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Olga Macedońska, Witold Tomaszewski

ON EXISTENCE OF FIXED POINTS FOR AUTOMORPHISMS OF ORDER TWO

In this note we use standard terminology and notations from group theory ([2]). For example $[a,b]=a^{-1}b^{-1}ab$ and $[a, {}_kb]=[a,\underbrace{b,b,\ldots,b}_{k}]$.

DEFINITION 1 [3]. An automorphism α in a group G, which leaves only the neutral element fixed is called **regular.**

In the case of abelian groups all regular automorphisms of order 2 are completely described by the following simple observation.

LEMMA 1. *Let G be abelian. Then the group $AutG$ contains a regular automorphism, say α, of order 2 if and only if G contains no elements of order 2. In this case α is given by $g^\alpha = g^{-1}$ for every $g \in G$.*

Proof. Let $\alpha \in AutG$ be a regular automorphism of order two. Since gg^α is a fixed point we have $g^\alpha = g^{-1}$ for any $g \in G$. By assumption it means that G has no elements of order two.

The converse implication is clear.

THEOREM 1. *Let G be an arbitrary group and let α be a regular automorphism of order 2 in G. Then G is abelian in any of the following cases:*

1. *G is finite;*
2. *G is locally nilpotent;*
3. *For every $g \in G$, the subgroup $gp(g, g^\alpha)$ is finite;*

F or every $g \in G$, the subgroup $gp(g, g^\alpha)$ is nilpotent.

Proof. The first case was established in [5] and the second in [1]. Since the automorphism α is of order two, any subgroup $gp(g, g^\alpha)$ is α-invariant. If $gp(g, g^\alpha)$ is finite (nilpotent), then by [5] ([1]) it is abelian. Now the third and

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fourth cases follow immediately from one of the two first cases respectively and from the above Lemma.

For a non-abelian group G with an automorphism α of order two G. Higman proved that if G is locally nilpotent, then α has non-trivial fixed points in G. We proved that if for every $g \in G$, the subgroup $gp(g, g^{\alpha})$ is nilpotent, then α has non-trivial fixed points in G. To see that our condition is weaker we show that if g and g^{α} commute, then the group G need not be locally nilpotent.

EXAMPLE 1. *The group* $G = \langle x, y | [x^2, y] \rangle$ *has an automorphism* α *of order two, such that every* $g \in G$ *commutes with* g^{α}, *while* G *is neither nilpotent nor finite.*

Proof. The group G is neither nilpotent nor finite, because it has as a quotient the infinite dihedral group $D = \langle x, y | \, x^2, y^2 \rangle$. The map $x \to x^{-1}$, $y \to y$ defines the required automorphism, because it maps the relation $[x^2, y] = 1$ into $[x^{-2}, y] = [x^2, y]^{-x^{-2}} = 1$. Since x^2 belongs to the center of G, we have $x^{-1} = x$ modulo center. Then also g^{α} equals g modulo center and hence g and g^{α} commute as required.

The next aim of this note is a simple proof of Theorem 1 without using the G. Higman's result, based on Lie rings methods.

We start with a Lemma, which by itself can be useful.

LEMMA 2. *If* $[a, b^2] = 1$, *then* $[a, b]^{(-2)^k} = [a, {}_{k+1}b]$.

Proof. We shall denote

$$t_i = [a, b]^{(-2)^i} \text{ for } i \geq 0.$$

The following properties of symbols t_i are obvious:

(i) t_i, t_j commute,

(ii) $t_i = t_{i-1}^{-2}$.

We have to prove the equality

$$t_k = [a, {}_{k+1}b].$$

For $k = 1$, since $[a, b^2] = 1$, the equality follows from the commutator identity $[a, b]^{-2} = [a, b, b][a, b^2]^{-1}$. To proceed by induction, we assume, that

$$t_i = [a, {}_{i+1}b], \text{ for } i < k. \tag{1}$$

It follows from the assumption that

$$[t_{k-2}, b] = t_{k-1}. \tag{2}$$

We need also

$$[t_{k-2}^{-1}, b] = t_{k-1}^{-1}, \tag{3}$$

which follows from the identity $[a^{-1}, b] = a[a, b]^{-1}a^{-1}$, (2) and property (i):

$$[t_{k-2}^{-1}, b] = t_{k-2}[t_{k-2}, b]^{-1}t_{k-2}^{-1} = t_{k-2}\, t_{k-1}^{-1}\, t_{k-2}^{-1} = t_{k-1}^{-1}.$$

To make the inductive step to $t_k = [a, {}_{k+1}b]$, we use the assumption (1), the property (ii), the identity $[a^2, b] = [a, b]^a[a, b]$, and the equality (3), then $[a, {}_{k+1}b] = [a, {}_k b, b] = [t_{k-1}, b] = [t_{k-2}^{-2}, b] = [t_{k-2}^{-1}, b]^{t_{k-2}^{-1}}[t_{k-2}^{-1}, b] = (t_{k-1}^{-1})^{t_{k-2}^{-1}}\, t_{k-1}^{-1} = t_{k-1}^{-2} = t_k$, which finishes the proof.

