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ON JAR-METRIC PRINCIPLE
A UNIFIED APPROARCH TO SOLVE OPTIMUM PATHS PROBLEMS ON MULTISTAGE DIRECTED GRAPH

Streszczenie. In dynamic programming,it is well known that there are some drawbacks in Bellman's principle of optimality,that there exist some gaps between the principle and related functional equations,and also that the computation for solving the problems of finite type is tedious and lack of mathematical beauty. In this paper we are 1)to give a mathematic system -Semi-field - and a computational tool - modi-matrix;2)to consider a multistage directed graph on which each link corresponds to an element of a semi -field ,called jar-metric of the link;to introduce two concepts:the optimum path from initial vertex to final one and optimum path of the graph; and to discuss their relationship;3)to set up jar-metric principle which is somewhat like Bellman's principle of optimality of finite type ; to give related computational formula which is equivalent to jar-metric principle;4)to solve optimum path problem on the graph mentioned above by jar-metric principle, to give an algebraic formula;and from which,to point out that,from computational point view,the forward process is not necessarily equivalent to the backward one 5)to solve two kinds of optimum path problems of N-th order in 3 and 4 to solve multi-object optimum path problem in 5 and 6 by jar-metric principle. Thus we can use our theory to solve all problems of finite type which can be solved by dynamic programming.But the basic of our theory will be firmer than that of Bellman's.And basic concept is geometric instead of dynamic. Some of algorithms in this paper might be known but they were not put into a unified fashion. Most material in this paper appeared in the papers: On Jar - metric Principle(I) , (II),(III),(IV) which are written in Chinese.

1.Semi-field and modi-matrix

Definition 1. A semi-field is a triple $\{S , \oplus , \otimes\}$ where S is a set with two operations:modi-addition $\oplus$ and modi-multiplication $\otimes$ satisfying laws of commutativity,associativity and distributivity and there exists a zero element z in S.

Definition 2. A semi-field with identity e is called to be optimizing if there is finite element in S and if a and b are in S,we have

$$a \oplus b = a \text{ or } b$$

In an optimizing semi-field ,if $a \oplus b = a$,we say that a is not worse than b ,denoted by $a \preccurlyeq b$. If $a \oplus b = a$ and $a \neq b$,we say a is better than b , or b is worse than a,denoted by $a \prec b$. If $a \prec e$, a is called a yin element, if $a \succ e$, a is called a yang element[1] ,and e itself ,the neutral element.Evidently,in an optimizing semi- field , S is a totally ordered set.

Theorem 1. In an optimizing semi-field $\{S, \oplus, \otimes\}$, we have

i) if $a \preccurlyeq b$, and $b \preccurlyeq a$, then $a=b$;

ii) if $a \preccurlyeq b$, and $b \preccurlyeq c$, then $a \preccurlyeq c$;

iii) if $a \preccurlyeq b$, and $c \preccurlyeq d$, then $a \oplus c \preccurlyeq b \oplus d$;

iv) if $a \preccurlyeq b$, then $a \otimes c \preccurlyeq b \otimes c$;

v) if $a \preccurlyeq b$ and $c \preccurlyeq d$, then $a \otimes c \preccurlyeq b \otimes d$;

vi) if $a \preccurlyeq b$ then for any non-negative integer k, $a^k \preccurlyeq b^k$;

vii) $e^k = e$;

viii) if a is a yang(yin,neutral) element, then, for any positive k, a^k is a yang (yin,neutral) element;

ix) if k is a positive integer, then $ka = a$;

x) if for every i, p_i and g_i are both equal to zero element or both positive integers not necessary equal, then

$$\sum_{i=0}^{k} p_i \cdot a^i = \sum_{i=0}^{k} g_i \cdot a^i$$

where $\sum$ means modi-addition.

Proof. By direct computation and matematical induction.

Definition 3. A semi-field is called to be strongly optimizing if it is optymizing and if $a \oplus b = b$ and $c \nsucc z$, we always have $a \otimes b \oplus b \otimes c = b \otimes c$.

Definition 4. A semi-field is called to be generalized optimizing if, for a, b in S,

$$(a \oplus b) \oplus a = a \oplus b,$$
$$(a \oplus b) \oplus b = a \oplus b.,$$

here $a \oplus b$ will not be necessary equal to a or b.

Thus in a generalized optimizing semi-field, for a and b being in S, we have $a \oplus b \preccurlyeq a$ and $a \oplus b \preccurlyeq b$. For a semi-field to be generalized optimizing, the necessary and suficient condition is that for all a in S,

$a \oplus a = a$

In the generalized optimizing semi-field $\{S, \oplus, \otimes\}$, if there is an element h which is not worse than a and b, that is to say, $a \oplus h = h$, and $b \oplus h = h$, then h is also not worse than $a \oplus b$, because

$$(a \oplus b) \oplus h = a \oplus (b \oplus h) = a \oplus h = h.$$

Hence $a \oplus b$ is the worst element among all those elements not worse than a and b.

*) Yin and yang are the alphabetic writing of two Chinese terms 阴 and 阳, borrowed from Chinese traditional Yin-yang analysis in an ancient book written by Laozi about more than two thousand years ago. Generally speaking, these two terms mean the two sides of any antitheses, such as positive and negative, good and bad, man and woman, sun and moon, and all such things.

$a \oplus b$ may be called the worst optimal bound of a and b . It is easy to generalize this assertion to any set with finite elements in the generalized optimizing semi-field.

It is evident that a generalized optimizing semi-field is a partially ordered set.

Now let us definite the concept of modi-matrix.

Let $X = \{x_1, x_2, \dots x_m\}$ and $Y = \{y_1, y_2, \dots y_n\}$ be two given sets and a_{ij} ($i=1,2, \dots m$; $j=1,2, \dots n$) are elements taken from a semi-field $\{S, \oplus, \otimes\}$.

An array A with m rows and n colums

$$A = \begin{array}{c} \\ x_1 \\ x_2 \\ \vdots \\ x_m \end{array} \begin{array}{c} \begin{array}{cccc} y_1 & y_2 & \cdots & y_n \end{array} \\ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & \cdots & \cdots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \end{array}$$

or $A = x_i\,[a_{ij}]$ or $A = [a_{ij}]$

is called a m x n modi-matrix over $\{S, \oplus, \otimes\}$ where $x_1, x_2, \dots x_m$ is called row margin ,X the row set , $y_1, y_2, \dots y_n$ the column margin and Y the column set.

This array determines such a correspondence that from row x_i to column y_j there corresponds an element a_{ij} or there is a weight from x_i to y_j .

Two modi-matrices A and B over the same semi-field are equal if they have the same row margin, the same column margin and the same correspondence.

If there is no ambiguity ,we may write the modi -matrix without writing out the row margin and column margin.

We define modi-addition $\oplus$ and modi-multiplication $\otimes$ between modi-matrices in the same way as the ones between ordinary matrices. It is easy to prove that commutative and associative laws of modi -addition,associative law of modi-multiplication and distributive law among modi-matrices hold true.

In paper [8],we develope the concept of modi-matrix in more general form but it will not be used in this paper.

2. Jar -metric principle

Let G be a direct simple graph with following special properties.

The vertex set V can be partitioned into n + 1 subsets

$$V = \bigcup_{i=0}^{n} V_i$$

$V^{(i)} = \{V_t^{(i)} \quad t=1, 2, \ldots t_i\}$,

$|V^{(i)}| = t_i$, $\quad i=0,1, \ldots n$

where $V^{(i)}$ is called the i-th state of G and $V_t^{(i)}$ is called the vertex in the i-th state ,and each link (directed edge) on G has the property that if it initiats from some vertex in $V^{(i-1)}$, then it must terminate at some vertex in $V^{(i)}$. For example ,we have a link $V_\lambda^{(i-1)} V_\mu^{(i)}$ $(1 \leq \lambda \leq t_{i-1}$, $1 \leq \mu \leq t_i)$.

$V^{(0)}$ is called the initial state ,$V^{(n)}$ the final state. If $t_0 = 1$ and $t_n = 1$, we usually write $V^{(0)} = \{V_0^{(0)}\}$, $V^{(n)} = \{V_0^{(n)}\}$ and call $V_0^{(0)}$ and $V_0^{(n)}$ initial and final vertices respectively. If $t_0 \neq 1$ or $t_n \neq 1$, we may write $V^{(0)} = \{V_t^{(0)} t = 1,2, \ldots t\}$. $V^{(n)} = \{V_t^{(n)} |$ $t = 1, 2, , \ldots t_n\}$.

