Groups Satisfying Semigroup Laws, and Nilpotent-by-Burnside Varieties

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We investigate the structure of groups satisfying a *positive law*, that is, an identity of the form $u \equiv v$, where u and v are positive words. The main question here is whether all such groups are nilpotent-by-finite exponent. We answer this question affirmatively for a large class $\mathcal{C}$ of groups including soluble and residually finite groups, showing that moreover the nilpotency class and the finite exponent in question are bounded solely in terms of the length of the positive law. It follows, in particular, that if a variety of groups is locally nilpotent-by-finite, then it must in fact be contained in the product of a nilpotent variety by a locally finite variety of finite exponent. We deduce various other corollaries, for instance, that a torsion-free, residually finite, n-Engel group is nilpotent of class bounded in terms of n. We also consider incidentally a question of Bergman as to whether a positive law holding in a generating subsemigroup of a group must in fact be a law in the whole group, showing that it has an affirmative answer for soluble groups. © 1997 Academic Press

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INTRODUCTION

Let F denote the free group on $X = \{x_1, x_2, \ldots\}$. A *positive word* in the x_i is a nontrivial element of F not involving the inverses of the x_i, that is, of the form $x_{i_1}^{m_1} x_{i_2}^{m_2} \cdots x_{i_k}^{m_k}$, where $k \geq 1$ and $m_j \geq 1$ for $j = 1, \ldots, k$. A *positive* (or *semigroup*) *law* of a group G is a nontrivial identity of the form $u \equiv v$ where u, v are positive words in F, holding under every substitution $X \to G$. The *degree* of such a law is the length of the longer of u, v.

In this paper we investigate a structure of groups satisfying a positive law. By a result of Mal'cev [13] (see also [15, 21]) a group which is an extension of a nilpotent group by a group of finite exponent satisfies a positive law. The main question of interest to us is whether the converse is true.

QUESTION 1. *If a group satisfies a positive law, must it be nilpotent-by-finite exponent?*

Ol'shanskiĭ and Storozhev [17] have shown that in this generality the question has a negative answer. They give an example of a 2-generator group satisfying a positive law which is nonetheless not nilpotent-by-finite. In contrast with this negative result, our main result (Theorem B) answers the question affirmatively for a large class $\mathcal{C}$ of groups including soluble and residually finite groups. This may be considered as a further illustration of the dichotomy, indicated, in particular, by the difference in methodologies and results associated with, on the one hand, the positive solution of the restricted Burnside problem and, on the other hand, the negative solution of the general Burnside problem, between groups built up in standard ways from soluble and locally finite groups and those not so constituted.

For each positive integer e we denote by $\mathfrak{B}_e$ the restricted Burnside variety of exponent e, that is, the variety generated by all finite groups of exponent e. We define an *SB-group* to be one lying in some product of finitely many varieties each of which is either soluble or a $\mathfrak{B}_e$ (for varying e).

THEOREM A. *There exist functions $c(n)$ and $e(n)$ of n only, such that any SB-group G satisfying a positive law of degree n is an extension of a nilpotent group of class $\leq c(n)$ by a locally finite group of exponent dividing $e(n)$, that is,*

$$G \in \mathfrak{N}_{c(n)} \mathfrak{B}_{e(n)},$$

where $\mathfrak{N}_c$ denotes the variety of all nilpotent groups of class $\leq c$.

For a closely related though possibly smaller class of groups than that of *SB*-groups a similar conclusion can be inferred from [6, Theorem C(ii)] except for the exclusive dependence of $c(n)$ and $e(n)$ on n. Compare also the result of Lewin and Lewin [11] that a finitely generated soluble group satisfying a positive law is nilpotent-by-finite.

The dependence in Theorem A of the parameters $c(n)$ and $e(n)$ uniquely on n has the consequence that if in a group G each finitely generated subgroup is an *SB*-group and satisfies a positive law (possibly depending on the subgroup) of degree $\leq n$, then $G \in \mathfrak{N}_{c(n)}\mathfrak{B}_{e(n)}$. Similarly, if G is a subcartesian product of *SB*-groups each of which satisfies a positive law of degree $\leq n$, then again $G \in \mathfrak{N}_{c(n)}\mathfrak{B}_{e(n)}$. Let $\mathcal{C}$ denote the class of groups obtained from the class of all *SB*-groups by repeated applications of the operations L and R, where for any group-theoretic class $\mathcal{X}$ of groups (see [19]), $L\mathcal{X}$ denotes the class of all groups locally in $\mathcal{X}$ and $R\mathcal{X}$ the class of groups residually in $\mathcal{X}$. In particular, residually finite and residually soluble groups are in $\mathcal{C}$. We thus have the following extension of Theorem A:

THEOREM B. *If a group G belongs to the class $\mathcal{C}$ and satisfies a positive law of degree n, then*

$$G \in \mathfrak{N}_{c(n)}\mathfrak{B}_{e(n)}.$$

For residually finite G this strengthens Theorem A of Shalev [21], which asserts in effect that a residually finite group satisfying a positive law is an extension of a "strongly locally nilpotent group" by a group of finite exponent. A *strongly locally nilpotent group* is one which generates a locally nilpotent variety; it follows readily from our Theorem A that in fact such a group belongs to $\mathfrak{N}_c\mathfrak{B}_e$ for some c and e:

THEOREM C. *A variety $\mathfrak{B}$ of groups is locally nilpotent-by-finite (i.e., has all of its finitely generated subgroups nilpotent-by-finite) if and only if*

$$\mathfrak{B} \subseteq \mathfrak{N}_c\mathfrak{B}_e$$

for some c, e.

