ZESZYTY NAUKOWE POLITECHNIKI ŚLĄSKIEJ 1986

Seria: AUTOMATYKA z. 81 Nr kol. 851

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A NEW APPROACH TO THE UNCERTAIN SYSTEMS MODELING[x)]

Abstract. An input-output model of an uncertain plant is proposed which maps an input metric space into an output Menger space. On the contrary a controller is modelled by a mapping of the Menger space into the metric space of control variables. A control objective is considered to be the reachability of the desired output $(\mathcal{E},r)$ neighborhood. Some notions of mathematical systems theory have been defined for the above model.

1. INTRODUCTION

The input-output models used in control systems synthesis and analysis are only approximations of real plant dynamics. The uncertainty resulting from the approximation can be included in modeling in a variety of ways. Stochastic (e.g. [1], [2]), fuzzy (e.g. [5], [12]), set-membership (e.g. [3], [4]), state-inequalities (e.g. [8], [9]) models are examples of the approaches to uncertain systems descriptions. However some generalization of the uncertain system modeling may be useful. This paper develops such unified approach proposed in [10] based on statistical metric spaces concept. The main idea of the method is to use different statistical distances to describe different types of uncertainty. However some general properties of statistical metric spaces [6], [7] give general properties of uncertain models.

The role of the notion of a "distance" in system analysis and design is crucial. In partly unknown systems the very association of a single number with a pair of elements is, realistically speaking, an over-idealization. In these situations it is appropriate to look upon the distance concept as an uncertain rather than a determinate one. However some useful properties of special metric spaces hold or may be transfered into statistical metric spaces and especially in Menger spaces. The fixed point theorem is the example of such useful result [10] - see Appendix.

[x)] Praca wykonana w ramach projektu badawczego RP. I. 02: Teoria sterowania i optymalizacji ciągłych układów dynamicznych i procesów dyskretnych.

2. MENGER SPACES REVISITED

In that part some important mathematical definitions are recalled [7].

Definition 1: A statistical metric space is an ordered pair $(S,\mathcal{F})$ where S is an abstract set whose elements will be called points and $\mathcal{F}$ is a mapping of $S \times S$ into the set of distribution functions - i.e., associates a distribution function $\mathcal{F}(p,q)$ with pair (p,q) of points in S. We shall denote the distribution function $\mathcal{F}(p,q)$ by F_{pq} whence $F_{pq}(x)$ will denote the value of F_{pq} for the real argument x. The functions F_{pq} are assumed to satisfy the following conditions:

I. $F_{pq}(x) = 1$ for all $x > 0$ iff $p = q$

II. $F_{pq}(0) = 0$

III. $F_{pq} = F_{qp}$

IV. If $F_{pq}(x) = 1$ and $F_{qr}(y) = 1$ then
$F_{pr}(x+y) = 1$

Definition 2: A Menger space is a statistical metric space in which a triangle inequality

V. $F_{pr}(x+y) \geq T(F_{pq}(x), F_{qr}(y))$ for all $x,y \geq 0$

holds universally for some choice of T satisfying following conditions:

T1: $T(a,1) = a, \quad T(0,0) = 0$

T2: $T(c,d) \geq T(a,b)$ for $c \geq a, \quad d \geq b$

T3: $T(a,b) = T(b,a)$

T4: $T(T(a,b),c) = T(a, \ T(b,c))$

T is a 2-place function on the unit square called a triangular norm.

The topology in a Menger space may be induced by the family of neighborhoods. The concept of neighborhood is also important for definition of convergence, completeness and other topological notions. It may be defined in several nonequivalent ways. However the useful definition is following [7]:

Definition 3: Let p be a point in the statistical metric space $(S,\mathcal{F})$. By an (ε,r) neighborhood of p, $(\varepsilon > 0, \ r > 0)$ we mean the set of all points q in S for which $F_{pq}(\varepsilon) > 1-r$.
We write:

$$N_p(\varepsilon,r) = \left\{q \ : \ F_{pq}(\varepsilon) > 1-r\right\}$$

A sequence of points $\{p_n\}$ in an $(S,\mathcal{F})$ space is convergent to a point p in S iff for every $\varepsilon > 0$, $r > 0$ there exists an integer M such that $p_n \in N_p(\varepsilon, r)$ whenever $n > M$. It is a Cauchy sequence iff for every ε and r there exists an integer M such that $p_m \in N_{pn}(\varepsilon, r)$ whenever $n, m > M$. Completeness of a Menger space is guaranteed by convergence of every Cauchy sequence to an element of this space.