The subgraph induced by vertex subset $V^{(i-1)} \cup V^{(i)}$ is called the i -th stage of graph. Thus our G may be called the directed simple graph of n stages or the n (multi -)stage directed (simple)graph.

Now, to each link on G, there corresponds to an element of a given semi--fields $\{S, \oplus, \otimes\}$.

For explicity ,the element corresponding to the link $V_\lambda^{(i-1)} V_\mu^{(i)}$ may be denoted by $J(V_\lambda^{(i-1)}, V_\mu^{(i)})$,called the jar-metric of the link*. The multistage directed graph in which each link has a jar-metric is called the jared graph, denoted by $G[0, n]$. Here ,we mainly discuss the jared graph with $t_0 = t_n = 1$,if it is not stated explicitly.

On the jared graph ,if there is no link from $V_u^{(k)}$ to $V_v^{(k)}$, we may imagine that it does have a link from $V_u^{(k)}$ to $V_v^{(k)}$, but its jar-metric $J(V_u^{(k)}, V_v^{(k)})$ equals zero element z of semi-field $\{S, \oplus, \otimes\}$.

Then the i -th stage can be represented by a $t_{i-1} \times t_i$ modi-matrix denoted by STAGE $(V^{(i-1)}, V^{(i)})$ or STAGE (i) :

$$\begin{array}{ll} & \text{from}\backslash\text{to} \quad V_\mu^{(i)} \\ \text{STAGE}(i) = & V_\lambda^{(i-1)} [J(V_\lambda^{(i-1)}, V_\mu^{(i)})] \end{array} \tag{1}$$

If $t_{i-1} = 1$, (1) will be a row modi-vector ,and if $t_i = 1$, a column modi-vector. In the $t_{i-1} \times t_i$ modi -matrix, the λ-th row is denoted by (STAGE (i) $)_\lambda$ and the ν-th column by(STAGE (i) $)^\nu$.

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* Jar-metric is a transliteration from the Chinese term 标量. The term originally means a kind of standard containers used in the Han Dynasty about two thousend years ago. The reproducts are still exhibited in the Palace Museum in Beijing, China. We interpret it as an abstract measure in our theory.

We define the jar-metric from $V_\lambda^{(i-1)}$ to $V_\nu^{(i+1)}$ via $V_\mu^{(i)}$, denoted by $J(V_\lambda^{(i-1)}, V_\mu^{(i)}, V_\nu^{(i+1)})$, to be

$$J(V_\lambda^{(i-1)}, V_\mu^{(i)}, V_\nu^{(i+1)}) = J(V_\lambda^{(i-1)}, V_\mu^{(i)}) \otimes J(V_\mu^{(i)}, V_\nu^{(i+1)})$$

and the jar-metric from $V_\lambda^{(i-1)}$ to $V_\nu^{(i+1)}$ denoted by $J(V_\lambda^{(i-1)}, V_\nu^{(i+1)})$ to be $J(V_\lambda^{(i-1)}, V_\nu^{(i+1)}) = \sum_{\mu=1}^{t_i} J(V_\lambda^{(i-1)}, V_\mu^{(i)}, V_\nu^{(i+1)}) =$

$$= \sum_{\mu=1}^{t_i} J(V_\lambda^{(i-1)}, V_\mu^{(i)}) \otimes J(V_\mu^{(i)}, V_\nu^{(i+1)}) = $$

$$= (\mathrm{STAGE}\ (i))_\lambda \otimes (\mathrm{STAGE}(i+1))^\nu \tag{2}$$

We have

$$\sum_{\nu=1}^{t_{i+1}} J(V_\lambda^{(i-1)}, V_\nu^{(i+1)}, V_\eta^{(i+2)}) = \sum_{\nu=1}^{t_{i+1}} J(V_\lambda^{(i-1)}, V_\nu^{(i+1)}) \otimes J(V_\nu^{(i+1)}, V_\eta^{(i+2)})$$

$$= \sum_{\nu=1}^{t_{i+1}} \Big(\sum_{\mu=1}^{t_i} J(V_\lambda^{(i-1)}, V_\mu^{(i)}) \otimes J(V_\mu^{(i)}, V_\nu^{(i+1)})\Big) \otimes J(V_\nu^{(i+1)}, V_\eta^{(i+2)}) \tag{3}$$

and
$$\sum_{\mu=1}^{t_i} J(V_\lambda^{(i-1)}, V_\mu^{(i)}, V_\eta^{(i+2)}) = \sum_{\mu=1}^{t_i} J(V_\lambda^{(i-1)}, V_\mu^{(i)}) \otimes J(V_\mu^{(i)}, V_\eta^{(i+2)})$$

$$\sum_{\mu=1}^{t_i} J(V_\lambda^{(i-1)}, V_\mu^{(i)}) \otimes \Big(\sum_{\nu=1}^{t_{i+1}} J(V_\mu^{(i)}, V_\nu^{(i+1)}) \otimes J(V_\nu^{(i+1)}, V_\eta^{(i+2)})\Big) \tag{4}$$

and by the operation laws on the semi-filed, the right hand sides of (3) and (4) are equal. We define the result to be the jar-metric from $V_\lambda^{(i-1)}$ to $V_\eta^{(i+2)}$:

$$J(V_\lambda^{(i-1)}, V_\eta^{(i+2)}) = \sum_{\mu=1}^{t_i} J(V^{(i-1)}, V^{(\ )}, V^{(i+2)}) =$$
$$= \sum_{\nu=1}^{t_{i+1}} J(V_\lambda^{(i-1)}, V_\nu^{(i+1)}, V_\eta^{(i+2)}) =$$
$$= (\mathrm{STAGE}(i))_\lambda \otimes \mathrm{STAGE}(i+1) \otimes (\mathrm{STAGE}\ (i+2))^\eta \tag{5}$$

If $t_0 = t_n = 1$, the jar-metric from $V_0^{(0)}$ to $V_0^{(n)}$ can be defined in the similar way and be obtained by a following formula

$$J(V_0^{(0)}, V_0^{(n)}) = \prod_{i=1}^{n} \mathrm{STAGE}\ (i) \tag{6}$$

If $t_0 \neq 1$, or $t_n \neq 1$, we still have a formula in the same form

$$J(V^{(0)}, V^{(n)}) = \prod_{i=1}^{n} \mathrm{STAGE}\ (i)$$

but this result is not an element but just a $t_0 \times t_n$ modi-matrix. This is called the jar-metric of the jared graph. We sometimes call the modi-sum of all elements of the modi-matrix $J(V^{(o)}, V^{(n)})$ to be the total jar-metric of the jared graph G.

On the directed subgraph induced by vertex subset $\{V_{\lambda}^{(i-1)}, V^{(i)}, V^{(i+1)}, \ldots V^{(k-1)}, V_{\tau}^{(k)}\}$ ($1 \le \lambda \le t_{i-1}$, $k \ge i+1$, $1 \le \tau \le t_k$) we have

$$J(V_{\lambda}^{(i-1)}, V_{\tau}^{(k)}) = (\mathrm{STAGE}(i))_{\lambda} \otimes \prod_{j=i+1}^{k-1} \mathrm{STAGE}(j) \otimes (\mathrm{STAGE}(k))^{\tau} \qquad (7)$$

If we fix an integer s ($i-1 < s < k$, by the associative law of modi-multiplication, we have

$$J(V_{\lambda}^{(i-1)}, V_{\tau}^{(k)}) = \sum_{\xi=1}^{t_s} J(V_{\lambda}^{(i-1)}, V_{\xi}^{(s)}) \otimes J(V_{\xi}^{(s)}, V_{\tau}^{(k)}) \qquad (8)$$

If $V^{(i-1)}$ is called the start vertex of the induced graph and $V_{\tau}^{(k)}$ the end vertex of it, we can express (8) in words:

Jar-metric principle: On a multistage directed graph with jar-metric, the jar metric from any start vertex to any end vertex equals the modi-sum of all modi-products of the jar-metric from the start vertex to all those vertices of some middle state and that from those vertices of the middle state mentioned to the end vertex. This result is independent of all those states before the start vertex and after the end vertex.

As special cases, the start vertex may be the initial vertex of the jared graph, the end vertex may be the final vertex, and the middle state may be just next to the state that the start vertex belongs to or just before the one the end vertex belongs to.

Jar-metric principle is a very simple and intuitive one, it is just a kind of statement of the associative law of modi-multiplication of some modi-matrices.