It is perhaps appropriate to mention here by way of contrast with this result Golod's celebrated construction for each $d \geq 3$ of a non-nilpotent, residually finite d-generator group with all of its $(d-1)$-generator subgroups nilpotent (see, e.g., [9, p. 132]).

Remark. By results of Gromov [5], Milnor [14], and Wolf [24], a finitely generated group has polynomial growth if and only if it is nilpotent-by-finite. Thus, in view of Theorem B, for finitely generated groups in the class $\mathcal{C}$, having polynomial growth is equivalent to satisfying a positive law.

That this equivalence does not always hold for groups outside $\mathcal{C}$ is shown by the result of Adian [1] according to which the relatively free groups of odd exponent ≥ 665 and of finite rank > 1 have exponential growth. This may be considered as furnishing yet another illustration of the aforementioned dichotomy.

Part of the proof of Theorem A requires only the weaker assumption that some generating subsemigroup of G satisfies a positive law. Hence we obtain an affirmative answer for the class of soluble groups (see Theorem D) to the following well-known question of Bergman [2]:

QUESTION 2. *Let G be any group and $S \subseteq G$ any subsemigroup generating G. Must any positive law satisfied by S (i.e., holding under all substitutions $X \to S$) actually be satisfied by G?*

(We have been informed by Ivanov that he and Rips have independently constructed examples showing that the answer in general to Bergman's question is negative.)

THEOREM D. *Bergman's question has an affirmative answer for soluble groups: if G is a soluble group (or, slightly more generally, an extension of a soluble group by a locally finite group of finite exponent), and $S \subseteq G$ is any generating subsemigroup satisfying a positive law, then that law holds in G (whence, by Theorem A, G is actually in $\mathfrak{N}_{c(n)}\mathfrak{B}_{e(n)}$, where n is the degree of the positive law).*

We conjecture that this result holds more generally for SB-groups, and hence for groups in the class $\mathcal{C}$. It is not difficult to show that to achieve this extension of Theorem D it suffices to establish it in the case that G is an extension of a locally finite group of finite exponent by a nilpotent group. Taken together with the (unpublished) counterexamples of Ivanov and Rips, such a result would provide a further illustration of the dichotomy mentioned previously.

Our proofs of Theorems A and D rely heavily on Shalev's results in [21] concerning finite groups satisfying a positive law, on various results of Zelmanov [25–29] on the restricted Burnside problem and Engel Lie rings, and on results of Lubotzky and Mann [12] on "powerful p-groups." We relegate those proofs to Sections 2 and 3. In the concluding Section 4 we deduce Theorem C and the following further consequences of these theorems:

COROLLARY 1. *There exist functions $\hat{c}(n)$ and $\hat{e}(n)$ depending only on n, such that any residually finite n-Engel group without elements of order dividing $\hat{e}(n)$ is nilpotent of class $\leq \hat{c}(n)$. In particular, a torsion-free, residually finite, n-Engel group is nilpotent of class $\leq \hat{c}(n)$.*

This result generalizes the result of Zelmanov [26, Problem 1′] that a torsion-free nilpotent n-Engel group has its nilpotency class bounded in terms of n alone. It may also be regarded as an analogue for residually finite groups of Gruenberg's result on soluble n-Engel groups [7].

In [10] Shirshov asked if a group satisfying the nth Engel condition can be defined by positive laws. The preceding corollary affords an affirmative answer for residually finite such groups.

COROLLARY 2. *The laws of a residually finite n-Engel group follow from a set of positive laws.*

Finally, from Theorem B and the result of Cossey [3] that a nilpotent-by-finite group has a finite basis for its laws, one deduces the following:

COROLLARY 3. *Every finitely generated group in the class $\mathcal{C}$ satisfying a positive law possesses a finite basis consisting of positive laws.*

In conclusion, we note the following interesting result of Point [18, Prop. 8], used in proving Theorems A and D, but of independent interest. (We include a self-contained proof in Section 3 for completeness.)

PROPOSITION. *In a group satisfying a positive law every finitely generated subgroup H has finitely generated commutator subgroup $[H, H]$ (and moreover the rank of $[H, H]$ is bounded above in terms of the minimal number of generators of H and the degree of the positive law).*

We do not know if the conclusion of this result continues to hold under the weaker assumption that some generating subsemigroup of the group satisfies a positive law. A result in this direction would be useful for extending Theorem D to SB-groups.

