verbal subgroups in a group, another interesting question is the coincidence of verbal subgroups defined by different sets of words as well as their corresponding verbal width. For $UT_n(K)$ and $T_n(K)$ this problem has been investigated in [1, 2, 7]. In particular, in [1, 2] it is determined with which terms of the lower central series of $UT_n(K)$ the verbal subgroups generated by outer commutator words and power words coincide and the respective verbal width is calculated. Additionally, in [1] it is proved that the width of verbal subgroups generated by simple commutators c_k in $T_n(K)$, where $char K = 0$, is equal to 1. Then in [7] the author calculates the width of verbal subgroups generated by outer commutator words and power words in $T_n(K)$ for the field of arbitrary characteristic. He proves that $wid_\omega(T_n(K)) = 1$ for every outer commutator word ω and $wid_{x^s}(T_n(K)) = 1$ for all but two cases of K being finite and the exponent s being divisible by $char K$ (see [7] for details).

The statements of Theorems 1 and 2 lead to the characterization of verbal subgroups generated by Engel words in groups $UT_n(K)$ and $T_n(K)$ along with their width. Thus in the last section of the paper we discuss our results in terms of verbal subgroups and their properties.

2. Solutions to Engel equations in the group of unitriangular matrices.

We shall start our considerations with a simple observation on some commutator equations in $UT_n(K)$. A following result will be useful:

Lemma 1. *Let B denote the matrix $\mathbf{1}_n + \sum\limits_{i=1}^{n-1} e_{i,i+1}$ from $UT_n(K)$. For every* *matrix $C \in UT_n^m(K)$, $m \geq 1$ the commutator equation*

$$C = [A, B]$$

has a solution $A \in UT_n^{m-1}(K)$.

Proof. We prove the result by induction on the matrix size n. At first we note that the smallest possible n is equal to $m+1$ and then $UT_n^m(K) = UT_{m+1}^m = \{\mathbf{1}_n\}$. Then $C = \mathbf{1}_n$ and for every matrix $A \in UT_n(K)$ we have $[A, B] = \mathbf{1}_n = C$. Thus, in particular, we may choose $A \in UT_n^{m-1}(K)$ satisfying the statement of the Lemma.

Now we assume that the Lemma holds for all matrices of sizes not greater than n. Let us choose $\bar{C} \in UT_{n+1}^m(K)$ and let $\bar{B}$ denote the matrix

$$\mathbf{1}_{n+1} + \sum_{i=1}^{n} e_{i,i+1} \in UT_{n+1}(K),$$

where $\bar{C}$ and $\bar{B}$ can be represented as

$$\bar{C} = \begin{pmatrix} C & \mathbf{c} \\ \mathbf{0} & 1 \end{pmatrix}, \qquad \bar{B} = \begin{pmatrix} B & \mathbf{b} \\ \mathbf{0} & 1 \end{pmatrix},$$

where $\mathbf{c}^T = (c_1, c_2, ..., c_{n-m}, 0, ..., 0) \in K^n$, $\mathbf{0}$ is a zero vector from K^n and $\mathbf{b}^T = (0, ..., 0, 1) \in K^n$.

Now, let $\bar{A} \in UT_{n+1}(K)$ be the matrix:

$$\bar{A} = \begin{pmatrix} A & \mathbf{a} \\ \mathbf{0} & 1 \end{pmatrix},$$

where $A \in UT_n(K)$. Then we have:

$$[\bar{A}, \bar{B}] = \begin{pmatrix} [A, B] & \mathbf{x} \\ \mathbf{0} & 1 \end{pmatrix},$$

where $\mathbf{x} = A^{-1}(B^{-1} - \mathbf{1}_n)\mathbf{a} + A^{-1}B^{-1}(A - \mathbf{1}_n)\mathbf{b}$. By our inductive assumption we can choose $A \in UT_n^{m-1}(K)$ such that $C = [A, B]$. Then we have:

$$\begin{aligned} &rank(A - \mathbf{1}_n) \leq n - m, \\ &rank(B^{-1} - \mathbf{1}_n) = rank(B - \mathbf{1}_n) = n - 1, \end{aligned}$$