3. MODELING OF PARTLY UNKNOWN PLANT

Consider plant given by input-output basic models of a form:

$$y = f(u) \tag{1}$$

where u is an element of input metric space (U, d) and y an element of output metric space $(\bar{S}, \rho)$.

Since (1) is only an approximation of the real plant the single valued prediction of the plant is unrealistic. So the plant should be modelled by the mapping from the input space (U, d) into a proper Menger space $(S,\mathcal{F})$. Then we wre led to the model (1) in which however y is now an element of a Menger space $(S,\mathcal{F})$ and f is understood as a mapping from (U, d) into $(S,\mathcal{F})$. If there is no uncertainty in the system the model is still in force setting

$$F_{yz}(x) = H(x - \rho(y,z)) = H_{\rho(y,z)}(x) \tag{2}$$

H is a unit step defined as:

$$H(x) = \begin{cases} 0 & x \leqslant 0 \\ 1 & x > 0 \end{cases}$$

So the input space is also a Menger space with $\mathcal{F} = \mathcal{H}$ where $\mathcal{H}(p,q) = H_{d(p,q)}$.

Different uncertainties lead to different functions F. However it is very important to consider not only the uncertainty resulting from an environment of the system but also the possibilities of realization of the control objective. This problem will be discussed more extensively in the next chapter. Here consider typical uncertainties and respective functions F. In the set-membership or inequality approach y may be interpreted as a set of possible responses and the distribution function may be defined as:

$$F_{yz}(x) = H(x - \rho_h(y,z)) \tag{3}$$

where $\varrho_h(\cdot,\cdot)$ is a Hausdorff distance between two sets. Other possibility is to consider y as the result of a selector for the point-set mapping and discuss more general distribution function for example defined as follows:

$$F_{yz}(x) = \begin{cases} G(x/\varrho(y,z)) & y \neq z \\ H(x) & y = z \end{cases} \tag{4}$$

where G is a distribution function different from H satisfying G(0) = 0.

The function F defined in (4) is also suitable for Hausdorff distances ϱ_h. This approach is in some sense generalization of (3) and (4). However it becomes similar to fuzzy set approach. Then S is understood as a set and interesting interpretation of function F follows. Consider relation R(x) defined by

$$R(x) = \left\{(y,z) \ : \ \varrho(y,z) < x\right\} \tag{5}$$

Then R(x) is fuzzy relation and $F_{yz}(x)$ may be defined as the grade of membership of (y,z) in R(x) i.e.:

$$F_{yz}(x) = \mu_{R(x)}(y,z) \tag{6}$$

Note that the usual definition of a fuzzy subset of S as a membership function $\mu_A(p)$ is equivalent to

$$F_{pA}(0+) = \mathrm{Sup}(F_{pq}(0+) \mid q \in A).$$

In all cases mentioned above triangular norm may be defined as:

$$T(a,b) = \min(a,b) \tag{7}$$

what may be convenient for further consideration.

Some difficulties must be met in the probabilistic case although the interpration of function F_{yz} is the most natural. Namely define $F_{yz}(x)$ as the probability that the distance of y to z is less then x. More precisely y must be interpreted as a random variable i.e. function from $(\Omega,\mathfrak{B},P)$ into R, or a stochastic process and we define:

$$F_{yz}(x) = P\left\{\omega \text{ in } \Omega \mid \varrho(y(\omega),\ z(\omega)) < x\right\} \tag{8}$$

where $y(\omega)$, $z(\omega)$ are realizations of y, z respectively. However triangular norm (7) does not ensure the triangle inequality V. In the case of the stochastic inpendency of the distances T may be chosen as a product i.e.

$$T = ab \tag{9}$$

This is the case of so called Wald spaces [7]. However in some cases for example when the normal spaces are considered T may be defined only as:

$$T = \max(a + b - 1, \quad 0) \tag{10}$$

For control purposes it may be more convenient to consider function F defined by a metric $m(\cdot,\cdot)$ based on an expectation operation. Then a function similar to that defined by (3) or (4) may be applied with ρ or ρ_h changed by m. This permits use of the T-norm given by (7).

The uncertainty in a plant is usually represented by uncertainty in parameters of its model. Denote by w such uncertain parameter and consider w to be an element of a Menger space $(W, \mathcal{F}_0)$ and the model (1) will be extended to the form:

$$y = f(u,w) \tag{11}$$

where now f is a mapping of a product of input metric space and parameter Menger space into output Menger space.