If we develope the result on right hand side of (6), we have

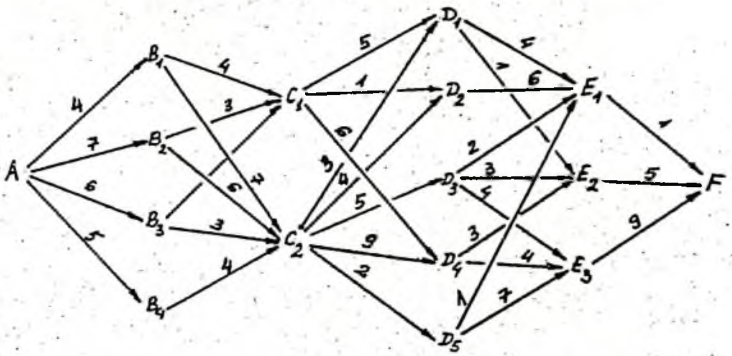
$$J(V_0^{(0)}, V_0^{(n)}) = \sum J(V_0^{(0)}, V_{i_1}^{(1)}) \otimes J(V_{i_1}^{(1)}, V_{i_2}^{(2)}) \otimes \ldots \otimes J(V_{i_{h-1}}^{(h-1)}, V_{i_h}^{(h)}) \otimes \ldots \otimes J(V_{i_{n-1}}^{(n-1)}, V_0^{(n)}) \qquad (9)$$

where under the modi-addition symbol Σ we refer to all possible combinations $i_1, i_2, \ldots i_{n-1}$ where $1 \le i_j \le t_j$ ($j=1,2, \ldots, n-1$).

Geometrically, if we define the jar-metric of a path to be the modi-product of jar-metrics of all links on the path. Then the result on (9) equals the modi-sum of jar-metrics of all path from initial vertex to final vertex. Of course, here, if there is no link from $V_r^{(k-1)}$ to $V_s^{(k)}$, that is to say, $J(V_r^{(k-1)}, V_s^{(k)}) = z$, then the jar-metric of each path which passes through $V_r^{(k-1)}$ and $V_s^{(k)}$ will be zero element.

We call the jar-metric $J(V_0^{(0)}, V_0^{(n)})$ the jar metric from the initial vertex on the graph G, that is the modi-sum of jar-metric of all paths from initial to final vertex.

Example 1. Find the shortest path(s) from A to F its length on the following graph.



Solution: We can solve the problem by finding the jar-metric taken from the semi-field $\{\overline{R}\ ,\ \wedge\ ,\ +\}$ the related path(s) in the graph. Let us write down the modi-matrices of stages

$$\text{STAGE }(A,B) = \begin{array}{c|cccc} \text{from}\backslash\text{to} & B_1 & B_2 & B_3 & B_4 \\ \hline A & 4 & 7 & 6 & 5 \end{array}$$

$$\text{STAGE }(A,B) = \begin{array}{c} \text{from}\backslash\text{to} \\ B_1 \\ B_2 \\ B_3 \\ B_4 \end{array} \begin{array}{c} \begin{array}{cc} C_1 & C_2 \end{array} \\ \begin{bmatrix} 4 & 7 \\ 3 & 6 \\ 6 & 3 \\ & 4 \end{bmatrix} \end{array}$$

$$\text{STAGE}(C,D) = \begin{array}{c} \text{from}\backslash\text{to} \\ C_1 \\ C_2 \end{array} \begin{array}{c} \begin{array}{ccccc} D_1 & D_2 & D_3 & D_4 & D_5 \end{array} \\ \begin{bmatrix} 5 & 1 & & 6 & \\ 3 & 4 & 5 & 9 & 2 \end{bmatrix} \end{array}$$

$$\text{STAGE}(D,E) = \begin{array}{c} \text{from}\backslash\text{to} \\ D_1 \\ D_2 \\ D_3 \\ D_4 \\ D_5 \end{array} \begin{array}{c} \begin{array}{ccc} E_1 & E_2 & E_3 \end{array} \\ \begin{bmatrix} 4 & 1 & \\ 6 & & \\ 2 & 3 & 4 \\ & 3 & 4 \\ 1 & & 7 \end{bmatrix} \end{array}$$

$$\text{STAGE }(E,F) = \begin{array}{c} \text{from}\backslash\text{to} \\ E_1 \\ E_2 \\ E_3 \end{array} \begin{array}{c} F \\ \begin{bmatrix} 1 \\ 5 \\ 9 \end{bmatrix} \end{array}$$

Now we define

$$\text{STAGE}(A,\ B)\otimes\text{STAGE}(B,\ C)\equiv\text{STAGE}(A,B,C,)\equiv\text{STAGE}(A,C)$$

$$\text{STAGE}(A,\ B)\otimes\text{STAGE}(B,\ C)\otimes\text{STAGE}(C,D,)$$
$$\equiv\text{STAGE}(A,B,C,D,)=\text{STAGE}(A,D)$$

and so on. We have

$$\text{STAGE}\ (A,C)=\begin{array}{cc} \text{from}\backslash\text{to} & \begin{array}{cc} C_1 & C_2 \end{array} \\ A & \left[\begin{array}{cc} \frac{8}{B_1} & \frac{9}{B_3\quad B_4} \end{array}\right] \end{array}.$$

$$\text{STAGE}\ (A,D)=\begin{array}{cc} \text{from}\backslash\text{to} & \begin{array}{ccccc} D_1 & D_2 & D_3 & D_4 & D_5 \end{array} \\ A & \left[\begin{array}{ccccc} \frac{12}{C_2} & \frac{9}{C_1} & \frac{14}{C_2} & \frac{14}{C_1} & \frac{11}{C_2} \end{array}\right] \end{array}$$

Here,we have made two convertions.The first is that all elements which ought to be written but not written out are the ∞'s in the semi-field $\{\bar{R},\wedge,+\}$ —— the positive infinity.The second is that the vertex under a number divided by a short line is the one where the shortest path passes through. For example ,on STAGE (A,C),we can read the paths from A to C_2 via B_3 or B_4, are the shortest among all (four) possible paths from A to C_2 , and the lenght will be 9. Again ,on STAGE (A, D) ,we can read that the path from A to D_4 via C_1 is the shortest with the lenght 14.

As for the shortest path from A to C_1 ,we can look at STAGE (A ,C) and find that it must pass through vertex B_1 . Similarly we have

$$\text{STAGE}(A,E)=\begin{array}{cc} \text{from}\backslash\text{to} & \begin{array}{ccc} E_1 & E_2 & E_3 \end{array} \\ A & \left[\begin{array}{ccc} \frac{12}{D_5} & \frac{13}{D_1} & \frac{18}{D_3,D_4,D_5} \end{array}\right] \end{array}$$

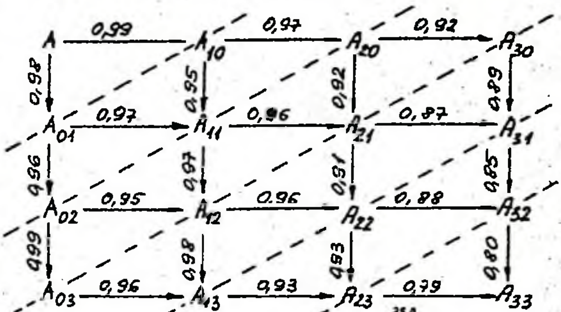
$$\text{STAGE}\ (A,F)=\begin{array}{cc} \text{from}\backslash\text{to} & F \\ A & \left[\frac{13}{E_1}\right] \end{array}$$

Therefore the shortest length from A to F is 14 and the shortest path can be found out from STAGE (A, F), STAGE (A,E),STAGE (A,D) and STAGE(A,C) successively ,we have

$$A\ \begin{matrix} B_3 \\ B_4 \end{matrix}\,[C_2\ \ D_5\ \ E_1\ \ F]$$

Hence we have two shortest paths with the length 14.

Example 2. A reconnaissance plane is going to carry out a bomb task from its base A to the object B of his enemy district. All possible flying paths are indicated in the following graph.The figure on each link represents the probability in favour that the plane passes through the link. Find the favourest path from A to B and its probability in favour.



