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All Automorphisms of the 3-Nilpotent Free Group of Countably Infinite Rank Can Be Lifted

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It is known from S. Andreadakis (*Proc. London Math. Soc. (3)* **15** (1965), 239–268) and S. Bachmuth (*Trans. Amer. Math. Soc.* **122** (1966), 1–17) that for a free 3-nilpotent group $\bar{F}$ of rank n the map Aut $F \to$ Aut $\bar{F}$ is not onto. It is proved here that for $\bar{F}$ countably infinitely generated the map is onto.

1. INTRODUCTION

Jacob Nielsen [11] investigated the mapping class group of closed 2-dimensional orientable manifolds. That paper is interesting due to its group theoretical contents. It was shown that every automorphism of some quotient group F/N of a free group F is induced by an automorphism of the free group. The investigation of similar situations has continued since 1962 when A. W. Mostowski [7] proved that all automorphisms of an abelianised relatively free group $\bar{F}/\bar{F}'$ are induced from the free group $\bar{F}$. In 1985 it was proved by Bachmuth and Mochizuki [2′] that for a finitely generated free metabelian group G of rank >3 all automorphisms of G are induced from the free group F. A similar statement is true for G free nilpotent of class 2, but is not true for the nilpotency class $n \geqslant 3$; i.e., not all automorphisms of G are induced from the free group [1, 2].

It was natural to extend these results for the groups infinitely generated. In 1978 it was proved by R. Swan in [3] and then in [5] that all automorphisms of an infinitely generated free abelian group are induced from the free group.

In this paper the same follows for a free infinitely generated 2-nilpotent group. Our main result concerns 3-nilpotent free groups for which unexpectedly all automorphisms are induced from the free group in the case of

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countably infinite rank. We give here sufficient conditions for the relatively free group of infinite rank to have all automorphisms induced from the free group.

2. Notations and Preliminaries

By F we denote an absolutely free group of a finite or countably infinite rank. If it is necessary to fix the rank we write F_q. The relatively free group in a variety $\underline{V}$ we denote by $F(\underline{V})$ or by $\bar{F}$. As usual $[x, y] = x^{-1}y^{-1}xy$, $G' = [G, G] = \gamma_2(G)$, $\gamma_n(G) = [\gamma_{n-1}(G), G]$.

If $\operatorname{Aut} G$ is the automorphism group of G and V a characteristic subgroup in G, then there exists an obvious homomorphism $\operatorname{Aut} G \to \operatorname{Aut} G/V$, and every $\alpha \in \operatorname{Aut} G$ induces an $\bar{\alpha} \in \operatorname{Aut} G/V$. We say that $\bar{\alpha}$ can be lifted to α. We are interested here in the case when G is free and $V \subseteq G'$. Since Nielsen, it is known that the map $\operatorname{Aut} F_q \to \operatorname{Aut}(F_q/F'_q)$ is onto.

Theorem 1 (cf. [3, 5]). *For a free group F of any infinite rank the map* $\operatorname{Aut} F \to \operatorname{Aut} F/F'$ *is onto.*

In the case when $V = \gamma_n(F)$ and only for F finitely generated the following is known.

Theorem 2 (cf. [1, 2]). *The map* $\operatorname{Aut} F \to \operatorname{Aut} F/\gamma_3(F)$ *is onto.*

Theorem 3 (cf. [1, 2]). *The map* $\operatorname{Aut} F \to \operatorname{Aut} F/\gamma_n(F)$ *is not onto for* $n \geqslant 4$.

We shall show that in the case when $n = 4$ and F is infinitely generated, the situation is entirely different.

Let $V \subset W$ be two characteristic subgroups in G and suppose that in the row $\operatorname{Aut} G \to^{\sigma} \operatorname{Aut} G/V \to^{\delta} \operatorname{Aut} G/W$ the maps $\sigma\delta$ and δ are onto. We formulate then the obvious fact.

