DEMONSTRATIO MATHEMATICA
Vol. XXVII No 3–4 1994

O. Macedońska, W. Tomaszewski

ON THE FORM OF FIXED POINTS FOR INVOLUTIONS IN A FREE ABELIAN GROUP

Dedicated to Professor Tadeusz Traczyk

If α is an automorphism of order two in an abelian group then each element of the form uu^α is a fixed point for α. The question is for which α each fixed point is of the form uu^α. There are examples of relatively free groups of rank two where the automorphism σ permuting generators has this property. We describe the automorphisms of order two with all fixed points of the form uu^α in a free abelian group of rank two.

Let F be a relatively free group of rank two generated by x, y, and σ be the automorphism of F, permuting the generators. If F is abelian, $w = x^s y^t$ then $w = w^\sigma$ implies $s = t$ and $w = (x^s)(x^s)^\sigma$, so each fixed point of σ has a form uu^σ, and hence the group of fixed points is cyclic, generated by xx^σ. It can be deduced from [5] that in a free two-nilpotent group of rank two the group of fixed points for σ is also cyclic, generated by (xy)(xy)$^\sigma$. In a free metabelian group of rank two the group of fixed points for σ also consists of elements uu^σ for all u in a commutator subgroup [3], and is infinitely generated, which also shows that the Gersten's Theorem, which says that in a free group of a finite rank each automorphism has a finitely generated group of fixed points [1], can not be extended to a two-generated free metabelian group. We describe here automorphisms α of order two in a free abelian group A generated by x and y, having all fixed points of the form uu^α. This property of fixed points is (or is not) satisfied for all conjugate automorphisms simultaneously. We show that there are three conjugacy classes for automorphisms of order two in A only one of which

This paper has been presented at the Conference on Universal Algebra and its Applications, organized by the Institute of Mathematics of Warsaw University of Technology held at Jachranka, Poland, 8–13 June 1993.

has all fixed points of the form uu^α (though in the free group of rank two there are four conjugacy classes of automorphisms of order two [2], none of which has all fixed points of the form uu^α).

Each automorphism α in A is given by $x^\alpha = x^k y^l$, $y^\alpha = x^m y^n$, and with respect to x, y, α is defined by the matrix $M = \begin{bmatrix} k & l \\ m & n \end{bmatrix}$. We assume that $\alpha^2 = id \neq \alpha$ then $M^2 = I$, and hence $k^2 = n^2 = 1 - ml$, $(k+n)l = (k+n)m = 0$. If $m = l = 0$ we get two not conjugate automorphisms of order two, $-id$ and δ defined by the matrix $D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Obviously, $-id$ has no fixed points ($\neq 1$) and δ has a cyclic groups of fixed points generated by x where only x^{2k} are of the form uu^α.

LEMMA 1. *In a free abelian group generated by x, y, an automorphism $\alpha\,(\alpha^2 = id \neq \alpha)$ is conjugate to either $-id$, or δ, or σ, respectively defined by matrices* $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, *and* $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Proof. We note first that the matrix of σ is conjugate to that of δ over $\mathbf{Z}[1/2]$ by $C = \begin{bmatrix} 1 & 1 \\ -1/2 & 1/2 \end{bmatrix}$, but not over $\mathbf{Z}$. Let now $\alpha \neq -id, \delta$, then m, l are not both zeros, $n = -k$ and α is defined by $M = \begin{bmatrix} k & l \\ m & -k \end{bmatrix}$, $k^2 = 1 - ml$. Hence $\det(M) = -1$. Since $M^2 = I$, the Jordan form of M is diagonal and $\det M = -1$ implies that M has eigenvalues 1 and -1. So α has a fixed point a, say. Since A is torsion free, it follows from the Theorem on Subgroups in Abelian Groups ([4], 3.5.2) that there exists b such that a, b form a base where α has the matrix $M_1 = \begin{bmatrix} 1 & 0 \\ t & -1 \end{bmatrix}$. If t is even, $t = 2n$ say, then for $C = \begin{bmatrix} 1 & 0 \\ n & -1 \end{bmatrix}$, $CM_1C^{-1} = D$ and hence α is conjugate to δ. If $t = 2n+1$, then $CM_1C^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ for $C = \begin{bmatrix} -n & 1 \\ 1+n & -1 \end{bmatrix}$ and α is conjugate to σ which finishes the proof. ∎