Our notation is as follows. Let G be a group. We denote by $Z(G)$ the center of the group, by G^n the subgroup generated by all nth powers, by $\gamma_n(G)$ the nth term of the lower central series, and by $G^{(n)}$ the nth term of the derived series. For $x, y, x_1, \ldots, x_n \in G$ we write

$$[x_1, x_2] = x_1^{-1} x_2^{-1} x_1 x_2, \qquad [x_1, \ldots, x_n] = [[x_1, \ldots, x_{n-1}], x_n],$$
$$[x, {}_n y] = [[x, {}_{n-1} y], y].$$

We say that a group G is n-Engel if it satisfies the law $[x, {}_n y] \equiv 1$. If A and B are subsets of G, we denote by $\langle A \rangle$ the subgroup generated by A, and by $[A, {}_n B]$ the subgroup of G generated by all $[x, y_1, \ldots, y_n]$ where $x \in A$ and $y_1, \ldots, y_n \in B$. Following Lubotzky and Mann [12], we say that a p-group is *powerful* if $[G, G] \leq G^p$ ($[G, G] \leq G^4$ if $p = 2$).

2. PROOFS OF THEOREMS A AND D IN THE FINITE
AND SOLUBLE CASES

We first establish the finite case of Theorem A.

THEOREM 1. *There exist functions $c(n)$ and $e(n)$ such that any finite group G satisfying a positive law of degree n is an extension of a nilpotent group of class $\leq c(n)$ by a finite group of exponent dividing $e(n)$.*

This strengthens [21, Theorem B]; the improvement over Shalev's theorem consists in the independence of $c(n)$ of the number of generators of G.

Proof. By [21, Prop. 2.4] G has a soluble characteristic subgroup H such that the exponent of G/H divides $h(n)$, some function of n only. Then by [21, Corollary 3.3] there is a function $k = k(n)$ such that H^k is nilpotent. Being finite and nilpotent, H^k is the direct product of its Sylow p_i-subgroups P_i say. By [21, Corollary 4.2] there is a function $l = l(n)$ such that $P_i^{k_i}$ is a powerful p_i-group for some k_i dividing l. Our task is thus reduced to establishing Theorem 1 under the

Additional assumption 1. G is a powerful p-group.

By [21, Theorem B] there exist functions $c_1 = c_1(n)$ and $e_1 = e_1(n)$ such that every 2-generator subgroup $K \leq G$ satisfies

$$\gamma_{c_1+1}(K^{e_1}) = \{1\}.$$

Hence the group G satisfies, in particular, the 2-variable law

$$\left[x^{e_1}, {}_{c_1} y^{e_1}\right] = 1.$$

Since G is a powerful p-group every element of G^{e_1} has the form g^{e_1} for some $g \in G$ [12, Props. 1.7 and 4.1.7]. We infer that G^{e_1} satisfies the c_1-Engel law

$$\left[x, {}_{c_1} y\right] = 1.$$

It follows from [12, Corollary 1.5] that G^{e_1} is also powerful. As before, we can work instead with G^{e_1}, which has the same properties as G (a finite powerful p-group satisfying a positive law of degree n) and is, in addition, c_1-Engel. Thus we may, without loss of generality, make the further

Additional assumption 2. G is c_1-Engel.

We shall require a corollary of the following result of Zelmanov [25] on Engel Lie algebras.

THEOREM (Zelmanov [25]).   *A Lie algebra over a field of characteristic zero satisfying the rth Engel condition $[x, {}_r y] = 0$ is nilpotent of class $< c_2(r)$ where $c_2(r)$ depends only on r.*

From this one deduces

COROLLARY 4.   *If a Lie ring L satisfies the rth Engel condition $[x, {}_r y] = 0$, then there exists a further function $e_2 = e_2(r)$ such that*

$$e_2 L^{c_2} = 0. \tag{1}$$

Proof.   Let $L = L_{\mathbb{Q}}$ denote the free Lie algebra on free generators $x_1, x_2, \ldots$ over the field $\mathbb{Q}$ of rational numbers, and $L_{\mathbb{Z}}$ the subring of L generated by $x_1, x_2, \ldots$. The subring $L_{\mathbb{Z}}$ is then a free Lie ring with free generators $x_1, x_2, \ldots$, and $L_{\mathbb{Q}} = \mathbb{Q} L_{\mathbb{Z}}$. Denote by I the ideal of $L_{\mathbb{Z}}$ generated by all elements of the form $[u, {}_r v]$, $u, v \in L_{\mathbb{Z}}$. The quotients $L_{\mathbb{Z}}/I$ and $L_{\mathbb{Q}}/\mathbb{Q}I$ then both satisfy the rth Engel condition. By the theorem of Zelmanov quoted previously, the rational Lie algebra $L_{\mathbb{Q}}/\mathbb{Q}I$ is nilpotent of class $< c_2 = c_2(r)$. Hence the commutator $[x_1, x_2, \ldots, x_{c_2}]$ lies in the ideal $\mathbb{Q}I$, and so has the form