and since multiplication by invertible matrices does not affect the matrix rank, then denoting $E = A^{-1}B^{-1}(A - \mathbf{1}_n)$ and $D = A^{-1}(B^{-1} - \mathbf{1}_n)$ we have

$$\begin{aligned} &rank(E) \leq n - m, \\ &rank(D) = n - 1. \end{aligned}$$

For the equality $\bar{C} = [\bar{A}, \bar{B}]$ to hold it is necessary that

$$\mathbf{c} = D\mathbf{a} + E\mathbf{b}.$$

Since E and $\mathbf{b}$ are already determined, then we observe that

$$E\mathbf{b} = \mathbf{e} = (e_1, e_2, ..., e_{n-m}, 0, ..., 0)^T$$

and we have to solve the system:

$$\mathbf{w} = D\mathbf{a},$$

where $\mathbf{w} = \mathbf{c} - \mathbf{e} = (w_1, w_2, ..., w_{n-m}, 0, ..., 0)^T$. Now, considering the rank of the augmented matrix of the system

$$rank(D|w) = \max\{n-1, n-m\} = n-1$$

we have that $rank(D|w) = rank(D)$ hence the system has a solution $\mathbf{a}$. Moreover, due to the rank of D we have $D_{1,2} \cdot D_{2,3} \cdot ... \cdot D_{n-1,n} \neq 0$ and hence:

$$\begin{array}{lll}
0 = w_{n-1} = D_{n-1,n} \cdot a_n & \Rightarrow & a_n = 0 \\
0 = w_{n-2} = D_{n-2,n-1} \cdot a_{n-1} + D_{n-2,n} \cdot a_n & \Rightarrow & a_{n-1} = 0 \\
\vdots & & \\
0 = w_{n-m+1} = D_{n-m+1,n-m+2} \cdot a_{n-m+2} + ... & \Rightarrow & a_{n-m+2} = 0.
\end{array}$$

Therefore $\bar{A} \in UT_{n+1}^{m-1}(K)$ and the Lemma is proved. □

Now we are ready to prove Theorem 1.

Proof of Theorem 1. We will show that for every matrix C from $UT_n^k(K)$ the k-Engel equation $e_k(A, B) = C$ has a solution: $A \in UT_n(K)$ and $B = \mathbf{1}_n + \sum_{i=1}^{n-1} e_{i,i+1}$.

The proof is inductive on k. For $k = 1$ we have $e_1 = [x, y]$ and hence $V_{e_1}(UT_n(K)) = UT_n^1(K)$. Moreover, by Lemma 1 we have that for every matrix $C \in UT_n^1(K)$ there exists a matrix $A \in UT_n(K)$ such that $C = [A, B]$ and the statement holds.

Now, assume that for all i-Engel words, $i < k$, and for every matrix C from $UT_n^i(K)$ the i-Engel equation $e_i(A, B) = C$ has a solution $A \in UT_n(K)$ and B defined as above. We consider the values of the word e_k. Let us take a matrix $C \in UT_n^k(K)$. Then, by Lemma 1, there exists solution $A \in UT_n^{k-1}(K)$ such that $C = [A, B]$. By the inductive assumption there exists a matrix $A' \in UT_n(K)$ being the solution to $A = e_{k-1}(A', B)$. Hence $C = [A, B] = [e_{k-1}(A', B), B] = [A',_{k-1}, B, B] = [A',_k, B] = e_k(A', B)$ and the theorem follows by induction. □

3. Engel values in the group of triangular matrices.

We begin the considerations with the following observation:

Lemma 2. *Let K be a field containing at least $n+1$ elements. Then for every matrix $A \in UT_n(K)$ the commutator equation*

$$A = [X, Y],$$

has a solution $X = U$, $Y = D$ such that $U \in UT_n(K)$ is a unitriangular matrix and $D = \sum_{i=1}^{n} d_i e_{i,i}$ is a diagonal matrix satisfying $d_i \neq d_j$ whenever $i \neq j$.

Proof. We prove the result by induction on the matrix size n. Take $n = 2$ and assume $|K| > 2$. Then K has at least two distinct invertible elements d_1 and d_2 and we may put

$$D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}.$$

Now, let $A = \mathbf{1}_2 + ae_{i,2}$, $a \in K$, be an arbitrary matrix from $UT_2(K)$. Take $U = \mathbf{1}_2 + ue_{i,2} \in UT_n(K)$, such that $u = (d_1^{-1}d_2 - 1)^{-1}a$. Then direct calculations show that

$$A = [U, D]$$

and the statement is true.