COROLLARY 1. *If α is a regular automorphism of order two in a group G and for some k, $[a, a^\alpha]^{(-2)^k} = 1$, then $[a, a^\alpha] = 1$.*

Proof. An automorphism α is regular if and only if the following holds:

$$(4) \qquad (g^\alpha g^{-1} = 1) \implies (g = 1), \qquad \forall g \in G.$$

In the above notation $t_i = [a, a^\alpha]^{(-2)^i}$, we have to show that if $t_k = 1$, then $t_0 = 1$. Since $[a, a^\alpha]^{-1} = [a, a^\alpha]^\alpha$, we have $t_{k-1}^{-1} = t_{k-1}^\alpha$, and by (ii), $t_k = (t_{k-1}^{-1})^2 = t_{k-1}^\alpha t_{k-1}^{-1}$. Now by (4), if $t_k = 1$ then $t_{k-1} = 1$. By repeating the step we get $t_0 = 1$.

NOTATION. *Let G be a group with a regular automorphism α of order two, then for every $g \in G$ we define a sequence of elements $c_1 = g$, $c_2 = c_1 c_1^\alpha$ and for $i > 2$:*

$$(5) \qquad c_i = c_{i-1} c_{i-1}^\alpha.$$

It follows for $i > 1$ that

$$(6) \qquad c_i^{-\alpha} c_i = [c_{i-1}, c_{i-1}^\alpha].$$

We need two more properties of elements c_i.

LEMMA 3. *Let α be a regular automorphism of order two in a group G. If $g \in G$, we denote $H = gp(g, g^\alpha)$. Then in the above notation the following holds:*

$$(7) \qquad c_i^{-\alpha} c_i \in \gamma_i(H),$$

$$(8) \qquad [c_{i+1}, c_{i-1}^{-1}] = [c_{i-1}, c_{i-1}^{2\alpha}].$$

Proof. If $i = 2$, then by (6), $c_2^{-\alpha} c_2 = [c_1, c_1^\alpha] \in \gamma_2(H)$. Now by (6), by the identity $[a, b] = [b^{-1}a,\ b]$, and by the inductive assumption we get

$$c_i^{-\alpha} c_i = [c_{i-1}, c_{i-1}^\alpha] = [c_{i-1}^{-\alpha} c_{i-1}, c_{i-1}^\alpha] \in [\gamma_{i-1}(H), H] \subseteq \gamma_i(H),$$

which proves (8).

To prove (8) we note, that by (5), c_{i+1} can be written as $c_{i+1} = c_{i-1} c_{i-1}^{2\alpha} c_{i-1}$, which, by the identity $[ab^2 a, a^{-1}] = [a, b^2]$, gives required (8).

THEOREM 2. *Let α be a regular automorphism of order two in a group G. If for every $g \in G$, the subgroup $gp(g, g^\alpha)$ is nilpotent then G is abelian.*

Proof. By assumption for every $g \in G$ there exists $n = n(g)$, such that the subgroup $H = gp(g, g^\alpha)$ is n-nilpotent and hence by (7), $c_{n+1}^{-\alpha} c_{n+1} \in \gamma_{n+1}(H) = 1$. Since α is regular, it follows by (4), that $c_{n+1} = 1$.

To show by induction that $c_{n+1} = 1$ implies $c_2 = 1$, we perform the inductive step. Let $c_{i+1} = 1$, then because of (8), we get $[c_{i-1}, c_{i-1}^{2\alpha}] = 1$. By Lemma 2, we obtain $[c_{i-1}, c_{i-1}^{\alpha}]^{(-2)^{n-1}} \in \gamma_{n+1}(H) = 1$, and since α is regular, it follows by Corollary 1, that $[c_{i-1}, c_{i-1}^{\alpha}] = 1$. Now by (6), $c_i^{-\alpha} c_i = 1$ and again, since α is regular, we get $c_i = 1$. By repeating this step we obtain $c_2 = 1$, which means that for every $g \in G$, $gg^\alpha = 1$, and hence G is abelian as required.

We note now that in spite of the fact that nilpotent non-abelian groups do not have regular automorphisms of order two, there exist soluble non-abelian groups with regular automorphisms of order two.

EXAMPLE 2. *The infinite dihedral group $D = \langle x, y \mid x^2, y^2 \rangle$ is metabelian, but not nilpotent. The automorphism permuting generators is of order two and regular.*

In [4] we gave an example of "the biggest" two-generator metabelian group G, where the automorphism permuting generators is regular. By "the biggest" we mean that any other group with the same properties is a quotient group of G. A natural question arises:

In which varieties non-abelian groups have no regular automorphisms?

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INSTITUTE OF MATHEMATICS
SILESIAN TECHNICAL UNIVERSITY
ul. Kaszubska 23
44-100 GLIWICE, POLAND

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