Solution. This graph can be considered as a graph of 6 stages if we look along dotted lines. Thus the probability in favour that the plane flying along the path will be the product of all those of each link on the path. And the path with the greatest probability will be the favourest one. Thus our problem will be to find the jar-metric of the graph on the semi-field $\{I,\wedge,\times\}$ where $I=[0,1]$. Now we write down the modi -matrices of the stages :

$$\text{STAGE (1)} = \begin{array}{c|cc} \text{from}\backslash\text{to} & A_{10} & A_{01} \\ \hline A & 0,99 & 0.98 \end{array}$$

$$\text{STAGE (2)} = \begin{array}{c|ccc} \text{from}\backslash\text{to} & A_{20} & A_{11} & A_{02} \\ \hline A_{10} & 0.97 & 0.95 & \\ A_{01} & & 0.97 & 0.96 \end{array}$$

$$\text{STAGE (3)} = \begin{array}{c|cccc} \text{from}\backslash\text{to} & A_{30} & A_{21} & A_{12} & A_{03} \\ \hline A_{20} & 0.92 & 0.92 & & \\ A_{11} & & 0.96 & 0.97 & \\ A_{02} & & & 0.95 & 0.99 \end{array}$$

$$\text{STAGE (4)} = \begin{array}{c|ccc} \text{from}\backslash\text{to} & A_{31} & A_{22} & A_{13} \\ \hline A_{30} & 0.89 & & \\ A_{21} & 0.87 & 0.91 & \\ A_{12} & & 0.96 & 0.98 \\ A_{03} & & & 0.96 \end{array}$$

$$\text{STAGE (5)} = \begin{array}{c|cc} \text{from}\backslash\text{to} & A_{32} & A_{23} \\ \hline A_{31} & 0.85 & \\ A_{22} & 0.88 & 0.93 \\ A_{13} & & 0.92 \end{array}$$

$$\text{STAGE (6)} = \begin{array}{c|c} \text{from}\backslash\text{to} & B \\ \hline A_{32} & 0.80 \\ A_{23} & 0.79 \end{array}$$

In calculating ,we will take four places after decimal point in order to distinguish which is the better one. Then we have

$$\begin{array}{lcccc} & \text{from}\backslash\text{to} & A_{20} & A_{11} & A_{02} \\ \text{STAGE }(1\ ,2\ ) = A & & \left[\dfrac{0.9603}{A_{10}}\right. & \dfrac{0.9506}{A_{01}} & \left.\dfrac{0.9408}{A_{01}}\right] \end{array}$$

$$\begin{array}{lccccc} & \text{from}\backslash\text{to} & A_{30} & A_{21} & A_{12} & A_{03} \\ \text{STAGE}(1,\ 2,) = A & & \left[\dfrac{0.8835}{A_{20}}\right. & \dfrac{0.9126}{A_{11}} & \dfrac{0.9221}{A_{11}} & \left.\dfrac{0.9314}{A_{02}}\right] \end{array}$$

$$\begin{array}{lcccc} & \text{from}\backslash\text{to} & A_{31} & A_{22} & A_{13} \\ \text{STAGE}(1,4) = A & & \left[\dfrac{0.7940}{A_{21}}\right. & \dfrac{0.8852}{A_{12}} & \left.\dfrac{0.9036}{A_{12}}\right] \end{array}$$

$$\begin{array}{lccc} & \text{from}\backslash\text{to} & A_{32} & A_{23} \\ \text{STAGE}(1,5) = A & & \left[\dfrac{0.7790}{A_{22}}\right. & \left.\dfrac{0.8313}{A_{13}}\right] \end{array}$$

and

$$\begin{array}{lcc} & \text{from}\backslash\text{to} & B \\ \text{STAGE }(1,6\ ) = A & & \left[\dfrac{0.6567}{A_{23}}\right] \end{array}$$

thus the favwrest path for the plane will be

$$A \quad A_{01} \quad A_{11} \quad A_{12} \quad A_{13}, \quad A_{23} \quad B$$

and its probability in favor is 0.6567.

We can use the jar-metric principle to calculate jar-metric of the optimum path on jared graph on different optimizing semi-fields. But the result will have some differences between those on optimizing semi-field and strongly optimizing one.

Suppose we have a multistage directed graph G. From initial vertex $V_0^{(0)}$ to final vertex $V_0^{(n)}$, there are several paths. Let

$$V_0^{(0)} \quad V_{i_1}^{(1)} \quad V_{i_2}^{(2)} \cdots \quad V_{i_k}^{(k)} \quad V_0^{(n)} \tag{10}$$

be any of them, and

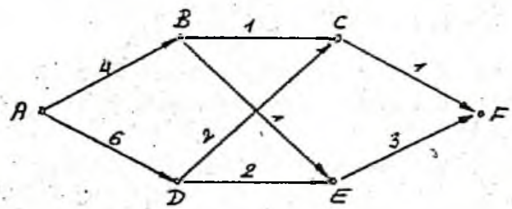
$$L\ (\ 0,n\ ) : V_0^{(0)} \quad V_{p_1}^{(1)} \quad V_{p_2}^{(2)} \cdots \ V_{p_k}^{(k)} \cdots \ V_0^{(n)} \tag{11}$$

be a fixed one. Besides the concept of the optimum path from $V_0^{(0)}$ to $V_0^{(n)}$ on G, we introduce

<u>Definition 5</u>. If L (0,n) is an optimum path from $V_0^{(0)}$ to $V_0^{(n)}$ and if any subpath, say L (h,k), i. e., a subpath from $V_p^{(k)}$ to $V_p^{(k)}$ on L (0,n) is an optimum path from $V_p^{(h)}$ to $V_p^{(k)}$ on the induced subgraph $[G\ h,p_h;k,p_k]$ of G, then we say L (0,n) is the optimum path of G.

There is a bit but quite important difference between the definition and everyday experience.Let us see the following example.

Example 3. On a 3-stage directed graph G,according to the following rules,discuss the shortest path from A to F and those of graph G respectively.



i) if length of a path is the sum of lengths of all links on it ;

ii) if length of a path equals the maximum length among those of all links on it;

iii) if length of a path equals the sum of length of all links on it taken mod 4.

Solutions. i) There are four paths from A to F.The length of the path ABCF is 6 and no one of the others is shorter than it,so it is the shortest path from A to F. And what is more ,we canprove without difficulty that any subpath of the path ABCF is the shortest on the corresponding induced graph.Hence the path ABCF is the shortest of graph G also .

ii) The langth of the path ABEF is 4. It is the shortest path from A to F because the length of any other path will not be shorter than it. But the subpath BEF on the corresponding induced subgraph is not the shortest from B to F. Therefore ,the path ABEF is not the shortest path of graph G. But the path ABCF is really the shortest path of the graph G. Of course,it is the shortest one from A to F also.

iii)The length of the path ABEF equals 0 (= 4 + 1 + 3 = 0 mod 4). This path is the shortest from A to F,but its subpath ABE is not the shortest on the corresponding induced graph.Actually,according to our rule ,there exists no shortest path of the graph G. ///

As discussed above,according to our definition ,on a multistage directed graph,it does not necessarily have optimum path of the graph.And ,even if there is an optimum from initial vertex to final vertex,it needs not be the optimum path of the graph.

Theorem 2. i) On a multistage directed graph G, the jar-metric of each link taken from an optmizing semi-field. If L(0,n)is an optimum path from initial vertex to final vertex,then its jar-metric equals $J(v_0^{(0)},v_0^{(n)})$ which can be computed by (6).

ii) On a multistage directed graph ,if the jar-metric of each link is taken from a strongly optimizing semi-field,then the optimum path from initial vertex to final vertex is same as that of the graph.

Proofs. i) This is the result of (6) and (9).

ii) By definition ,it is evident that the optimum path of G is that from initial vertex to final vertex.

In contrast, let L (0,n), as (10), be an optimal path $V_0^{(0)}$ to $V_0^{(n)}$, and

$$L(h,k): V_{p_h}^{(h)} \quad V_{p_{h+1}}^{(h+1)} \cdots \quad V_{p_k}^{(k)}$$

be any subpath of L (0 ,n). Suppose L (0, h), L (h , k) and L (k, n) have jar-metric P,Q and R respectively. Then the jar-metric of L (0,n) equals $P \otimes Q \otimes R$.

On L (0 ,n), if we delet the subpath L (h,k) and join $V_{p_h}^{(h)}$, $V_{i_{h+1}}^{(k+1)}$... $V_{i_{k-1}}^{(k-1)}$ $V_{p_k}^{(k)}$ where vertexes $V_{i_{h-1}}^{(k-1)}$,... $V_{i_{k-1}}^{(k-1)}$ may be any in the (h+1)... ...(k-1)- th state respectively. The path from $V_0^{(0)}$ to $V_0^{(n)}$ thus constructed will not be better than L (0 , n). Then

$$P \otimes Q \otimes R = \sum P \otimes J(V_{p_h}^{(h)}, V_{i_{h+1}}^{(h+1)}) \otimes \ldots \otimes J(V_{i_{k-1}}^{(k-1)}, V_{p_k}^{(k)}) \otimes R =$$

$$= P \otimes \left(\sum J(V_{p_h}^{(h)}, V_{i_{h+1}}^{(h+1)}) \otimes \ldots \otimes J(V_{i_{k-1}}^{(k-1)}, V_{p_k}^{(k)})\right) \otimes R$$

Then we may assert that the quantity in the brackets must be Q. That is to say, L (h , k) is an optimum path from $V_{p_h}^{(h)}$ to $V_{p_k}^{(k)}$ on the corresponding induced subgraph.