LEMMA 2. *Each fixed point for α is of the form uu^α if and only if α is conjugate to σ.*

Proof. We assume that $\alpha \neq -id$, then as in Lemma 1, α is conjugate to an automorphism β with the matrix $M_1 = \begin{bmatrix} 1 & 0 \\ t & -1 \end{bmatrix}$, that is $x^\beta = x, y^\beta = x^t y^{-1}$. Let $w = x^r y^s$ be a fixed point for β, then $w = w^\beta = x^r x^{st} y^{-s}$, implies $s = 0$ and each fixed point is $x^r, r \in \mathbf{Z}$. So β would have all fixed points of the form uu^β, if and only if there exists u such that $x = uu^\beta$. If

$u = x^\mu y^\nu$ then $x = uu^\beta = (x^\mu y^\nu)(x^\mu y^\nu)^\beta = x^\mu y^\nu x^\mu (x^t y^{-1})^\nu$ and hence $\{\mu, \nu\}$ should be an integer solution of the equation $1 = 2\mu + t\nu$ which exists if and only if t is coprime to 2. It follows from Lemma 1 that β, and hence α, is conjugate to σ as required. ∎

The situation with fixed points for $-id$ and δ is clear, so we consider α defined by $M = \begin{bmatrix} k & l \\ m & -k \end{bmatrix}$, $k^2 = 1 - ml$, where m, l are not both zeros.We can assume that $m \neq 0$, and if m, l are not both even we assume that m is odd, because othervise we consider the automorphism $\sigma\alpha\sigma$ (obtained by interchanging x and y), with the matrix $\begin{bmatrix} k' & l' \\ m' & -k' \end{bmatrix} = \begin{bmatrix} -k & m \\ l & k \end{bmatrix}$, where $m' = l$, and use the fact that w is a fixed point for α if and only if w^σ is a fixed point for $\sigma\alpha\sigma$. In the following theorem we denote $d = \gcd(m, 1-k)$, $e = \gcd(m, l)$.

THEOREM. *In a free abelian group of rank two an automorphism α, given by $x^\alpha = x^k y^l$, $y^\alpha = x^m y^{-k}$, $k^2 + ml = 1$, $m \neq 0$, has the cyclic group of fixed points generated by $a = x^{m/d} y^{(1-k)/d}$, which is of the form $a = uu^\alpha$ if and only if e is odd. Then $u = x^\mu y^\nu$ for μ, ν satisfying $\mu(k+1) + \nu m = m/d$.*

Proof. The fixed points for α in A correspond to eigenvectors, with integer coordinates, for $\lambda = 1$. In our notation $x^\alpha = x^k y^l$ corresponds to $[0, 1]\begin{bmatrix} k & l \\ m & -k \end{bmatrix} = (k, l)$. So, to find an eigenvector for $\lambda = 1$ we solve a system $\begin{cases} x(k-1) + ym = 0 \\ xl - y(k+1) = 0 \end{cases}$ which has an integer solution $[m, 1-k]$. To find an eigenvector for $\lambda = 1$ we solve a system with the determinant $\begin{vmatrix} k-1 & l \\ m & -k-1 \end{vmatrix} = 0$, which has an integer solution $\{m, 1-k\}$. So the group of fixed points is cyclic, generated by $a = x^{m/d} y^{(1-k)/d}$. If take $b = x^r y^s$, where s, r satisfies $sm - r(1-k) = d$, then since for $p = m/d$, $q = (1-k)/d$, we get $\begin{vmatrix} p & q \\ r & s \end{vmatrix} = ps - qr = 1$, the elements a, b form a new base, where α has the matrix $M_1 = CMC^{-1} = \begin{bmatrix} 1 & 0 \\ t & -1 \end{bmatrix}$ for $C = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$. By calculation we get $t = 2krs + s^2 m - r^2 l$. By Lemma 2, α has all fixed points of the form uu^α, if and only if t is odd, which is not so if m and l are both even. Let now m be odd, then d is odd. We denote $t_1 = s^2 m - r^2 l$ and consider $t_1 m = (sm)^2 - r^2(ml)$. As we know, $sm = r(1-k) + d$, and $ml = 1 - k^2$. This implies $t_1 m \equiv d^2 \bmod 2$ and hence t_1 (and t) is odd. So by Lemma 1, α is conjugate to σ, and hence, by Lemma 2, it has all fixed points of the form uu^α. We conclude that α has all fixed points of the form uu^α, if and only if m, l are not both even, that is e is odd. If now $a = uu^\alpha$, $u = x^\mu y^\nu$