Lemma 1. *The map σ is onto if and only if for every $\bar{\alpha} \in \operatorname{Ker} \delta$, $\bar{\alpha}$ can be lifted to an $\alpha \in \operatorname{Aut} G$.*

We denote $\operatorname{Ker}(\operatorname{Aut} F_q \to \operatorname{Aut} F_q/F'_q)$ by K_q.

Theorem 4 (Nielsen–Magnus [10, 6]). *If in F_q we fix the base $x_1, x_2, ..., x_q$, then K_q is generated by the automorphisms of the types K^i_{jk}, K^i_{ij}.*

$$K^i_{jk}\colon \begin{cases} x_i \to x_i[x_j, x_k], & i \neq j \neq k \neq i, \\ x_t \to x_t, & t \neq i, \end{cases}$$

$$K^i_{ij}\colon \begin{cases} x_i \to x_i[x_i, x_j], & i \neq j, \\ x_t \to x_t, & t \neq i. \end{cases}$$

The K^i_{jk}, K^i_{ij} we call Nielsen–Magnus automorphisms. The kernel of the map $\operatorname{Aut} F_q \to \operatorname{Aut} F_q/\gamma_3(F_q)$ we denote by K'_q because of

THEOREM 5 (cf. [1, 2]). *K'_q is the commutator subgroup of K_q.*

THEOREM 6 (cf. [7]). *In a free nilpotent group G every endomorphism which induces the identity map in G/G' is an automorphism.*

COROLLARY 1. *If F is of finite rank then* $\operatorname{Ker}(\operatorname{Aut} F/\gamma_3(F) \to \operatorname{Aut} F/F')$ *is generated by the maps given below which change only one letter, say $\bar{x}_i$, leaving $\bar{x}_t$, $t \neq i$, fixed*:

$$\bar{K}^i_{jk}\colon \bar{x}_i \to \bar{x}_i[\bar{x}_j, \bar{x}_k], \qquad i \neq j \neq k \neq i,$$

$$\bar{K}^i_{ij}\colon \bar{x}_i \to \bar{x}_i[\bar{x}_i, \bar{x}_j], \qquad i \neq j,$$

$$\bar{K}^i_{ji}\colon \bar{x}_i \to \bar{x}_i[\bar{x}_j, \bar{x}_i], \qquad i \neq j.$$

It is worth noting here that taking the bars off gives us the automorphisms K^i_{jk} and K^i_{ij} in F, but K^i_{ji} is not an automorphism in F. Still the automorphism $\bar{K}^i_{ji}$ of $F/\gamma_3(F)$ can be lifted to the automorphism $(K^i_{ij})^{-1}$: $x_i \to x_i[x_i, x_j^{-1}]$, since modulo $\gamma_3(F)$ this is equal to $x_i[x_j, x_i]$. Because of Lemma 1 this proves the Theorem 2.

COROLLARY 2. *If F is of finite rank then* $\operatorname{Ker}(\operatorname{Aut} F/\gamma_4(F) \to \operatorname{Aut} F/\gamma_3(F))$ *is generated by all maps of the form*

$$R^i_{rst}\colon \bar{x}_i \to \bar{x}_i[\bar{x}_r, \bar{x}_s, \bar{x}_t], \qquad r \neq s,$$

$$\bar{x}_j \to \bar{x}_j, \qquad j \neq i.$$

Those automorphisms which can be lifted to F build a subgroup which is generated, by Theorems 5 and 4, by commutators of the Nielsen–Magnus automorphisms. This subgroup we denote by $\bar{K}'_q$ as the image of K'_q.

3. LI-PROPERTY

The purpose of the next two paragraphs is to show that the variety of 3-nilpotent groups satisfies two special properties. We consider F finitely generated with the fixed basis $x_1, x_2, ..., x_q$. In a variety $\underline{\underline{V}}$ the corresponding free group $\bar{F}$ has the basis $\bar{x}_1, \bar{x}_2, ..., \bar{x}_q$.