By contradiction ,if not so ,we put the quantity in the brackets to be T ,that is

$$T = J(V_{p_h}^{(h)}, V_{q_{h+1}}^{(h+1)}) \otimes J(V_{q_{h+1}}^{(h+1)}, V_{q_{h+2}}^{(h+2)}) \otimes \ldots \otimes J(V_{q_{k-1}}^{(k-1)}, V_{p_k}^{(k)})$$

then there exists path, say ,

$$L'(h,k): V_{p_h}^{(n)} \quad V_{q_{h+1}}^{(h+1)} \cdots V_{q_{k-1}}^{(k-1)} V_{p_k}^{(k)}$$

which would be better than L (h,k). Then we would have

$$T + Q \neq Q, \text{ and } T + Q = T$$

Since the semi-field is strongly optimizing, we would have

$$P \otimes T \otimes R + P \otimes Q \otimes R \neq P \otimes Q \otimes R$$

The path L (0,h) L'(h, k) L (k,n) would be better than L (0,n). This is contrary to hypotesis.

Thus every uppath L (h,k) on L (0,n) is an optimum path from V_{p_h} to V_{p_k} on the corresponding induced subgraph. Therefore L (0,n) is an optimum path of G.

At the end of the section, we'd like to make some comments. To a chain of ordinary matrices , the problem of finding its best association has been discussed by some scholars. Some of their results can be transplanted to our theory and make something clear.

For example ,there is a directed graph of 4 stage with the vertex set

$$\{v_i^{(0)} \mid i = 1,2,\dots 10\} \cup \{v_i^{(1)} \mid i = 1,2,\dots 20\} \cup \{v_i^{(2)} \mid i=1,2..50\}$$
$$\cup \{v_1^{(3)}\} \cup \{v_i^{(4)} | i= 1,2,\dots 100\}$$

to each link there is a jar-metric taken from some optimizing semi-field. Now we want to find the jar-metric of the optimum one among all 1.000.000(= 10 x 20 x 50 x 1 x 100) paths, i.e. , the total jar-metric of the graph. If the modi-matrices of the four stages be M_1 , M_2 , M_3 , M_4 ,with orders 10 x 20, 20 x 50 , 50 x 1 ,1 x 100 .We must first calculate

$$M = M_1 \otimes M_2 \otimes M_3 \otimes M_4$$

and then we search for the optimum one among all elements on M.Now,how do we calculate M ? In dynamic programming ,we do not make any difference between forward procedure and backward one.But,actually ,things are not quite so.We can easily calculate by backward procedure

$$M_1 \otimes (M_2 \otimes (M_3 \otimes M_4))$$

here we must do 117,000 (= 50 x 1 x 100 + 20 x 50 x 100 + 10 x 50 x 100 $\otimes$'s and (118,000 -1) $\oplus$'s.If we calculate M by forward procedure

$$((M_1 \otimes M_2) \otimes M_3) \otimes M_4$$

we will do 11,500 $\otimes$'s and 10,989 $\oplus$'s only. Moreover ,it is easily to check that the best association of the chain will be

$$(M_1 \otimes (M_2 \otimes M_3)) \otimes M_4$$

In this case we need only to 10, 2,200 $\otimes$'s and 2,169 $\oplus$'s.Thus ,if we consider the number of $\otimes$'s only ,those of the best association will be 19 % of those by the backward procedure ,and 1,76 % of those by the forward one. forward and backward procedures will not be the same in the sense of com - putation complexity.

The second point is about R.Bellman's principle of optimality. It is well known that some optimum processes do not have such a property mentioned in the principle and also processes which have the property mentioned above need not be optimum. In general,there is no universal equivalent relation between the principle and the formula. May be the result obtained by forward formula will not necessarily be the same as those obtained by backward one.Example 3 shows the matter. Here , we'd like to take jar-metric principle as a basis instead of Bellman's principle of optimality. We know that jar-metric principle will be held true on some strict basis and it is equivalent to formula (6) which has an effective algebraic structure.Moreover ,the formula can be stated to solve some other complicate problem which will be discussed in following sections.

In dynamic programming,people like to consider as a basis,all those problems depending on time ,and put all problems which can be converted into multi-stage graph into those depending on time.In our theory ,we'd like to discuss all those geometrical problems as a basis and then put those problems depending on time into geometrical ones.Thus ,in our theory,"dynamic"feature disappears.

3.Semi -field N-THOPT and the first N-th order optimum paths of first kind

Pansystem Analysis,motivated and developed by professor Wu Xuemou and his colleagues,has been obtained a great deal of results and theorems . One of those is so-called optimum principle of N-th order. Putting his word into our framework,it says that : There are many paths from the initial vertex to the final vertex on a multistage directed graph,with jar-metric taken from a strongly optimizing semi-field. To each path,there corresponds an element of the jar-metric of strongly optimizing semi-field.The optimum path in the sense as mentioned in section 2 , is called that of zero or -der .Of course such a path will not necessarily be unique.In this graph ,we pay no attention to all those optimum paths mentioned,there will be some optimum paths among the remaining ones.We call those the optimum path of first order of the graph. Similary,if we pay no attention to all paths of all first N-1'th order ,we may find the optimum among the remaining path which will be called the optimum path of N-th order. Then we have

Optimum principle of N-th order (Wu Xuemou) [4]. If L (0,n) is an optimum path of N-th order in a multistage directed graph G and if the sub-path L (h,k) of L (0,n) is the optimum path of m-th order in the related induced subgraph,then we have:

$$m \leq N$$

Theorem 3. (Qin Koukaung) [5]. If L (0,n),L (0, h) and L(h,n) are the optimum paths of N-th , m -th and m -th order respectively on the related (induced sub-) graphs,then we have

$$m_1 + m_2 \leq N$$

Corollary 1. If $0 = h_0 < h_1 < h_2 < \ldots < h_{s-1}$

L (0,n) and L (h_{i-1} , h_i) are the optimum paths of N-th and m_i-th order on related (induced sub-) graphs respectively,then we have

$$\sum_{i=1}^{l+q} m_i \leq N \ (0 \leq l < l+q \leq s)$$

Particulary ,we have

$$\sum_{i=1}^{s} m_i \leq N$$

Corollary 2. If $0 < m_{i_0} < N$,then for all i ,we have $m_i < N$. If $m_{i_0} = N$,then for all $i \neq i_0$,we have $m_i = 0$.

Now us a these result to develope our theory.

On strongly optimizing semi-field $\{S , \oplus , \otimes\}$, we take (N + 1) yang elements or identity elements to form a sequence. If it satisfies the conditions

$$a_0 \prec a_1 \prec \cdots \prec a_k \prec a_{k+1} = z = \ldots = z$$

where $0 \leq k \leq N + 1$ and if we define that $z = z$ can be written as $z \prec z$, then we call this sequence with N + 1 elements to be strictly monotonic to bad and write as

$$\{a_0 , a_1 , \ldots a_{k+1}, z, \ldots z\} \qquad (1)$$

where the 0-th term a_0 is called the optimum element of 0-th order of the sequence,the k-th term a_k is the optimum element of k-th order. a_1 is called suboptimum element also.

The family which contains all strictly monotonic to bad sequences like (1) is denoted by N-th and the sequence will be called the element of the family.

Let $A = \{a_0, a_1 \ldots a_N\}$, $B = \{b_0, b_1 \ldots b_N\}$ belong to N-th.We call them to be equal,if and only if $a_i = b_i (= 0, 1, \ldots N)$.

Given two elements A and N-Th ,we rearrange all those 2N + 2 terms monotonic to bad and take the first N + 1 non-repeared(except zero) elements to form a new sequence which is unique and is an element of N-Th. We define this to be modi-sum $A \oplus B$ of A and B.