$$[x_1, x_2, \ldots, x_{c_2}] = \sum_i q_i [u_i, {}_r v_i] w_i,$$

where $u_i, v_i \in L_{\mathbb{Z}}$, w_i is an operator made up of a succession of right multiplications by elements from $L_{\mathbb{Z}}$, and the q_i are rational numbers. Since $x_1, x_2, \ldots$ are free generators of $L_{\mathbb{Q}}$, the preceding equation consti- tutes an identity in $L_{\mathbb{Q}}$, and therefore certainly continues to hold after replacing the x_j (on both sides) by any elements of $L_{\mathbb{Z}}$. Hence if e_2 is the least common multiple of the denominators of the q_i, then $e_2[u_1, u_2, \ldots, u_{c_2}]$ belongs to I for all $u_1, u_2, \ldots, u_{c_2} \in L_{\mathbb{Z}}$, whence $e_2 L_{\mathbb{Z}}^{c_2} \subseteq I$, or, equivalently,

$$e_2 (L_{\mathbb{Z}}/I)^{c_2} = 0.$$

The desired conclusion now follows from the fact that any (countable) Lie ring satisfying the rth Engel condition is a homomorphic image of $L_{\mathbb{Z}}/I$. ∎

Returning to the proof of Theorem 1, we recall that the problem has been reduced to the case where G is a powerful, finite, c_1-Engel p-group, where $c_1 = c_1(n)$ depends on n only. Write

$$e_3(n) = (c_1!)^{c_2(c_1)} e_2(c_1),$$

where e_2 and c_2 are as in (1). Since G is powerful it follows by repeated applications of [12, Props. 1.6 and 4.1.6] that

$$\left[G^{p^{i_1}}, G^{p^{i_2}}, \ldots, G^{p^{i_k}}\right] \le \gamma_k(G)^{p^{(i_1+i_2+\cdots+i_k)}}.$$

Since $G^m = G$ for integers m not divisible by p, we infer that

$$\left[G^{n_1}, G^{n_2}, \ldots, G^{n_k}\right] \le \gamma_k(G)^{n_1 n_2 \cdots n_k} \tag{2}$$

for any positive integers $n_1, \ldots, n_k$. In particular, this gives

$$\left[G^{e_3}, G^{e_3}\right] \le [G, G]^{e_3^2} \le G^{p e_3^2} = \left(G^{e_3}\right)^{p e_3}.$$

By absorbing e_3 into the exponent we may now take G^{e_3} in place of G; we may thus, without loss of generality, make our final

Additional assumption 3. $[G, G] \le G^{p e_3}$.

Consider the associated Lie ring $L = L(G)$ determined by the lower central series of G:

$$L = G/\gamma_2(G) \oplus \gamma_2(G)/\gamma_3(G) \oplus \cdots .$$

Since G is c_1-Engel we have (see [26, Lemma 6]) that the Lie ring L satisfies the linearized Engel condition

$$\sum_{\sigma \in \mathrm{Sym}(c_1)} \left[x, y_{\sigma(1)}, \ldots, y_{\sigma(c_1)}\right] = 0$$

for all $x, y_1, \ldots, y_{c_1} \in L$. Taking $y = y_1 = \cdots = y_{c_1}$, we infer that the subring $(c_1!)L$ satisfies the c_1-Engel condition $[x, {}_{c_1}y] = 0$. Hence by (1)

$$e_2(c_1)(c_1!L)^{c_2(c_1)} = 0,$$

that is,

$$(c_1!)^{c_2(c_1)} e_2(c_1) L^{c_2(c_1)} = e_3 L^{c_3} = 0, \tag{3}$$

where $e_3 = e_3(n)$ and $c_3 = c_3(n) = c_2(c_1(n))$ are functions of n only. Back in the group G, (3) yields

$$\left(\gamma_{c_3}(G)\right)^{e_3} \le \gamma_{c_3+1}(G).$$

Invoking (2) and Assumption 3, we infer from this that

$$\left(\gamma_{c_3}(G)\right)^{e_3} \le \gamma_{c_3+1}(G) = \left[[G, G], {}_{c_3-1}G\right] \le \left[G^{p e_3}, {}_{c_3-1}G\right] \le \left(\gamma_{c_3}(G)\right)^{p e_3}.$$

Since G is a finite p-group we must therefore have

$$\left(\gamma_{c_3}(G)\right)^{e_3} = \{1\}.$$

By (2) we then have

$$\gamma_{c_3}(G^{e_3}) \leq \left(\gamma_{c_3}(G)\right)^{e_3 c_3} \leq \left(\gamma_{c_3}(G)\right)^{e_3} = \{1\},$$

completing the proof. ∎

Proof of Theorem D, *and Theorem* A *in the case* $G \in \mathfrak{S}_l \mathfrak{B}_m$. Let G be an extension of a soluble group of derived length $\leq l$, by a locally finite group of exponent dividing m, and S be a subsemigroup satisfying a positive law $u \equiv v$ of degree n, and generating G. We shall show that for any subgroup H of G generated by a finite subset $A \subseteq S$, we have