DEFINITION. The variety $\underline{\underline{V}}$ has LI (lifting identity)-property if for every natural number q and every $\bar{\alpha} \in \operatorname{Aut} \bar{F}$, which is induced by an $\alpha \in \operatorname{Aut} F$ the following implication holds: if $\bar{\alpha}: \bar{x}_i \to \bar{x}_i$ for $i \leqslant k$, then α can be chosen to satisfy $\alpha: x_i \to x_i$ for $i \leqslant k$.

The variety of abelian groups satisfy the LI-property because every matrix of the form

$$\begin{pmatrix} E_k & * \\ 0 & * \end{pmatrix}$$

can be changed into E_q by elementary transformations, which leave the first k columns fixed.

The relation similar to that given in Lemma 1 holds also for LI-property:

LEMMA 2. *If $V \leqslant W \leqslant F$ and $\underline{\underline{W}}$ satisfies the LI-property, then the variety $\underline{\underline{V}}$ satisfies the LI-property if for every $\bar{\alpha} \in \operatorname{Ker}(\operatorname{Aut} F/V \to \operatorname{Aut} F/W)$ which is induced by an $\alpha \in \operatorname{Aut} F$ the implication holds: if $\bar{\alpha}: \bar{x}_i \to \bar{x}_i$, $i \leqslant k$, then α can be chosen to satisfy $\alpha: x_i \to x_i$, $i \leqslant k$.*

Proof. Let $\bar{\beta} \in \operatorname{Aut} F/V$ and $\bar{\beta}$ be identical for $\bar{x}_i$, $i \leqslant k$. Then $\bar{\beta}$ induces $\bar{\bar{\beta}} \in \operatorname{Aut} F/W$ which, because of the LI-property can be lifted to $\beta \in \operatorname{Aut} F$, identical on x_i, $i \leqslant k$. Now β induces $\bar{\beta}' \in \operatorname{Aut} F/V$ so that $\bar{\beta} = \bar{\beta}'\bar{\alpha}$ for an $\bar{\alpha} \in \operatorname{Ker}(\operatorname{Aut} F/V \to \operatorname{Aut} F/W)$ which is also identical for x_i, $i \leqslant k$. This gives the result.

LEMMA 3. *The variety of 2-nilpotent groups satisfies the LI-property.*

Proof. Because of Lemma 2 we consider $\bar{\alpha} \in \operatorname{Ker}(\operatorname{Aut} F/\gamma_3(F) \to \operatorname{Aut} F/F')$. By the Corollary 1 $\bar{\alpha}$ is a product of the automorphisms which change only one generator and hence need not change $\bar{x}_i$, $i \leqslant k$, if $\bar{\alpha}$ does not. Since their liftings do the same we get the result.

To prove the LI-property for the variety of 3-nilpotent groups we need to find generators of the subgroup $\bar{K}'_q$ of induced automorphisms in $\operatorname{Ker}(\operatorname{Aut} F/\gamma_4(F) \to \operatorname{Aut} F/\gamma_3(F))$.

LEMMA 4. *The subgroup $\bar{K}'_q$ is generated, for $q \geqslant 3$, by automorphisms of the given five types $a(ijkr) = R^i_{jkr}$, $b(ijr) = R^i_{jri}$, $c(ijk) = R^i_{jkj}$, $d(ijr) = R^i_{rii}R^j_{jir}$, $e(ijr) = R^i_{irr}R^j_{rjr}$.*