For example,in a strongly optimizing semi-field $\{\bar{R} , \wedge , +\}$, there are two strictly monotonic to bad sequences with 4 terms $\{1 , 3, 4, 6\}$ and $\{2,3,4,7\}$ Rearranging these 8 elements : 1,2,3,3,4,4,6,7, we have a new sequence $\{1 , 2 , 3 , 4\}$ and denote $\{1 , 3 , 4 , 6\} \oplus \{2, 3, 4, 7\} = \{1, 2 , 3, 4\}$ The modi -addition thus defined satisfies laws of commutativity. and associativity.

To A and B ,we rearrange $(N + 1)^2$ modi-products $a_i \otimes b_j$ $(0 \leq i , j \leq N)$ monotonic bad.

Then taking the first N + 1 non -repeated (except zero) elements to form a new sequence,we define this by $A \otimes B$.

For example,we have two sequences $\{1 , 3 , 4 , 6\}$ and $\{1 , 3, z , z\}$ on $\{\bar{R}, \wedge , +\}$. Doing the all modi-products,we have

$$\begin{matrix} 2, & 4, & 5, & 7, \\ 4, & 6, & 7, & 9 \\ z, & z, & z, & z \\ z, & z, & z, & z \end{matrix}$$

the first 4 non- repeated elements are 2 , 4 , 5 , 6, then we have

$$\{1, 3, 4, 6\} \otimes \{1,3, z,z\} = \{2,4,5,6\}$$

The law of commutativity is evidently true for the modi -multiplication thus defined.

Now, we are going to discuss the law of associativity. If $a_i \prec a_{i+1}$ and $a_i \ne z$, we have

$$a_i \oplus a_{i+1} = a_i \quad \text{and} \quad a_i \oplus a_{i+1} \ne a_{i+1}$$

by the strong optimal, for any $h \ne z$, we have

$$a_i \otimes h \oplus a_{i+1} \otimes h = a_i \otimes h$$
$$a_i \otimes h \oplus a_{i+1} \otimes h \ne a_{i+1} \otimes h$$

Thus we have $a_i \otimes h \prec a_{i+1} \otimes h$. If $a_i = z$ or $h = z$, by our convention on symbol $z \prec z$, we still have $a_i \otimes h \prec a_{i+1} \otimes h$. Thus, for

$$(e \preceq) \; a_0 \prec a_1 \prec \dots \prec a_N \prec a_{N+1}$$

$$(e \otimes h \preceq) \; a_0 \otimes h \prec a_1 \otimes h \prec \dots \prec a_N \otimes h \prec a_{N+1} \otimes h$$

That is to say, if a_{N+1} is worse than all a_i ($i=0,1,\dots N$), then $a_{N+1} \otimes h$ is worse also than all $a_i \otimes h$ ($i = 0, 1, \dots N$).

Suppose $A \otimes B = \{a_{i_0} \otimes b_{j_0}, a_{i_1} \otimes b_{j_1}, \dots a_{i_N} \otimes b_{j_N}\}$ and $a_t \otimes b_s$ be an element of $\{a_i \otimes b_j \mid i,j = 0,1,\dots N\}$ which is worse than all those terms in $A \otimes B$. Then, to any element h, $(a_t \otimes a_s) \otimes h$ must be worse than any term in $A \otimes B$ modi-multiplied by h. Therefore, $(A \otimes B) \otimes C$ is a sequence in which each term is taken from the first N optimum modi -product of some term of $A \otimes B$ and c_k of C, also those modi-product of some terms $(a_i \otimes b_j) \otimes c_k$. Since $(a_i \otimes b_j) \otimes c_k = a_i \otimes (b_j \otimes c_k)$. Therefore we have

$$(A \otimes B) \otimes C = A \otimes (B \otimes C)$$

Similary we can prove that the law of distributivity holds also.

Elements $E = \{e, z, \dots z\}$ and $Z = \{z, z, \dots z\}$ are identity and zero element of the final N-th.

Therefore the family N-th is a semi-field[d] with identity. We denote it by N-THOPT or $\{$N-Th, $\oplus, \otimes\}$ or more clearly, $\{S, \oplus, \otimes\}$- N-THOPT.

When $N = 0$, N-THOPT will reduce to the strongly optimizing semi-field itself.

In semi-field N-THOPT, $A \oplus B$ equals, in general, neither A nor B. But it has the following properties

$$(A \oplus B) \oplus A = A \oplus B, \quad (A \oplus B) \oplus B = A \oplus B$$

thus N-THOPT is a generalized optimizing-field, called Shier semi-field [6]

If a sequence like (1) contains some zero elements, we can omit those terms, for simplicity. For example $\{a_0, a_1, a_2, z, \dots z\}$ may be written as $\{a_0, a_1, a_2\}$; $\{b_0, z, \dots z\}$ as $\{b_0\}$ or b_0. Of course, for $\{z, \dots z\}$, it would be better to write as z.

Suppose the jar -metric of each link on a multistage directed graph be a yang element or e taken from a strongly optimizing semi-field $\{S, \oplus, \otimes\}$

If there are links with different jar-metric from V_i to V_j, we can arrange these in a monotonic to bad order. If there are more than N + 1 terms, we taken the first N + 1 terms. If there are only k ($\leq$ N) terms, we can add N + 1 -k zero elements to them. Thus, in short, we can write the first N + 1 jar-metrics, from V_i to V_j, as an element $A = \{a_0, a_1, \ldots a_N\}$ which belongs to N-THOPT. We may say a being a jar-metric taken from N-THOPT. If there are two groups of links from V_i to V_j, their jar-metric are A and B respectively. Then $A \oplus B$ will be the jar-metric of these two groups of links and, geometrically, it represents the jar-metrics of the first N + 1 non-repeated optimum links from these two groups of links.

If the jar-metric from V_i, to V_j be A, and that V_j to V_k be C, then the jar-metrics from V_i to V_k via V_j will be $A \otimes C$.

For a n-stage directed graph G, if each link corresponds to jar-metric taken from N-THOPT, then the jar-metric from the initial vertex $V_0^{(0)}$ to the final vertex $V_0^{(n)}$ of the graph G can be calculated by jar-metric principle. In this result, we can find the optimum paths of 0th, 1th, ... and Nth orders. We refer to this as a problem of finding optimum paths of first N - order of first kind.

When N = 0, it is our fundamental result obtained in [2] and when N = 1, we have established an algoritm in the paper [3].

Since on semi-field N-THOPT, the computational complexities of calculating $A \oplus B$ and $A \otimes B$, are two numbers depending only on N. Thus we have:

Theorem 4. The computational complexity of calculating the jar-metrics of optimum paths of the first N order of first N order kind is the same as that of zero order.

Example 4. On the 5-stage directed graph shown in example 1, every link corresponds to a real number, as its length. To find the shortest path of the 0-th, 1-th, 2-th, 3-th order (i.e., the shortest, second, third and fourth shortest) and their lengths.

Solution. We may consider the length of each link being an element of the generalized optimizing semi-field $\{\bar{R}, \wedge, +\}$ -3-THOPT. Then our problem has been converted into that of finding the jar-metric from A to B of the graph. We may write the modi-matrices of these five stages as those in example 1.

Let us find the jar-metric from A to C_1. Calculating $4 \otimes 4$, $7 \otimes 3$, $6 \otimes 6$ and $5 \otimes z$, we have $\{8, 10, 12, z\}$. Note that, for example 10, it is the jar-metric of the path from A to C_1 via B_2 and which is modi-product of 7, taken from the optimum of 0-th order of $\{7, z, z, z\}$, and 3, taken from $\{3, z, z, z\}$, thus we can write $\frac{10}{B_2^{\{0\}}}$, and so on.

Thus we can write the jar-metric from A to C as $\left\{\frac{8}{B_1^{\{0\}}}, \frac{10}{B_2^{\{0\}}}, \frac{12}{B_3^{\{0\}}}\right\}$

Similary, the jar-metric from A to C_2 is equal to $\left\{\frac{9}{B_3^{\{0\}}B_4^{\{0\}}}, \frac{10}{B_1^{\{0\}}}, \frac{12}{B_2^{\{0\}}}\right\}$.

Thus we have

$$\text{STAGE }(A,C) = \text{STAGE }(A,B) \otimes \text{STAGE }(B,C) =$$

$$\begin{array}{lll} \text{from}\backslash\text{to} & C_1 & C_2 \\ = A & \left[\left\{\frac{8}{B_1^{\{0\}}}, \frac{10}{B_2^{\{0\}}}, \frac{12}{B_3^{\{0\}}}\right\}\right. & \left.,\left\{\frac{9}{B_3^{\{0\}}B_4^{\{0\}}}, \frac{11}{B_1^{\{0\}}}, \frac{13}{B_2^{\{0\}}}\right\}\right] \end{array}$$

Notice that alphabets a under bars will not participate in any operations henceforth.