Now, let us assume that for a given n the statement of the Lemma holds. Take an arbitrary matrix $\bar{A} \in UT_{n+1}(K)$, and assume $|K| > n+1$. Observe that

$$\bar{A} = \begin{pmatrix} A & \underline{a} \\ \underline{0} & 1 \end{pmatrix},$$

where $A \in UT_n(K)$, $\underline{a}^T = (a_1, a_2, ..., a_n) \in K^n$ and $\underline{0} \in K^n$ is a zero vector. By our inductive assumption there exists a matrix $U \in UT_n(K)$ such that $A = [U, D]$ and $d_i \neq d_j$ for $i \neq j$. Define $\bar{U} \in UT_{n+1}(K)$ and $\bar{D} \in D_{n+1}(K)$ as follows:

$$\bar{U} = \begin{pmatrix} U & \underline{u} \\ \underline{0} & 1 \end{pmatrix}, \qquad \bar{D} = \begin{pmatrix} D & \underline{0}^T \\ \underline{0} & d_{n+1} \end{pmatrix},$$

where $\underline{u}^T = (u_1, u_2, ..., u_n) \in K^n$. As $|K| > n+1$ then there exists at least one invertible element k from K, different from all diagonal entries of matrix D. Put $d_{n+1} = k$. Then we have:

$$[\bar{U}, \bar{D}] = \begin{pmatrix} [U, D] & U^{-1}(D^{-1}d_{n+1} - \mathbf{1}_n)\underline{u} \\ \underline{0} & 1 \end{pmatrix},$$

and for the equality $\bar{A} = [\bar{U}, \bar{D}]$ it suffices that

$$\underline{a} = U^{-1}(D^{-1}d_{n+1} - \mathbf{1}_n)\underline{u}.$$

Since $D^{-1}d_{n+1} = \sum_{i=1}^{n} d_i^{-1}d_{n+1}e_{i,i}$ and by our assumptions $d_i^{-1}d_{n+1} \neq 1$ for all $i \in \{1, 2, ..., n\}$, then the matrix $U^{-1}(D^{-1}d_{n+1} - \mathbf{1}_n)$ is invertible and hence we may define

$$\underline{u} = \left(U^{-1}(D^{-1}d_{n+1} - \mathbf{1}_n)\right)^{-1}\underline{a}.$$

Then $\bar{A} = [\bar{U}, \bar{D}]$ and the Lemma follows by induction. $\square$

In the proof of Lemma 2 we constructed a matrix $D \in T_n(K)$, such that every element of the derived subgroup $UT_n(K)$ of $T_n(K)$ is a value of a commutator involving D. The sufficient condition for the existence of matrix D is that field K contains at least $n+1$ elements. In the following Lemma we prove that this condition is also a necessary one.

Lemma 3. *If $T = \sum_{i=1}^{n}\sum_{j=i}^{n} t_{i,j} \in T_n(K)$ is a triangular matrix with the property that for every unitriangular matrix $A \in UT_n(K)$ the equation*

$$A = [U, T]$$

has a solution $U \in UT_n(K)$, then all diagonal entries of matrix T are pairwise distinct, i.e. $t_{i,i} \neq t_{j,j}$ whenever $i \neq j$.

Proof. We prove the Lemma by induction on matrix size n. Let $T \in T_2(K)$

$$T = \begin{pmatrix} t_{1,1} & t_{1,2} \\ 0 & t_{2,2} \end{pmatrix}, \quad t_{i,i} \in K^*\text{for } i = 1, 2, \quad t_{1,2} \in K$$

be the matrix with the property assumed in the lemma. We take an arbitrary matrix $A \in UT_2(K)$, say

$$A = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \quad a \in K,$$

and consider the matrix equation $A = [U, T]$. Simple calculations show that for every matrix $U = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \in UT_2(K)$ we have:

$$[U, T] = \begin{pmatrix} 1 & u(t_{1,1}^{-1}t_{2,2} - 1) \\ 0 & 1 \end{pmatrix},$$

and the equality $A = [U, T]$ implies

$$a = u(t_{1,1}^{-1}t_{2,2} - 1). \tag{1}$$

As a is arbitrary element of K, it is clear that equation (1) has solution u if and only if $(t_{1,1}^{-1}t_{2,2} - 1)$ is invertible in K, that is $t_{1,1} \neq t_{2,2}$. Thus, for $n = 2$ the Lemma holds.