Proof. We need to calculate essentially different commutators not equal to 1 of the Nielsen–Magnus generators modulo $\gamma_4(F)$. We place them into 7 sets according to the position of the top index at the bottom. $[\bar{K}^i_{ij}, \bar{K}^i_{rs}] = R^i_{rsj}$, $[\bar{K}^i_{jr}, \bar{K}^j_{st}] = R^i_{str}$, $[\bar{K}^i_{jr}, \bar{K}^s_{ti}] = R^s_{jrt}$, $[\bar{K}^i_{jr}, \bar{K}^r_{st}] = R^i_{tsj}$. These commutators give automorphisms of type a without the top letter at the bottom. $[\bar{K}^i_{ij}, \bar{K}^i_{ir}] = R^i_{jri}$, $[\bar{K}^i_{ij}, \bar{K}^j_{jr}] = R^i_{rji}$, $[\bar{K}^i_{ij}, \bar{K}^r_{ri}] = R^r_{ijr}$, $[\bar{K}^i_{ij}, \bar{K}^j_{rs}] = R^i_{sri}$ are the automorphisms of the b. $[\bar{K}^i_{ij}, \bar{K}^i_{jr}] = R^i_{jrj}$, $[\bar{K}^i_{jr}, \bar{K}^j_{rs}] = R^i_{rsr}$, $[\bar{K}^i_{jr}, \bar{K}^s_{ij}] = R^s_{rjj}$, $[\bar{K}^i_{jr}, \bar{K}^s_{ir}] = R^s_{rjr}$ are type c. We get also $d(ijr) = [\bar{K}^i_{ij}, \bar{K}^j_{ir}] = R^i_{rii} R^j_{jir}$, $e(ijr) = [\bar{K}^i_{jr}, \bar{K}^j_{ir}] = R^i_{irr} R^j_{rjr}$, $f(ij) = [\bar{K}^i_{ij}, \bar{K}^j_{ji}] = R^i_{iji} R^j_{ijj}$ and $g(ijrs) = [\bar{K}^i_{jr}, \bar{K}^j_{is}] = R^i_{isr} R^j_{rjs}$.

Since we calculate modulo $\gamma_4(F)$, we have $R^{-1}_{ijr} = R_{jir}$ and $R_{irj} R_{rji} R_{jir} = 1$, then $f(ij) = R^i_{iji} R^j_{ijj} = R^r_{irj} R^i_{iji} R^j_{ijj} R^r_{rji} R^r_{jir} = d(irj)^{-1}\, d(jri)\, b(rji)$ and $g(ijrs) = R^i_{isr} R^j_{rjs} = (R^i_{rsi} R^i_{irs})\, R^j_{rjs} = R^i_{rsi}(R^r_{srr} R^i_{irs})(R^r_{rsr} R^j_{rjs}) = b(irs)\, d(ris)\, d(rjs)^{-1}$ which finishes the proof.

Our next aim is to show that the variety of 3-nilpotent groups satisfies the LI-property. So we are interested to know if it is possible for a non-trivial product of automorphisms a, b, c, d, e each of which changes $\bar{x}_i$ (i is fixed), not to change $\bar{x}_i$. Such products exist and we shall show that they are equal to other products of automorphisms where the factors also do not change $\bar{x}_i$.