To be well behaved the product must be M-product [11] defined as the pair $(U \times W, \mathcal{H} \mathcal{M} \mathcal{F}_0)$ where $\mathcal{H} \mathcal{M} \mathcal{F}_0$ is defined by:

$$\mathcal{H} \mathcal{M} \mathcal{F}_0(p,q) = \min\left(\mathcal{H}(u_1, u_2), \mathcal{F}_0(w_1, w_2)\right) \tag{10}$$

where

$$\mathcal{H}(u_1, u_2) = H_{d(u_1,u_2)}$$

$$\mathcal{F}_0(w_1, w_2) = F_{ow_1 w_2}$$

and

$$\mathrm{Min}(H_{d(u_1,u_2)}, F_{o\, w_1\, w_2})(x) = \mathrm{Min}(H(x - d(u_1,u_2)), F_{ow_1 w_2}(x)) \tag{11}$$

4. MODELING OF CONTROL OBJECTIVE AND CONTROLLER

Although a variety of approaches to system design is reasonable both for deterministic and uncertain systems including optimal control design, it seems that the objective lies almost always in achieving desired response of a system for given sets of parameters, reference signals etc. This objective is however unrealistic for uncertain systems. Moreover even in the case when we are able to get complete information about a plant to be controlled the desired response may be unachievable because of the constraints imposed on the control variables. The Menger space approach provides great flexibility in modeling the control objective which can be stated as reachability of $(\mathcal{E},r)$ neighborhood of the desired response. Denote the desired output by y^o in S. Then the control objective may be written in the form:

$$y \in N_{\{y^o\}}(\mathcal{E},r) \tag{12}$$

$\{y_o\}$ is y_o treated as an element of S.

The type of neighborhood is defined by the Menger space in which outputs are considered. However, it now seems obvious that the proper Menger space may be chosen not only on the base of the type of uncertainty introduced by the environment of the system, but also taking into account some design needs. In some sense we may regard the objective (12) as the target set control with given degree of risk, $\mathcal{E}$ is interpreted as a diameter of the target set and r as the given degree of risk (in the sense of the Menger space $(S,\mathcal{F})$). The input-output model of the system and the model of control objective imply a model of controller which may be useful in control system design.

Therefore we are led to a controller defined by a mapping of Menger space into metric space i.e.:

$$u = g(y) \tag{13}$$

The controller must place the outputs of the system in the $(\mathcal{E},r)$ neighborhood of the desired response. Combining the models (1) and (13) we are led to a model of the overall system in the form:

$$y = f(g(y)) \tag{14}$$

The model (14) is given by the mapping $f(g(y))$ of the Menger space $(S,\mathcal{F})$ into itself. Therefore some properties of the feedback systems may be considered by regarding properties of specially defined mappings in Menger space.

5. SYSTEMS THEORY CONCEPTS

For systems given by the models considered above some mathematical systems theory concepts may be defined. We shall give examples of such definitions.

<u>Definition 4</u>: The system (1) is (c,V) Lipschitzian if there exist a positive real constant c and a distribution function V(x) with V(0) = 0 such that:

$$F_{f(u_1)\,f(u_2)}(c\,x) \geq V(x - d(u_1,u_2)) \tag{15}$$

for every $u_1, u_2 \in U$ and every $x > 0$.

Definition 5: The system (1) is $(\mathcal{E},r)$ controllable to y^o if there exists u such that

$$f(u) \in N_{\{y^o\}}(\mathcal{E},r) \tag{16}$$

$\{y^o\}$ is y^o treated as an element of S.

Definition 6: The system (1) is $(\mathcal{E},r)$ stabilizable at y^o if there exists a controller $u = g(y)$ such that the output of the overall system is unique and lies in $N_{\{y^o\}}(\mathcal{E},r)$

Definition 7: The system (11) is completely identifiable if there exists an inverse mapping:

$$w = f^{-1}(\cdot\,,y) \tag{17}$$

with values in metric space $(\overline{W},\varrho)$

Definition 8: The system (12) is $\mathcal{F}_o$ - identifiable if there exists an inverse mapping

$$w = f^{-1}(\cdot,y)$$

with values in statistical metric space $(W,\mathcal{F}_o)$

Definition 9: A solution y of the feedback system (14) is stable if y is the attractive fixed point of the mapping $f(g(\cdot))$.

The last definition shows that one way of finding conditions for the uncertain system to posses some desired properties is the use of fixed point theorem in Menger space.
An example of such a result may be given in the form of the following sufficient condition of the $(\mathcal{E},r)$ stabilizability.