Now, assume that the Lemma holds for matrices of size less than or equal to n. Let $\bar{T} \in T_{n+1}(K)$ be the triangular matrix with the property assumed in the lemma and take an arbitrary matrix $\bar{A} \in UT_{n+1}(K)$. Then

$$\bar{A} = \begin{pmatrix} A & \underline{a} \\ \underline{0} & 1 \end{pmatrix}, \qquad \bar{T} = \begin{pmatrix} T & \underline{t} \\ \underline{0} & t_{n+1,n+1} \end{pmatrix}$$

where $\underline{a}^T, \underline{t}^T \in K^n$, $A \in UT_n(K)$, $T \in T_n(K)$ are fixed.

Again, direct calculations show that for every matrix $\bar{U} = \begin{pmatrix} U & \underline{u} \\ \underline{0} & 1 \end{pmatrix}$, such that $U \in UT_n(K)$ and $\underline{u}^T \in K^n$, we have:

$$[\bar{U}, \bar{T}] = \begin{pmatrix} [U,T] & B\underline{t} + C\underline{u} \\ \underline{0} & 1 \end{pmatrix},$$

where $B = U^{-1}T^{-1}(U - \mathbf{1}_n)$ and $C = U^{-1}(T^{-1}t_{n+1,n+1} - \mathbf{1}_n)$. Thus, the equation $\bar{A} = [\bar{U}, \bar{T}]$ is equivalent to

$$A = [U,T] \tag{2}$$

$$\underline{a} = B\underline{t} + C\underline{u}. \tag{3}$$

From our inductive assumption and (2) we have that $\bar{T}_{i,i} = T_{i,i} \neq T_{j,j} = \bar{T}_{j,j}$ for $i \neq j$, $1 \leq i,j \leq n$. In (3) the first part $B\underline{t}$ is fixed as $\bar{T}$ is fixed and U is determined by (2). Hence it is clear that if (3) has solution for every $\underline{a}^T \in K^n$ then C must be invertible. Thus $(T^{-1}t_{n+1,n+1} - \mathbf{1}_n) \in T_n(K)$ and hence $t_{n+1,n+1} \neq \bar{T}_{i,i}$ for all $i = 1, 2, ..., n$. The Lemma follows by induction. □

The statement of Lemma 3 implies that for the case of small fields (i.e. the fields containing less then n invertible elements), the search of solutions to the Engel equations $A = e_k(X,Y)$ for a given unitriangular matrix A and $k > 1$ require searching for both triangular matrices X and Y in parallel (none of them can be fixed in the general case).

Proof of Theorem 2. Assume that $k \geq 1$ and $|K| > n$. We will show that for every matrix C from $UT_n(K)$ the k-Engel equation $e_k(U,D) = C$ has a solution, where $D = \sum\limits_{i=1}^{n} d_i e_{i,i}$ and $d_i \neq d_j$ for all $i \neq j$.

The proof is inductive on k. For $k = 1$ we have $e_1 = [x,y]$ and hence, by results of [1], $V_{e_1}(T_n(K)) = UT_n(K)$. Then by Lemma 2 for every matrix $C \in UT_n(K)$ there exists a matrix $U \in UT_n(K)$ which is the solution to the equation $C = [U,D]$ with

$$D = \sum_{i=1}^{n} d_i e_{i,i}, \qquad d_i \neq d_j \quad \text{for all} \quad i \neq j. \tag{4}$$

Now, assume that for all i-Engel words, $i < k$, and for every matrix C from $UT_n(K)$ the i-Engel equations $e_i(U_i, D) = C$ are solvable in $UT_n(K)$, with D

being defined as in (4). Let us take a matrix $C \in UT_n(K)$ and consider the Engel equation $e_k(U,D) = C$. Again, by Lemma 2, there exists a matrix $U \in UT_n(K)$ such that $C = [U,D]$. By the inductive assumption there exists a matrix $U' \in UT_n(K)$ such that $U = e_{k-1}(U',D)$. Hence $C = [U,D] = [e_{k-1}(U',D),D] = [U',_{k-1},D,D] = [U',_k,D] = e_k(U',D)$ and the Theorem follows by induction. $\square$

Proof of Theorem 3.