We recall now that the automorphism $a(ijkr)$ multiplies $\bar{x}_i$ by $[\bar{x}_j, \bar{x}_k, \bar{x}_r]$ which is either a basic commutator or its inverse or product of two basic commutators by Jacobi identity. In any case the basic commutators have 3 different letters not equal to $\bar{x}_i$. The automorphism $b(ijk)$ multiplies $\bar{x}_i$ by $[\bar{x}_j, \bar{x}_k, \bar{x}_i]$ which can be written through basic commutators with three different letters, one of which is necessarily $\bar{x}_i$. The automorphism $c(ijk)$ multiplies $\bar{x}_i$ by $[\bar{x}_j, \bar{x}_k, \bar{x}_j]$ which is the basic commutator or its inverse on two letters and without $\bar{x}_i$. So no cancellation between a, b, c is possible. An automorphism of type d can multiply $\bar{x}_i$ by two different commutators, namely, $d(jir)$ gives $[\bar{x}_r, \bar{x}_i, \bar{x}_i]$ which is basic or its inverse on two letters, with two $\bar{x}_i$. The second factor of $d(ijr)$ multiplies $\bar{x}_i$ by $[\bar{x}_i, \bar{x}_j, \bar{x}_r]$. The basic commutator of this type on three letters with one $\bar{x}_i$ we had previously, as a factor of $b(ijr)$. This leads to the fact that the product $d(rij)^{-1}\, d(jir)\, b(ijr)$ leaves $\bar{x}_i$ fixed. We need to find all such situations when the product of automorphisms leaves $\bar{x}_i$ fixed. Since between the basic commutators which differ by number of the entries equal to $\bar{x}_i$ or by arity no cancellations are possible, we need to consider the last automorphism $e(ijr)$ or $e(sir)$. The first multiplies $\bar{x}_i$ by $[\bar{x}_i, \bar{x}_r, \bar{x}_r]$ and the second by $[\bar{x}_r, \bar{x}_i, \bar{x}_r]$. They are basic commutators or inverses which did not occur before and the only possibility we have is $e(ijr)\, e(sir)$: $\bar{x}_i \to \bar{x}_i$, which is a product of automorphisms that leaves $\bar{x}_i$ fixed.

THEOREM 7. *The variety of* 3-*nilpotent groups satisfies LI-property.*

Proof. Because of Lemmas 3 and 2 it is enough to show that if $\bar{\alpha} \in \bar{K}'$

and $\bar{\alpha}: \bar{x}_i \to \bar{x}_i$ for $i \leqslant k$ then $\bar{\alpha}$ is induced by an $\alpha \in \operatorname{Aut} F$, such that $\alpha: x_i \to x_i$ for $i \leqslant k$. Since the $\bar{\alpha}$ is, by Lemma 4, a product of automorphisms of types a, b, c, d, e we need to consider the situation when the factors change $\bar{x}_i$ but their product $\bar{\alpha}$ leaves $\bar{x}_i$ fixed. It follows from the previous discussion that the only possibilities are $\bar{\alpha} = d(rij)^{-1} d(jir) b(ijr)$ and $\bar{\alpha} = e(ijr) e(sir)$. We can see that in the first case $\bar{\alpha} = f(rj) = [\bar{K}^r_{rj}, \bar{K}^j_{jr}]$ as follows from the proof of Lemma 4. This means that $\bar{\alpha}$ is induced by $[K^r_{rj}, K^j_{jr}]$ which also leaves x_i fixed. In the second case $\bar{\alpha} = e(sjr) = [\bar{K}^s_{jr}, \bar{K}^i_{sr}]$ which also can be lifted to $[K^s_{jr}, K^j_{sr}]$, which leaves x_i fixed. This finishes the proof.

4. EA-Property

Let $\bar{F}$ be a free group in a variety $\underline{\underline{V}}$ with the base $\bar{x}_1, \bar{x}_2, \ldots$ and $\bar{F}_r$ be the free group with the base $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_r$.

Definition. The variety $\underline{\underline{V}}$ has the EA (extended automorphism)-property if for every map $\bar{\alpha}: \bar{x}_i \to \bar{x}_i \bar{c}_i$, $\bar{c}_i \in \bar{F}'_s$, $i \leqslant q \leqslant s$, there exists an r, $q \leqslant s \leqslant r$, such that $\bar{\alpha}$ can be extended to an automorphism of $\bar{F}_r$ induced by an automorphism of F_r.

The varieties of abelian and 2-nilpotent groups satisfy the EA-property even for $r = q$ because by Theorem 6 the map $\bar{\alpha}$ is always an automorphism and by Theorems 1 and 2 every automorphism can be lifted.