$$H \in \mathfrak{N}_{c(n)}\mathfrak{B}_{e(n)},$$

where $c(n)$ and $e(n)$ are as in Theorem 1. Since S generates G, it will then follow that, in fact, $G \in \mathfrak{N}_{c(n)}\mathfrak{B}_{e(n)}$, establishing Theorem A for G. Furthermore, since any subgroup H (as before) is finitely generated and nilpotent-by-finite, it is residually finite. Since Bergman's question has an obvious affirmative answer for finite, and hence for residually finite, groups, it follows that for each subgroup H generated by a finite subset of S, Bergman's question has an affirmative answer. However, this must then be the case also for G since if the law $u \equiv v$ failed to hold in G, it would fail to hold in some such subgroup H. Hence Theorem D will also follow once we have established that $H \in \mathfrak{N}_{c(n)}\mathfrak{B}_{e(n)}$.

With H as given previously, note first that by a well-known result of Hall (see [19]) the quotient

$$\overline{H} = H/[\gamma_k(H^m), \gamma_k(H^m)]$$

is residually finite for all k. Hence there is a chain

$$\overline{H} \geq \overline{H}_1 \geq \overline{H}_2 \geq \cdots$$

of normal subgroups of $\overline{H}$ such that $|\overline{H} : \overline{H}_i| < \infty$ and $\cap_i \overline{H}_i = \{1\}$.

Let ϕ_i be the natural epimorphism of $\overline{H}$ onto $\overline{H}/\overline{H}_i$. Denote by S_0 the subsemigroup generated by the finite subset $A \subseteq S$; then $S_0 \subseteq H$. Since $\overline{H}/\overline{H}_i$ is finite we have $\phi_i(\overline{S}_0) = \phi_i(\overline{H})$, so that the finite group $\phi_i(\overline{H})$ satisfies the given positive law $u \equiv v$. Hence by Theorem 1, already proven, we have $\phi_i(\overline{H}) \in \mathfrak{N}_{c(n)}\mathfrak{B}_{e(n)}$, where $c = c(n)$ and $e = e(n)$ are as in that theorem. Since $\overline{H}$ is a subcartesian product of the $\phi_i(\overline{H})$, it then follows that $\overline{H} \in \mathfrak{N}_{c(n)}\mathfrak{B}_{e(n)}$, or, equivalently,

$$\gamma_c(H^e) \leq [\gamma_k(H^m), \gamma_k(H^m)] \qquad (4)$$

for all k. We now construct inductively a subnormal chain

$$H = N_0 \geq N_1 \geq \cdots \tag{5}$$

of finite-index subgroups of H, and a chain

$$S \supseteq S_0 \supseteq S_1 \supseteq \cdots \tag{6}$$

of subsemigroups such that S_i satisfies the given law $u \equiv v$ and generates N_i. Let $N_0 = H$, and let S_0 be as already defined. Define $N_1 = H^{me}$. Since H is soluble-by-finite, the quotient H/N_1 is finite (or order r say), whence N_1 is finitely generated (by s elements say). It is a well-known and relatively easy consequence of the fact that S_0 satisfies a positive law that every element h of H can be expressed in the form $h = ab^{-1}$ where $a, b \in S_0$. (From the positive law in the form $xw_1 \equiv yw_2$ it follows that $y^{-1}x \equiv w_2 w_1^{-1}$, whence every element of the form $y^{-1}x$, $x, y \in S_0$, can be replaced by one of the form $w_2 w_1^{-1}$, where $w_1, w_2 \in S_0$. By successively replacing the former by the latter in any string of elements of S_0 and their inverses, one obtains the desired result.) Hence there are elements $x_1, y_1, x_2, y_2, \ldots, x_s, y_s \in S_0$ such that

$$N_1 = \langle x_1 y_1^{-1}, \ldots, x_s y_s^{-1} \rangle.$$

Since $|H/N_1| = r$, we have that $y_i^r \in N_1 \cap S_0$, so that

$$N_1 = \langle x_1 y_1^{r-1}, y_1^r, x_2 y_2^{r-1}, y_2^r, \ldots, x_s y_s^{r-1}, y_s^r \rangle.$$

We define S_1 to be the subsemigroup of S_0 generated by the $2s$ elements $x_i y_i^{r-1}, y_i^r$ for $i = 1, \ldots, s$. Since $S_1 \subseteq S_0$, S_1 also satisfies the given law $u \equiv v$. In view of (4) we have

$$\gamma_c(N_1) \leq [\gamma_k(H^m), \gamma_k(H^m)] \leq [\gamma_k(N_0), \gamma_k(N_0)]$$

for all k. Note also that since $N_1 = H^{me} \leq H^m$, the group N_1 is soluble of derived length $\leq l$. We now consider

$$\overline{N}_1 = N_1/[\gamma_k(N_1), \gamma_k(N_1)]$$

in place of $\overline{H}$, and arguing as before deduce that

$$\gamma_c(N_1^e) \leq [\gamma_k(N_1), \gamma_k(N_1)]$$

for all k. We then set $N_2 = N_1^e$, and use the fact that N_1 is finitely generated and N_1/N_2 finite to find a finite subset of S_1 generating N_2, denoting by S_2 the subsemigroup generated by that finite subset. Continuing inductively, we construct the chains (5) and (6) with the property that, for all $i \geq 0$,

$$\gamma_c(N_{i+1}) \leq [\gamma_k(N_i), \gamma_k(N_i)]$$

for every k. It follows that

$$\gamma_c(N_{l+1}) \le \gamma_c(N_l)^{(1)} \le \gamma_c(N_{l-1})^{(2)} \le \cdots \le \gamma_c(N_1)^{(l)} = \{1\}.$$