Recall that $UT_\infty^m(K) = \varinjlim_n (UT_n^m(K),\varphi_n)$, hence for every infinite matrix $\widetilde{A} \in UT_\infty^m$ there exists such $N \in \mathbb{N}$ that for $i > N$ or $j > N$ the respective entry $\widetilde{A}_{ij}$ is either zero (if $i \neq j$) or 1 (if $i = j$). From Theorem 1 it follows that for every matrix $A \in UT_N^m(K)$ there exist matrices $B, C \in UT_N(K)$ such that $A = e_m(B,C)$. Define $\widetilde{B}, \widetilde{C} \in UT_\infty(K)$ such that

$$\widetilde{B}_{ij} = \begin{cases} B_{ij} & \text{if} \quad i \leq N \text{ and } j \leq N, \\ 1 & \text{if} \quad N < i = j, \\ 0 & \text{otherwise.} \end{cases} \quad , \quad \widetilde{C}_{ij} = \begin{cases} C_{ij} & \text{if} \quad i \leq N \text{ and } j \leq N, \\ 1 & \text{if} \quad N < i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\widetilde{A} = e_m(\widetilde{B},\widetilde{C})$ and the first statement of Theorem 3 follows (for a detailed reasoning in this matter check [1], Lemma 2 and proofs of Theorem 3 and 4).

For the proof of statement 2 we observe that if K is infinite field, then by Theorem 2 we have that for all $n \in \mathbb{N}$ every matrix $A \in UT_n(K)$ is a value of any Engel word $e_k(x,y)$. Hence, by the same arguments as above for an arbitrary matrix $\widetilde{A} \in UT_\infty$ we find the appropriate N and use Theorem 2 to solve the given Engel equation. Having the solution of the finite case, we construct the finitary solution as in the proof of statement 1. The proof is complete. $\square$

As a summary of Sections 1 and 2 we prove Corollary 1, which follows easily from the results of Theorems 1, 2 and 3.

Proof of Corollary 1. For any group G and an elements $g, a, b \in G$ observe that if a and b are solutions of the Engel equation $g = e_m(a,b)$, then

$$g = e_m(a,b) = e_{(k_1,k_2,\ldots,k_s)}(a,b,b,\ldots,b)$$

whenever $m = k_1 + k_2 + \ldots + k_s$. Thus a and b are solutions to the generalized Engel equation $e_{(k_1,k_2,\ldots,k_s)}(x,y_1,y_2,\ldots,y_s) = g$ in G. Now all statements of Corollary 1 follow from Theorems 1, 2 and 3. $\square$

4. Discussion

In this section we discuss briefly the implications of Theorems 1, 2 and 3 to verbal subgroups generated by Engel words in the considered linear groups.

By Theorems 1 and 2 we have that all elements of the group $UT_n^m(K)$ are m-Engel values in $UT_n(K)$ and , if K is sufficiently large, all elements for $UT_n(K)$

are m-Engel values in $T_n(K)$ for all $m \in \mathbb{N}$. In other words we proved the inclusions:

$$V_{e_m}(UT_n(K)) \supseteq UT_n^m(K), \qquad V_{e_m}(T_n(K)) \supseteq UT_n(K). \tag{5}$$

On the other hand, observe that in any group G

$$V_{e_k}(G) \subseteq V_{c_k}(G),$$

where $c_2(x_1,x_2) = e_1(x_1,x_2)$ and $c_k(x_1,x_2,...,x_k) = [c_{k-1}(x_1,x_2,...x_{k-1}),x_k]$ are basic commutator words, in which $x_i \neq x_j$ for $i \neq j$. Then the inclusions converse to (5) follow from the characterization of verbal subgroups in $UT_n(K)$ and $T_n(K)$ given in [1] and we have the following