Theorem 8. *The variety of 3-nilpotent groups satisfies the EA-property for $r = s + 1$.*

Proof. Let $\bar{\alpha}: \bar{x}_i \to \bar{x}_i \bar{c}_i$, $\bar{c}_i \in \bar{F}'_q$, $i \leqslant q$. Every $\bar{c}_i$ is a product of commutators of weights 2 and 3. The element $\bar{x}_i \bar{c}_i$ can be also achieved as the image of $\bar{x}_i$ under a product of automorphisms of types $\bar{K}^i_{jk}$, $\bar{K}^i_{ij}$, $\bar{K}^i_{ji}$, and R^i_{jks} for j, k, s possibly equal to i. By the Corollary 1 each $\bar{K}^*_{**}$ can be lifted to F. If R^i_{jks} has a form a, b, or c from Lemma 4, it also can be lifted; if not, we use $\bar{x}_r$, $r = q + 1$, to lift R^i_{jks} as d, e or g by means of multiplying it by R^r_{***}, namely $R^i_{jii} R^r_{rij} = d(irj)$, $R^i_{ijj} R^r_{rij} = e(irj)$, $R^i_{ijk} R^r_{krj} = g(irkj)$. The proof is complete.

5. The Main Result

In this paragraph F_∞ is the countably generated absolutely free group with the fixed base $x_1, x_2, \ldots$. For a given variety $\underline{\underline{V}}$, $(V \subseteq F')$ we write $\bar{F}_\infty$ for F_∞ / V and fix the base $\bar{x}_1, \bar{x}_2, \ldots$. Our main result gives sufficient conditions for the map $\operatorname{Aut} F_\infty \to \operatorname{Aut} \bar{F}_\infty$ to be onto. This will imply that every automorphism of the 3-nilpotent free group $\bar{F}_\infty$ can be lifted, which is not true in the case of $\bar{F}_n$.

Following H. Neumann [8] we represent an $\alpha \in \operatorname{Aut} F$ by a vector $\alpha = (a_1, a_2, ...)$, where $\alpha: x_i \to a_i$. For $\bar{\alpha} \in \operatorname{Aut} \bar{F}$ we use bars.

THEOREM. *The map* $\operatorname{Aut} F_\infty \to \operatorname{Aut} \bar{F}_\infty$ *is onto if* $\underline{\underline{V}}$ *satisfies the LI- and EA-properties.*

Proof. We consider $\operatorname{Aut} F_\infty \to \operatorname{Aut} \bar{F}_\infty \to \operatorname{Aut} \bar{F}_\infty/\bar{F}'_\infty$. By Theorem 1 and Lemma 1 it is enough to show that if $\bar{\alpha} \in \operatorname{Ker}(\operatorname{Aut} \bar{F}_\infty \to \operatorname{Aut} \bar{F}_\infty/\bar{F}'_\infty)$, then $\bar{\alpha}$ can be lifted. So we consider $\bar{\alpha} = (\bar{x}_1 \bar{c}_1, \bar{x}_2 \bar{c}_2, ...)$ for $\bar{c}_i \in \bar{F}'$. We denote also $\bar{x}_i \bar{c}_i = \bar{a}_i$, $\bar{\alpha} = (\bar{a}_1, \bar{a}_2, ...)$. To show that $\bar{\alpha}$ can be lifted we need to choose such representatives $a_i \in \bar{a}_i$, to get $\alpha = (a_1, a_2, ...) \in \operatorname{Aut} F_\infty$.

Having $\bar{\alpha} = (\bar{a}_1, \bar{a}_2, ...)$ fixed we shall build in $\bar{F}_\infty$ a sequence of subgroups

$$\bar{X}_{t_1} \subset \bar{A}_{T_1} \subset \bar{X}_{t_2} \subset \cdots \subset \bar{X}_{t_k} \subset \bar{A}_{T_k} \subset \bar{X}_{t_{k+1}} \subset \cdots, \tag{1}$$

where $\bar{X}_q = \operatorname{gp}(\bar{x}_1, \bar{x}_2, ..., \bar{x}_q)$, $\bar{A}_q = \operatorname{gp}(\bar{a}_1, \bar{a}_2, ..., \bar{a}_q)$, and $t_1 = 1$. Then we lift the sequence to F_∞ taking corresponding contraimages

$$X_{t_1} \subset A_{T_1} \subset X_{t_2} \subset \cdots \subset X_{t_k} \subset A_{T_k} \subset X_{t_{k+1}} \subset \cdots. \tag{2}$$

This will help us to choose the required representatives $a_i \in \bar{a}_i$.