Theorem: Consider (c,V) Lipschitzian uncertain system $(\mathcal{E},r)$ controllable to y^o with u in the input metric space (U,d) and y in the complete Menger space $(S,\mathcal{F},T)$ with T continuous and satisfying an inequality $T(x,x) \geqslant x$. Then the system is $(\mathcal{E},r)$ stabilizable at y^o.

Proof: The $(\mathcal{E},r)$ controllability implies an existence of u^o such that $f(u^o) = y^* \in N_{\{y^o\}}(\mathcal{E},r)$.
Assume that a controller g(y) is chosen such, that for $y = y^*$ gives u^o, i.e.

$$g(y^*) = u^o = f^{-1}(y^*) \tag{18}$$

Then y^* is a fixed point of the mapping $f(g(\cdot))$ i.e.

$$f(g(y^*)) = y^* \tag{19}$$

To make y^* an attractive fixed point one must ensure a contraction in the sense of the Menger space $(S,\mathfrak{F},T)$ of the mapping $f(g(.))$. It suffices to impose the following condition for the controller. Let $l > 0$, be the constant such that $cl < 1$. For every y,z in S and every $x > 0$ the controller $g(\cdot)$ must satisfy the relation:

$$V(lx - d(g(y),\ g(z))) \geqslant F_{yz}(x) \tag{20}$$

Now combining the (c,V) Lipschitzian property of the system with the property (20) of the controller we have:

$$F_{f(g(y))f(g(z))}(c\ x) \geqslant F_{y\ z}\left(\frac{c\ x}{k}\right) \tag{21}$$

where $k = c\ l < 1$ for every y,z in S and every $x > 0$ (21) is a contraction condition, and y^* is therefore, on the base of the fixed point theorem ([10] - see Appendix), unique attractive fixed point. That completes the proof.

6. CONCLUSION

A unified approach for uncertain modeling has been proposed. It implies a convenient way of control objective and controller modeling. Moreover it may be useful for control systems design and makes it possible to find common properties for systems characterized by different types of uncertainty. The criterion proposed for systems theory concepts could be compared with various notions considered elsewhere e.g. [1], [3] for some types of uncertainty. For example the notion of (ε,r) controllability considered for set-membership model of uncertainty coincides with the reachability of target tubes (min-max reachability [3] if $r = 0$).

Appendix: Contraction mappings in statistical spaces

Definition: A mapping M: $(S,\mathfrak{F}) \rightarrow (S,\mathfrak{F})$ is said to be a contraction if there exists a real number k : $0 < k < 1$, such that

$$F_{Mp\ Mq}(k\ x) \geqslant F_{pq}(x) \quad \text{for all } p,\ q \text{ in } S \text{ and each } x > 0$$

Fixed point theorem: Let $(S,\mathfrak{F},T)$ be a complete Menger space where T is a continuous triangular norm satisfying the condition $T(x,\ x) \geqslant x$ for each x in $[0,\ 1]$.

If M is a contraction mapping in S then exists a unique point p in E such that Mp = p. Moreover, for all q in S, every sequence of iterates of M converges to this fixed point (M^n q→p).

For proof see [10].

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Recenzent: Prof. dr inż. Tadeusz Puchałka

Wpłynęło do Redakcji 20.02.1984 r.

НОВЫЙ ПОДХОД К МОДЕЛИРОВАНИЮ СИСТЕМ С НЕУВЕРЕННОСТЬЮ

Р е з ю м е

В статье предлагается входно-выходная модель системы с неуверенностью, в которой входное метрическое пространство преобразовано в менгеровское выходное пространство. С другой стороны регулятор моделируется через отображение пространства Менгера в метрическое пространство управлений. Цель управления принята в виде достижения области соответствующей требуемому отклику системы. Для указанной модели введены определения некоторых математических понятий из теории систем.

NOWE PODEJŚCIE DO MODELOWANIA UKŁADÓW Z NIEPEWNOŚCIĄ

S t r e s z c z e n i e

Proponuje się wejściowo-wyjściowy model obiektu w warunkach niepewności, który jest odwzorowaniem wejściowej przestrzeni metrycznej w wyjściową przestrzeń Mengera. Z kolei regulator modelowany jest w postaci odwzorowania przestrzeni Mengera w przestrzeń metryczną sterowań. Cel sterowania jest przyjęty w postaci osiągalności $(\mathcal{E},r)$ otoczenia pożądanej odpowiedzi układu. Dla powyższego modelu wprowadza się definicje niektórych pojęć matematycznej teorii systemów.