Corollary 2. *Let K be an arbitrary field.*

1. *The verbal subgroup of group $UT_n(K)$ generated by the Engel word e_k coincides with $\gamma_{k+1}(UT_n(K))$ and has width equal to 1.*

2. *If K contains at least $n+1$ elements, then the verbal subgroup of group $T_n(K)$ generated by the Engel word e_k coincides with $\gamma_2(T_n(K))$ and has width equal to 1.*

3. *The verbal subgroup of group $UT_\infty(K)$ generated by the Engel word e_k coincides with $\gamma_{k+1}(UT_\infty(K))$ and has width equal to 1.*

4. *If K is infinite, then the verbal subgroup of group $T_\infty(K)$ generated by the Engel word e_k coincides with $\gamma_2(T_\infty(K))$ and has width equal to 1.*

We note also the nilpotency classes of the Engel subgroups in $T_n(K)$:

Corollary 3. *$UT_{n+1}(K)$ is a n-Engel group of nilpotency class n.*

Similar consequences are obtained for verbal subgroups generated by the generalized Engel words:

Corollary 4. *Let K be an arbitrary field.*

1. *The verbal subgroup of group $UT_n(K)$ generated by the generalized Engel word $e(k_1,k_2,...,k_s)$ coincides with $\gamma_m(UT_n(K))$ for $m = k_1+k_2+...+k_s+1$ and $wid_{e(k_1,k_2,...k_s)}(UT_n(K)) = 1$,*

2. *If K contains at least $n+1$ elements, then the verbal subgroup of group $T_n(K)$ generated by the generalized Engel word $e(k_1,k_2,...,k_s)$ coincides with $\gamma_2(T_n(K))$ and $wid_{e(k_1,k_2,...k_s)}(UT_n(K)) = 1$.*

Proof. For the proof of the first statement take $m = k_1 + k_2 + ...k_s$ and observe a pair of obvious inclusions:

$$V_{e_m}(UT_n(K)) \subseteq V_{e(k_1,k_2,...,k_s)}(UT_n(K)) \subseteq V_{c_{m+1}}(UT_n(K)) = UT_n^m(K).$$

By Corollary 2 we even have the equality

$$V_{e_m}(UT_n(K)) = V_{c_{m+1}}(UT_n(K))$$

which implies $V_{e(k_1,k_2,...,k_s)}(UT_n(K)) = V_{c_{m+1}}(UT_n(K))$. Moreover, every element of $V_{c_{m+1}}(UT_n(K))$ is a value of the word $e_m = [x,_m, y]$ for certain matrices $A, B \in UT_n(K)$, hence it is also the value of the word $e(k_1, k_2, ..., k_s) = [x,_{k_1}, y_1,_{k_2}, y_2, ...,_{k_s}, y_s]$ which we obtain by substituting x by A and y_1,y_2,..., y_s by B. Hence, $V_{e(k_1,k_2,...,k_s)}(UT_n) = \gamma_{m+1}(UT_n(K))$ and $wid_{e(k_1,k_2,...,k_s)}(UT_n(K)) = 1$.

The proof of the second statement is analogous. □.

[1] Bier, A., "Verbal subgroups in the group of triangular matrices over field of characteristic 0", J. Algebra 321, 2 (2009), p.483-494

[2] Bier, A., "The width of verbal subgroups in groups of unitriangular matrices over a field", Int. J. Alg. Comput. 22, 3 (2012), 1250019, 20 pp.

[3] Dokuchaev, M., Kirichenko, V., Polcino Milies, C., "Engel subgroups of triangular matrices over local rings", J. Algebra 290 (2005), p.433-446

[4] Garaščuk, M. S., Suprunenko, D. A., "Linear nilgroups", Dokl. Akad. Nauk BSSR 4 (1960), p.407-408

[5] Gupta, C.K., Hołubowski, W., "Commutator subgroup of Vershik-Kerov group", Linear Algebra and its Applications 436, 11 (2012), p.4279-4284

[6] Słowik, R. "The lower central series of subgroups of the Vershik–Kerov group", Linear Algebra and its Applications 436, 7 (2012), p.2299-2310

[7] Sosnovsky, Yu. V., "On the width of verbal subgroups of the groups of triangular matrices over a field of arbitrary characteristic", arXiv:1201.6513v1 [math.GR]