To get the sequence (1) we need to define only the numbers t_k and T_k for $k \geqslant 1$. We introduce $\bar{\alpha}^{-1} = (\bar{b}_1, \bar{b}_2, ...)$ and apply it to the sequence (1), then we get

$$\bar{B}_{t_1} \subset \bar{X}_{T_1} \subset \bar{B}_{t_2} \subset \cdots \subset \bar{B}_{t_k} \subset \bar{X}_{T_k} \subset \bar{B}_{t_{k-1}} \subset \cdots. \tag{3}$$

Let $t_1 = 1$. If t_k is defined we take T_k to be a minimal natural number such that $t_k < T_k$ and the map $\mu: \bar{x}_i \to \bar{b}_i$, $i \leqslant t_k$, can be extended to an automorphism of $\bar{X}_{T_k}$, which can be lifted. Such a T_k exists by the EA-property. Similarly, if T_k is defined we choose minimal t_{k+1}, $T_k < t_{k+1}$, such that the map $v: \bar{x}_i \to \bar{a}_i$, $i \leqslant T_k$, can be extended to an automorphism of $\bar{X}_{t_{k+1}}$. So, the sequence (1) is defined.

Since by the EA-property the automorphisms μ and v can be lifted, there exist Nielsen transformations M_k and N_{k+1}, $k \geqslant 1$, such that

$$M_k(\bar{x}_1, ..., \bar{x}_{t_k}, \bar{x}_{t_k+1}, ..., \bar{x}_{T_k}) = (\bar{b}_1, ..., \bar{b}_{t_k}, \bar{u}'_{t_k+1}, ..., \bar{u}'_{T_k}), \tag{4}$$

$$N_{k-1}(\bar{x}_1, ..., \bar{x}_{T_k}, \bar{x}_{T_k+1}, ..., \bar{x}_{t_{k-1}}) = (\bar{a}_1, ..., \bar{a}_{T_k}, \bar{w}_{T_k+1}, ..., \bar{w}_{t_{k+1}}). \tag{5}$$

If necessary, we may suppose that M_k acts identically on $\bar{x}_i$, $i > T_k$, and similarly for N_{k+1}.

By applying $\bar{\alpha}$ to (4) we have

$$M_k(\bar{a}_1, ..., \bar{a}_{t_k}, \bar{a}_{t_k+1}, ..., \bar{a}_{T_k}) = (\bar{x}_1, ..., \bar{x}_{t_k}, \bar{u}_{t_k+1}, ..., \bar{u}_{T_k}). \tag{6}$$

This helps to see that

$$M_k N_{k+1}(\bar{x}_1, ..., \bar{x}_{t_{k+1}}) = (\bar{x}_1, ..., \bar{x}_{t_k}, \bar{u}_{t_k+1}, ..., \bar{u}_{T_k}, \bar{w}_{T_k+1}, ..., \bar{w}_{t_{k-1}}). \tag{7}$$

Similarly, from (6) and (5) we have

$$N_{k+1} M_{k+1}(\bar{a}_1, ..., \bar{a}_{T_{k-1}}) = (\bar{a}_1, ..., \bar{a}_{T_k}, \bar{w}_{T_k+1}, ..., \bar{w}_{t_{k-1}}, \bar{u}_{t_{k-1}+1}, ..., \bar{u}_{T_{k-1}}). \tag{8}$$

Because of the LI-property $M_k N_{k+1}$ is identical for $i \leqslant t_k$ and $N_{k+1} M_{k-1}$ is identical for $i \leqslant T_k$.