For simplicity, we stipulate that all $\{0\}$ on the upper-right corner will be deleted, for example $B_3 = B_3$. Therefore we have

$$\begin{array}{lll} \text{from}\backslash\text{to} & C_1 & C_2 \\ \text{STAGE}(A,C) = A & \left[\left\{\frac{8}{B_1}, \frac{10}{B_2}, \frac{12}{B_3}\right\}\right. & \left., \left\{\frac{9}{B_3, B_4}, \frac{11}{B_1}, \frac{13}{B_2}\right\}\right] \end{array}$$

We can obtain

$$\text{STAGE }(A,D) \equiv \text{STAGE }(A,C) \otimes \text{STAGE}(C,D)$$

$$\begin{array}{lll} \text{from}\backslash\text{to} & D_1 & D_2 \\ = A & \left[\left\{\frac{12}{C_2}, \frac{13}{C_1}, \frac{14}{C_2^{\{1\}}}, \frac{15}{C_1^{\{1\}}}\right\}\right., & \left\{\frac{9}{C_1}, \frac{11}{C_1^{\{1\}}}, \frac{13}{C_1^{2}, C_2}, \frac{15}{C_2^{\{1\}}}\right\}, \end{array}$$

$$\begin{array}{lll} D_3 & D_4 & D_4 \\ \left\{\frac{14}{C_2}, \frac{16}{C_2^{\{1\}}}, \frac{18}{C_2^{\{2\}}}\right\}, & \left\{\frac{14}{C_1}, \frac{16}{C_1^{\{1\}}}, \frac{18}{C_1^{\{2\}}, C_2}, \frac{20}{C_2}\right\} & \left.,\left\{\frac{11}{C_2}, \frac{13}{C_2^{\{1\}}}, \frac{15}{C_2^{\{2\}}}\right\}\right], \end{array}$$

$$\text{STAGE }(A,E) \equiv \text{STAGE }(A,D) \otimes \text{STAGE }(D,E)$$

$$\begin{array}{lll} \text{from}\backslash\text{to} & E_1 & E_2 \\ = A & \left[\left\{\frac{12}{D_5}, \frac{14}{D_5^{\{1\}}}, \frac{15}{D_2}, \frac{16}{D_1\ D_3\ D_5^{\{2\}}}\right\}\right., & \left\{\frac{13}{D_1}, \frac{14}{D_1^{\{1\}}}, \frac{15}{D_1^{\{2\}}}, \frac{16}{D_1^{\{3\}}}\right\}, \end{array}$$

$$\begin{array}{c} E_1 \\ \left\{\frac{18}{D_3, D_4, D_5}, \frac{20}{D_3^{\{1\}}, D_4^{\{1\}} D_5^{\{1\}}}, \frac{22}{D_3, D_4^{\{1\}} D_5^{\{2\}}}, \frac{24}{D_4^{\{3\}}}\right\} \end{array}$$

$$\text{STAGE }(A,F) \equiv \text{STAGE }(A,E) \otimes \text{STAGE }(E,F)$$

$$\begin{array}{ll} \text{from}\backslash\text{to} & F \\ = A & \left[\left\{\frac{13}{E_1}, \frac{15}{E_1^{\{1\}}}, \frac{16}{E_1^{\{2\}}}, \frac{17}{E_1^{\{3\}}}\right.\right. \end{array}$$

Therefore ,there are two optimum (i.e., the shortest) paths with length 13.They are

$$AB_3C_2D_5E_1F \quad \text{and} \quad AB_4C_2D_5E_1F$$

There is an optimum path of first order with length 15. That is

$$A\ B_1C_2D_5E_1F$$

We have optimum path of second order with length 16. That is

$$AB_1C_1D_2E_1F$$

And ,finally we have 5 optimum paths of third order with length 17. They are

$$AB_3C_2D_1E_1F \qquad AB_3C_2D_2E_1F \qquad AB_4C_2D_1E_1F$$
$$AB_4C_2D_2E_1F \qquad AB_2C_2D_5E_1F$$

4.Semi -field and optimum paths of first N order of second kind

In this section,it is supposed that all letters a , b , a_{ij} , b_i and p_i are non-zero elements of an optimizing semi-field $\{S , \oplus , \otimes\}$. Suppose we have a_k , a parameter t and a non-negative integer k. We call the formal product $a_k t^k$ a term of k power with coefficient a_k.We define

$$a_0\ t^0 = a_0 \quad , \quad et^k = t^k \quad , \quad zt^k = z$$

We say two terms are equal , $a_i t^i = a_j\ t^j$

if and only if $a_i = a_j$ and $i = j$

We define the modi-sum of at^r and bt^s , where $r \neq s$,to be $at^r \oplus bt^s$ or $bt^s \oplus at^r$ .If $r = s$, we define $at^r \oplus bt^s$ to be $(a \oplus b)\ t^s$. Again $at^r \otimes bt^s$ is defined to be $(a \otimes b)\ t^{r+s}$ .Thus our definition are the same in form as those in ordinary sense.

If p_i belongs to $\{S , \oplus , \otimes\}$ ($i = 0 ,1 , \dots n$) and $p_0 \neq z$,we call

$$p_0 t^n \oplus p_1 t^{n-1} \oplus \dots \oplus p_{n-1}\ t \oplus p_n$$

a modi -polynomial of degree n . Two modi-polynomials are equal if and only if all corresponding terms of the same power are equal.

We can define modi-addition $\oplus$ and modi -multiplication $\otimes$ between modi-polynomials in same they way like those in ordinary sense. Now we construct a set which contains all modi-polynomials with non-yin elements as coefficients on an optimizing semi-field with identity and contains also z and e of the semi-field as identity and zero elements respectively . This set is a semi-field called a semi -field of modi-polynomial on $\{S, \oplus , \otimes\}$, denoted by $\{P(t) , \oplus , \otimes\}$ or $\{S , \oplus , \otimes\} - \{P(t) , \oplus , \otimes\}$.

If in the modi -polynomial

$$a_0t^n \oplus a_1t^{n-i_1} \oplus a_2t^{n-i_2} \oplus \ldots \oplus a_pt^{n-i_p} (0 < i_1 < i_2 < \ldots < i_p < n)$$

the coefficients $a_1, a_2, \ldots, a_p$ are strictly monotonic to bad: $a_1 \prec a_2 \prec \ldots \prec a_p$, we call it an essential modi-polynomial and denote by $\vec{P}(t)$. We write the symbol $\rightarrow$ above to emphasize that when an essential modi-polynomial is written in decreasing power, the coefficient sequence will be strictly monotonic to bad. Let the set off all essential modi-polynomials be denoted by $\vec{P}(t)$. Evidently, identity e and zero element z in $\{S, \oplus, \otimes\}$ belong to $\vec{P}(t)$ and $\vec{P}(t) \subset P(t)$.

To any element P (t) of $\{P(t), \oplus, \otimes\}$, we can use so-called badinizing process $[\![\ ]\!]$ to construct an essential modi-polynomial $[\![P(t)]\!]$. The process is defined as follows: For mononomial a_it^i we have

$$[\![a_it^i]\!] = a_it^i$$

and, particularly,

$$[\![z]\!] = z \quad \text{and} \quad [\![e]\!] = e$$

For binomial $a_it^i \oplus a_jt^j$ and $i \geqslant j$, we have

$$[\![a_it^i \oplus a_jt^j]\!] = \begin{cases} a_it^i \oplus a_j t^j, & i > j, \ a_i \prec a_j \\ a_jt^j & , i > j, \ a_i \succ a_j \\ (a_i \oplus a_j) t^j, & i = j \end{cases}$$

Then after total check, we can prove that

$$[\![[\![a_it^i \oplus a_jt^j]\!] \oplus a_kt^k]\!] = [\![a_it^i \oplus [\![a_jt^j \oplus a_kt^k]\!]]\!]$$

Thus we can write the result to be $[\![a_it^i \oplus a_jt^j \oplus a_kt^k]\!]$.

To a modi -polynomial of degree n

$$P(t) = a_0t^n \oplus a_1t^{n-i_1} \oplus a_2 t^{n-i_2} \oplus \ldots \oplus a_pt^{n-i_p}$$

where $0 < i_1 < i_2 < \ldots < i_p \leq n$, $a_0 \neq z$, if the sequence of the coefficients $a_0, a_1, \ldots a_p$ is strictly monotonic to good, i.e., $a_{i-1} \nsucc a_i$ $(i = 1, 2, \ldots p)$ then we have $[\![P(t)]\!] = a_pt^{n-i_p}$. If the sequence of the coefficients is not strictly monotonic to good, we can partition the sequence into several subsequences and each of them is strictly monotonic to good.