Thus H is a finite extension of a group of class c, namely N_{l+1}. Hence H is certainly residually finite, and therefore, as noted at the beginning of the proof, Bergman's question has an affirmative answer for H; that is, the law $u \equiv v$ holds in H. Hence by Theorem 1 (which extends immediately to residually finite groups) we have $H \in \mathfrak{N}_{c(n)}\mathfrak{B}_{e(n)}$ as claimed.   ∎

3. PROOF OF THE PROPOSITION AND COMPLETION OF THE PROOF OF THEOREM A

Proof of the proposition.   Let G be any group satisfying a positive law. We wish to show that if H is any finitely generated subgroup of G then H' is also finitely generated. (It will be clear from the proof that, in fact, the minimal number of generators of H' is bounded above in terms of the number of generators of H and the degree of the positive law.)

We first show that for any two elements a, b of G, the subgroup $\langle b \rangle^{\langle a \rangle}$ generated by all conjugates $a^i b a^{-i} = b_i$ is finitely generated. It is easy to see that G satisfies some 2-variable positive law, which moreover may be assumed to be homogeneous. (If the law is not homogeneous, then it implies one of the form $x^n \equiv 1$, which implies in turn the homogeneous law $(xy)^n \equiv (yx)^n$.) Thus we may suppose our positive law to have the form

$$x^{m_1} y^{r_1} \cdots \equiv y^{s_1} x^{n_1} \cdots, \, m_i, n_i, s_j, n_j > 0,$$

where $\Sigma_i m_i = \Sigma_j n_j$ and $\Sigma_i r_i = \Sigma_j s_j$, and where the two sides of the identity end in different symbols. The substitution of a for x and ab for y yields the relation

$$a^{m_1}(ab)^{r_1} \cdots = (ab)^{s_1} \cdots .$$

Rewriting both sides as products of conjugates of b and canceling residual powers of a, we obtain

$$b_{m_1+1} b_{k_1} \cdots b_{k_t} = b_1 b_{l_1} \cdots b_{l_t}, \tag{7}$$

where $1 < m_1 + 1 < k_1 < \cdots < k_t$, $1 < l_1 < \cdots < l_t$, and $k_t \ne l_t$. Hence

$$b_1 \in \langle b_2, \ldots, b_k \rangle, \qquad k = \max\{k_t, l_t\}. \tag{8}$$

Repeated conjugation of (8) by a yields $b_i \in \langle b_2, \ldots, b_k \rangle$ for $i \le 1$. Similarly, by (7)

$$b_k \in \langle b_1, \ldots, b_{k-1} \rangle,$$

and repeated conjugation of this by a^{-1} yields $b_i \in \langle b_1, \ldots, b_{k-1} \rangle$ for $i \geq k$. Hence $\langle b \rangle^{\langle a \rangle}$ is generated by $b_2, \ldots, b_k$. It follows that if H is any finitely generated subgroup of G and $g \in G$, then

$$H^{\langle g \rangle} \text{ is finitely generated.} \qquad (9)$$

We next show that, given any two elements $a, b \in G$, the commutator subgroup $\langle a, b \rangle'$ is finitely generated. The crucial fact allowing this is that $\langle a, b \rangle'$ is generated by the elements of the form $[a, b]^{a^m b^n}$ where m and n are integers. This follows in turn from the well-known fact that $\langle a, b \rangle'$ is generated by all commutators of the form $[a^r, b^s]$, r and s integers, via repeated application of the identities

$$a^{-i}[a^r, b^s]a^i = [a^{r+i}, b^s][b^s, a^i],$$

$$b^{-i}[a^r, b^s]b^i = [b^i, a^r][a^r, b^{s+i}],$$

starting with $r = s = 1$. Now $\langle [a, b] \rangle^{\langle a \rangle}$ is finitely generated as before, whence by (9) $(\langle [a, b] \rangle^{\langle a \rangle})^{\langle b \rangle}$ is finitely generated, as required.