We now lift the sequence (1) to the group F_∞. The Nielsen transformation $M_k N_{k+1}$ applied to the base $x_1, x_2, ...$ in F_∞ lets us, because of (7), choose representatives $u_i \in \bar{u}_i$ for $t_k + 1 \leqslant i \leqslant T_k$, and representatives $w_i \in \bar{w}_i$ for $T_k + 1 \leqslant i \leqslant t_{k+1}$, $k \geqslant 1$. Because of (7) in the group $X_{t_{k-1}} = \mathrm{gp}(x_1, ..., x_{t_{k-1}})$ we have another base $x_1, ..., x_{t_k}$, $u_{t_k+1}, ..., u_{T_k}$, $w_{T_k+1}, ..., w_{t_{k-1}}$. If we now take $A_{T_k} = \mathrm{gp}(x_1, ..., x_{t_k}, u_{t_k+1}, ..., u_{T_k})$ then the sequence (2) is defined as a lifting for (1).

Now we can choose inductively the required representatives $a_i \in \bar{a}_i$, such that $a_1, ..., a_{T_k}$ generate A_{T_k} freely. In A_{T_1} we define these generators by $M_1^{-1}(x_1, u_2, ..., u_{T_1}) = (a_1, ..., a_{T_1})$. Let the base $a_1, ..., a_{T_k}$ be chosen. Then $X_{t_{k-1}}$ has a base $a_1, ..., a_{T_k}, w_{T_k+1}, ..., w_{t_{k-1}}$ and $A_{T_{k-1}}$ has the base as in the right side of (8), without bars. By applying $(N_{k+1} M_{k+1})^{-1}$ we get $a_1, ..., a_{T_k}, ..., a_{T_{k-1}}$, where the first T_k elements are the same as were chosen earlier because $N_{k+1} M_{k+1}$ acts identically for $i \leqslant T_k$. So the vector $\alpha = (a_1, a_2, ...)$ gives the lifting of $\bar{\alpha}$ which finishes the proof.

Now since the variety $\underline{\underline{\gamma_4(F)}}$ has, by Theorems 7 and 8, LI- and EA-properties we have proved

THEOREM 9. *Every automorphism of the* 3-*nilpotent countably generated free group is induced by an automorphism of the absolutely free group.*

It is shown in [1, 2] that the same is not true for finitely generated free 3-nilpotent group. So we give an example of an automorphism $\bar{\alpha} \in \mathrm{Aut}\, \bar{F}_2$ which cannot be lifted to F_2 but can be lifted to F_∞. According to [1, Theorem 3.1], the automorphism $\bar{\alpha} = R^1_{121}$ in $\bar{F}$ of a finite rank cannot be lifted to F of the same rank. Since $R^1_{121} R^4_{142} = [K^4_{12}, K^1_{14}]$ can be lifted, it is enough to lift R^4_{142} or its inverse. We note that R^4_{412} can be written as an infinite product $R^4_{412} = R^4_{214}(R^4_{421} R^5_{152}) R^5_{215}(R^5_{521} R^6_{162}) \cdots =$

$b(421)\ g(4512)\ b(521)\ g(5612) \cdots = [K^4_{43}, K^3_{12}][K^4_{51}, K^5_{42}][K^5_{53}, K^3_{12}]$ $[K^5_{61}, K^6_{52}] \cdots$. We consider three infinite products of every third commutator $U = \prod [K^i_{i3}, K^3_{12}]$, $i = 4, 5, \ldots$, $S = \prod [K^{2i}_{2i+1\ 1}, K^{2i+1}_{2i\ \ 2}]$, $i = 2, 3, \ldots$, $T = \prod [K^{2i-1}_{2i\ \ 1}, K^{2i}_{2i-1\ 2}]$, $i = 3, 4, \ldots$. These products are automorphisms in the free group F of infinite rank because they can be written as products of eight simultaneous elementary automorphisms [4] which explains the required lifting of R^1_{121}.

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