then $a = uu^\alpha = (x^\mu y^\nu)(x^k y^l)^\mu (x^m y^n)^\nu$ and $\{\mu, \nu\}$ is an integer solution of a system which is equivalent to one equation $\mu(k+1) + \nu m = m/d$. ∎

COROLLARY. *In a free abelian group of rank two an automorphism* α $(\neq id, -id)$, *given by* $x^\alpha = x^k y^l$, $y^\alpha = x^m y^{-k}$, $k^2 + ml = 1$, *is conjugate to* δ *if* m, l *are both even, othervise it is conjugate to* σ. ∎

To give examples we denote $\alpha' = \sigma\alpha\sigma$. An element w is a fixed point for α if and only if w^σ is a fixed point for α' and $w = uu^\alpha$ if and only if $w^\sigma = (u^\sigma)(u^\sigma)^{\alpha'}$.

EXAMPLE 1. Let $x^\alpha = x^3 y^{-2}, y^\alpha = x^4 y^{-3}$. Here $m = 4, k = 3, d = 2$, so the subgroup of fixed points is generated by $a = x^{m/d} y^{(1-k)/d} = x^2 y^{-1}$ which is not of the form uu^α because m, l are both even. ∎

EXAMPLE 2. Let $x^\alpha = x^6 y^{-7}$, $y^\alpha = x^5 y^{-6}$. Here $m = k - 1 = d = 5$, so the subgroup of fixed points is generated by xy^{-1}. Since m is odd $xy^{-1} = uu^\alpha$. To find u we solve $7\mu + 5\nu = 1$, so $u = x^3 y^{-4}$. ∎

EXAMPLE 3. Let $x^\alpha = xy^3, y^\alpha = y^{-1}$. Since $m = 0$, we consider the automorphism $\alpha' = \sigma\alpha\sigma$ which maps $x \to x^{-1}, y \to x^3 y$. Here $m' = 3, k' = -1$, $d' = \gcd(3, -2) = 1$. The subgroup of fixed points for α' is generated by $a' = x^3 y^2$, which is of the form $uu^{\alpha'}$ where $u = y$. So for the initial automorphism α the subgroup of fixed points is generated by $a = x^2 y^3 = xx^\alpha$. ∎

EXAMPLE 4. Let $x^\alpha = xy^2$, $y^\alpha = y^{-1}$. Since $m = 0$, we consider the automorphism $\alpha' = \sigma\alpha\sigma$ which maps $x \to x^{-1}, y \to x^2 y$. Here $m' = 2, k' = -1, d' = 2$, so the subgroup of fixed points for α' is generated by xy (the same for α), which is not of the form uu^a because m, l are both even. ∎

References

[1] S. M. Gersten, *On fixed points of automorphisms of finitely generated free groups*, Bull. Amer. Math. Soc. 8 (1983), 451–454.

[2] R. C. Lyndon, P. E. Schupp, *Combinatorial Group Theory*. Dover, New York, 1977.

[3] O. Macedońska, D. Solitar, *On binary σ-invariant words in a group*, Contemporary Math., 169 (1994), 431–449.

[4] W. Magnus, A. Karrass, D. Solitar, *Combinatorial Group Theory*. Dover, New York, 1970.

[5] E. Płonka, *Symmetric words in nilpotent groups of class ≤ 3*, Fund. Math. 97 (1977), 95–103.

INSTITUTE OF MATHEMATICS,
SILESIAN TECHNICAL UNIVERSITY,
Kaszubska 23
44-100 GLIWICE, POLAND

Received November 12, 1993.