This establishes the 2-generator case. Now assume inductively that the claim is valid for subgroups of G which can be generated by $\leq n$ elements, and suppose that H requires $n + 1 > 2$ generators, say $h_1, \ldots, h_{n+1}$. Write H_i for the subgroup generated by

$$\{h_1, \ldots, h_{n+1}\} \setminus \{h_i\}, \qquad i = 1, \ldots, n + 1.$$

Then by the inductive hypothesis $[H_i, H_i]$ is finitely generated, whence so is $[H_i, H_i]^{\langle h_i \rangle}$. The conclusion now follows from the fact that $[H, H]$ is generated by the set-theoretical union of the $[H_i, H_i]^{\langle h_i \rangle}$. For this it suffices to show that the subgroup generated by this union, that is, by

$$U = \bigcup_i [H_i, H_i]^{\langle h_i \rangle},$$

is normal in H. For instance,

$$\left([H_1, H_1]^{h_1} \right)^{h_2} = [H_1, H_1]^{h_2 h_1 [h_1, h_2]} = [H_1, H_1]^{h_1 [h_1, h_2]},$$

and since $[H_1, H_1]^{h_1} \subseteq U$ and $[h_1, h_2] \in [H_3, H_3]$, we have $[H_1, H_1]^{h_1 h_2} \leq \langle U \rangle$. ∎

Completion of the proof of Theorem A. Let G be an SB-group, that is, $G \in \mathfrak{V}_1 \mathfrak{V}_2 \cdots \mathfrak{V}_t$, where each variety $\mathfrak{V}_i$ is either soluble or a $\mathfrak{B}_e$ for some e. Suppose G satisfies a positive law of degree n. The group G contains a normal subgroup H such that $H \in \mathfrak{V}_1$ and $G/H \in \mathfrak{V}_2 \cdots \mathfrak{V}_t$. Arguing by induction on t, we assume that Theorem A is valid for G/H, so that

$$G/H \in \mathfrak{N}_{c_1} \mathfrak{B}_{e_1}$$

for some c_1 and e_1. If $\mathfrak{V}_1 = \mathfrak{N}_{c_2}$ for some c_2, then $G \in \mathfrak{N}_{c_2}\mathfrak{N}_{c_1}\mathfrak{B}_{e_1} \subseteq \mathfrak{S}_{c_1 + c_2}\mathfrak{B}_{e_1}$, whence by the first case considered previously, we have $G \in \mathfrak{N}_{c(n)}\mathfrak{B}_{e(n)}$.

Now suppose that $\mathfrak{V}_1 = \mathfrak{B}_{e_2}$ for some e_2. Then $G \in \mathfrak{B}_{e_2}\mathfrak{N}_{c_1}\mathfrak{B}_{e_1}$. The group G contains a normal subgroup K from $\mathfrak{B}_{e_2}\mathfrak{N}_{c_1}$. It clearly suffices to show that $K \in \mathfrak{N}_{c(n)}\mathfrak{B}_{e(n)}$. We prove this by induction on c_1.

If $c_1 = 0$ we have nothing to prove. Consider the case $c_1 > 0$. It is enough to prove that every finitely generated subgroup of K lies in $\mathfrak{N}_{c(n)}\mathfrak{B}_{e(n)}$; hence without loss of generality we may assume that K is finitely generated. By Proposition 1 the commutator subgroup K' is finitely generated. The finitely generated subgroup $K' \in \mathfrak{B}_{e_2}\mathfrak{N}_{c_1 - 1}$, and therefore, by the inductive assumption, $K' \in \mathfrak{N}_{c(n)}\mathfrak{B}_{e(n)}$. Since K' is finitely generated, $(K')^{e(n)}$ has finite index in K'. The group K acts on the finite quotient $K'/(K')^{e(n)}$ by conjugation. Let C be the centralizer of this action. The group C has a finite index in K and $[C, C, C] \leq (K')^{e(n)}$ which is nilpotent. Hence C is a soluble subgroup of finite index in K, and by the first case of Theorem A, already proven, we have $K \in \mathfrak{N}_{c(n)}\mathfrak{B}_{e(n)}$.   ∎

4. DEDUCTION OF THEOREM C AND THE COROLLARIES

Proof of Theorem C.   Sufficiency is clear. For the necessity let $\mathfrak{V}$ be, as in the statement of the theorem, a locally nilpotent-by-finite variety of groups. We wish to show that $\mathfrak{V} \subseteq \mathfrak{N}_c\mathfrak{B}_e$ for some c, e.

The relatively free group of $\mathfrak{V}$ of rank 2 is by assumption nilpotent-by-finite, and therefore, according to Mal'cev [13], satisfies some positive law and hence some positive 2-variable law, of degree n say, which will then hold in every group in $\mathfrak{V}$. Since the finitely generated groups of $\mathfrak{V}$ are nilpotent-by-finite, they are certainly *SB*-groups, and so by Theorem A lie in $\mathfrak{N}_{c(n)}\mathfrak{B}_{e(n)}$. Hence by Theorem B every group of $\mathfrak{V}$ lies in $\mathfrak{N}_{c(n)}\mathfrak{B}_{e(n)}$.   ∎

Proof of Corollary 1.   We need two lemmas, the first of which is a result of Wilson [23]. We give a proof different from Wilson's, using a result of Zelmanov on Lie rings.

LEMMA 1 (Wilson [23]).   *A k-generator, residually finite n-Engel group K, is nilpotent of class bounded above by some function of k and n only.*

Proof.   Since a finite Engel group is nilpotent (by the well-known result of Zorn), we have that K is residually nilpotent. Consider the associated Lie ring

$$L = L(K) = \gamma_1(K)/\gamma_2(K) \oplus \gamma_2(K)/\gamma_3(K) \oplus \cdots,$$

determined by the lower central series of K. Since K is n-Engel by assumption, we have in L that $[x, {}_n y] = 0$ for all $x \in L$ and $y \in \gamma_i(K)/\gamma_{i+1}(K)$, and, as noted earlier, the less obvious linearized Engel identity (see [26, Lemma 6])

$$\sum_{\sigma \in \mathrm{Sym}(n)} [x, y_{\sigma(1)}, \ldots, y_{\sigma(n)}] = 0$$

for all $x, y_1, \ldots, y_n \in L$. By a result of Zelmanov [29] these two conditions imply that the k-generator Lie ring L is nilpotent of class bounded by a function of k and n only. Since K is residually nilpotent, the same function of k and n will then bound its class. ∎

LEMMA 2. *Let K be an n-Engel group and H be a normal subgroup of K without nontrivial elements of order dividing k. If for some element g of K we have $[H, g^k] = \{1\}$, then $[H, g] = \{1\}$.*

Proof. Assuming $[h, g^k] = 1$ for all $h \in H$, we prove by induction on i going from n to 1 that for all $h \in H$ we have $[h, {}_i g] = 1$. The initial proposition is $[h, {}_n g] = 1$ for all $h \in H$, which is valid since K is n-Engel. Suppose inductively that $[h, {}_{i+1} g] = 1$ for all $h \in H$. By using the identity

$$[a, bc] = [a, c][a, b][a, b, c]$$

repeatedly, one finds that for all $h \in H$,

$$1 = [h, {}_{i-1}g, g^k] = [h, {}_i g]^k \prod_j [h, {}_{i+1}g]^{n_j h_j}$$

for various integers n_j and elements $h_j \in H$. From the inductive hypothesis it follows that $[h, {}_i g]^k = 1$ for all $h \in H$, whence $[h, {}_i g] = 1$ since K has no k-torsion. This completes the induction. Hence $[h, g] = 1$ for all $h \in H$. ∎

We are now ready to prove Corollary 1. Let G be a residually finite and n-Engel group, and let K be the rank-2 relatively free group of the variety generated by G. Since G is residually finite, so is K, whence, by Lemma 1, K is nilpotent of class bounded by some function of n alone. It follows that K, and therefore every group in the variety generated by G, satisfies a positive 2-variable law of degree bounded in terms of n only. Hence, by Theorem B,

$$G \in \mathfrak{N}_{\hat{c}(n)} \mathfrak{B}_{\hat{e}(n)},$$

where $\hat{c}(n), \hat{e}(n)$ depend only on n.

Assuming now that G has no $\hat{e}$-torsion, we shall deduce that, in fact, $G \in \mathfrak{N}_{\hat{c}(n)}$. More specifically, we shall show by induction on m that if a group K is n-Engel, $\hat{e}$-torsion free, and belongs to $\mathfrak{N}_m \mathfrak{B}_{\hat{e}}$, then, in fact,

$K \in \mathfrak{N}_m$. If $m = 1$, then $K^{\hat{e}} = \{1\}$, whence $K = \{1\}$ since K has no $\hat{e}$-torsion. Suppose inductively that the claim is valid for m and that $K \in \mathfrak{N}_{m+1}\mathfrak{B}_{\hat{e}}$. Setting $H = \gamma_{m+1}(K^{\hat{e}})$, we have $[H, K^{\hat{e}}] = \{1\}$, whence by Lemma 2 we have $H \leq Z(K)$. We wish to show that the central quotient $\bar{K} = K/Z(K)$ has no $\hat{e}$-torsion so that we can apply the inductive hypothesis to it. If $\bar{g} = gZ(K) \in \bar{K}$ satisfied $\bar{g}^{\hat{e}} = 1$, we should have $g^{\hat{e}} \in Z(K)$, that is, $[K, g^{\hat{e}}] = 1$, whence by Lemma 2 $[K, g] = 1$, that is, $g \in Z(K)$, so that indeed $\bar{K}$ is $\hat{e}$-torsion free. Since $H \leq Z(K)$ we have that $\bar{K}^{\hat{e}}$ is nilpotent of class m, whence $\bar{K} \in \mathfrak{N}_m\mathfrak{B}_{\hat{e}}$. By the inductive hypothesis we then have $\bar{K} \in \mathfrak{N}_m$, whence $K \in \mathfrak{N}_{m+1}$, completing the proof. ∎

Proof of Corollary 2. Let G be a residually finite n-Engel group. As in the proof of Corollary 1, it follows that G is an extension of a nilpotent group by a group of finite exponent. Hence G satisfies a positive law, $u \equiv v$ say. A well-known argument shows that every law $w \equiv 1$ in G is a consequence of the particular law $u \equiv v$ and some other positive law $u_1 \equiv v_1$ holding in G, whence the result.

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