This partition may not be necessarily unique and some subsequences may contain only one term. To each subsequence, we retain the term with the optimum coefficient. Thus we obtain a new sequence of coefficients called the first badinized sequence and the related modi-polynomial. If the sequece is not strictly monotonic to bad, we can do the same process as above and so on. After doing finitely many operations, we will at last obtain a strictly monotonic to bad sequence and a related modi-polynomial -the essential modi -polynomial. For example, we have a modi-polynomial on the semi-field $\{\bar{R}, \wedge, +\}$:

$$S(t) = 3t^{10} \oplus 2t^9 \oplus t^8 \oplus 7t^7 \oplus 5t^6 \oplus 4t^5 \oplus 6t^4 \oplus 3t^3 \oplus 4t^2 \oplus 8t$$

we have

i	0	1	2	3	4	5	6	7	8	9
a_i	3	2	1	7	5	4	6	3	4	8
first badinization			1			4		3	4	8
second badinization			1					3	4	8

therefore $[\![R(t)]\!] = t^8 \oplus 3t^3 \oplus 4t^2 \oplus 8t$.

Evidently, to a given modi-polynomial, the coefficient sequence and its first, second, ... badinized sequences and related modi-polynomial will correspond to a fixed essential modi-polynomial.

The badinization process makes each modi-polynomial of $\mathcal{P}(t)$ correspond to an essential modi-polynomial and each essential modi-polynomial corresponds to a subset of modi-polynomials in $\mathcal{P}(t)$.

$\mathcal{P}(t)$ will be devided into several disjoint subsets, and subset corresponds to an essential modi-polynomial. All those modi-polynomials form a set $\vec{\mathcal{P}}(t)$. z is a special essential modi-polynomial to which there corresponds only one modi-polynomial z itself in $\mathcal{P}(t)$, and e is another special essential modi-polynomial to which there correspond all modi-polynomials with the constant term (i. e., the coefficient of t^0)e.

Now, we can define the modi-addition $\oplus_0$ and modi-multiplication $\otimes_0$ in the set $\vec{\mathcal{P}}(t)$. If $F(t)$ and $G(t)$ belong to $\vec{\mathcal{P}}(t)$, evidently, we have

$$[\![F(t)]\!] = F(t) \quad \text{and} \quad [\![G(t)]\!] = G(t)$$

and

$$F(t) \oplus_0 G(t) = [\![F(t)]\!] \oplus_0 [\![G(t)]\!] \equiv [\![F(t) \oplus G(t)]\!]$$

We define

$$F(t) \otimes_0 G(t) \equiv [\![F(t) \otimes G(t)]\!]$$

We can prove without difficulty that the $\{\vec{\mathcal{P}}(t), \oplus_0, \otimes_0\}$ is a generalized optimizing semi-field.

A traveller may take quite different ways by different trafic tool from city V_i to city V_j,

For example, by ship along a river, it will take him n days to complete the travel and will cost him a_0 dollars; by express train it will take him $n-i_1$ days and will cost him a_1 dollars and so on.

For simplicity, we make a stipulation that the time consumed is denoted by a positive integer. Of course, if there are several ways to complete the travelling with the same cost, then he must like to take that way with shorter time. Thus, if $0 < i_1 < i_2 < \dots < i_p \leqslant n$ then $a_0 \prec a_1 \prec \dots \prec a_p$. Then we may denote the matter happened on the way from V_i to V_j as an essential modi-polynomial, called the cost polynomial from V_i to V_j:

$$\vec{A}(t) = a_0 t^n \oplus a_1 t^{n-i_1} \oplus a_2 t^{n-i_2} \oplus \dots \oplus a_p t^{n-i_p}$$

Suppose we have another cost polynomial for the other way from V_i to V_j : $\vec{B}(t) = b_0 t^m \oplus b_1 t^{m-j_1} \oplus b_2 t^{m-j_2} \oplus \dots \oplus b_s t^{n-j_s}$

If the traveller likes to spend (n-i) days to complete the travel ,then the least cost will be found in $\vec{A}(t) \oplus \vec{B}(t)$,the coefficient of t^k which is the nearest non-zero one before the term.

If from V_i to V_j ,we have $\vec{A}(t)$ and from V_j to V_k, we have $\vec{B}(t)$, then from V_i to V_k via V_j ,the cost and time will be found in $\vec{A}(t) \oplus \vec{B}(t)$.

On a multistage directed graph ,each link corresponds to a jar -metric taken from $\{S, \oplus, \otimes\} - \{\vec{P}(t), \oplus, \otimes\}$ .Then we can find the jar-metric from the initial vertex to the final vertex by our jar-metric principle from which we can find out the ways to complete the path in prescribed time with least cost or prescribed cost with least time.

We call it the problem of finding optimum paths of all first N-order of second kind.

5. Semi-field R and Generalized optimizing operator

Suppose that there are l semi-fields

$$\{S_i, \oplus_i, \otimes_i\} \qquad (i=1,2, \dots l)$$

and $R^l = S_1 \times S_2 \times \dots S_l$ is the l dimensional direct product of S_i . Particulary,if $l=1$,we put $R^1 = S_1$. We call

$$a = [a_1, a_2, \dots a_l] \qquad (a_i \in S_i)$$

to be a vector or an element of R^l and a_i to be the i -th component of the vector a . Of course,we may define operations between such vectors :

$$a \oplus b = [a_1, a_2, \dots a_l] \oplus [b_1, b_2, \dots b_l]$$
$$= [a_1 \oplus_1 b_1, a_2 \oplus_2 b_2, \dots a_l \oplus_l b_l]$$
$$a \otimes b = [a_1, a_2, \dots a_l] \otimes [b_1, b_2, \dots b_l]$$
$$= [a_1 \otimes_1 b_1, a_2 \otimes_2 b_2, \dots a_l \otimes_l b_l]$$

It is easy to verify that R^l is a semi-field with zero element $Z = [z_1, z_2, \dots z_l]$ where z_i is zero element of S_i. If in each semi - field S ,there exists identity element e_i,then R^l has identity element $E = [e_1, e_2, \dots e_l]$.

Generally speaking ,even if all semi-field are strongly optimizing , the modi-sum of two vectors a and b in R^l ($l \neq 1$) is not necessarily equal to a or b. If $a \oplus b \neq a$ or b ,we say a and b to be incomparable, and we denote that by $a \nsim b$.

From now ,we shall confine ourselves to study our problems on strongly optimizing semi-fields S_i ($i = 1, 2, \ldots, l$).

Evidently , R^l is a generalized optimizing semi-field ,and in R^l,we have

i) $a \precsim a$;

ii) if $a \precsim b$ and $b \precsim a$,then $a = b$;

iii) if $a \precsim b$ and $b \precsim c$, then $a \precsim c$;

if $a \prec b$ and $b \precsim c$,then $a \prec c$.

In R^l , given a finite set Y of vectors:

$$Y = \{ y^{(i)} \mid i = 1,2, \ldots h \} \tag{1}$$

The family formed by all such sets like Y is denoted by SET.

In this and next section ,we put that small latin letter ,such as a, b , x , y , represent vector in R^l ,and that small latin letter with subscript,such as a_i , a_j ,always represents some component of the vector denoted by the same letter ,such as a, b , and that capital letter such as A , B , always represent element (the finite sets of vectors) of SET.

Now we suppose that Y is a set in R^l,say ,it is (1).

Definition 6. If q in Y and there exist no such vector x in Y that $x \prec q$ holds ,we call q an extreme vector (point) in Y or non-worse element in Y . We denote Y^* the set formed by all such non-repeated extreme vectors in Y and call it the extreme set of Y ,or the non -worse set of Y.

If $Y^* = Y$,we call Y an elementary of SET.

If $l = 1$, $S_1 = R^1 = \{S , \oplus , \otimes\}$,the extreme vector in $Y (\neq \emptyset)$ is its optimum element (1 dimensional vector) and Y contains only one vector. In this case,the process from Y to Y^* is an optimization operation.

If $l \neq 1$,we can still consider this process which makes Y correspond to Y^* being an optimization operation.Thus the symbol is a kind of generalized optimizing operator.

Theorem 5. Suppose that Y is in SET ,

i) if $Y \neq \emptyset$, then $Y^* \neq \emptyset$;

ii) if $|Y| = 1$,then $Y^* = Y$;

iii) if each two elements of Y are incomparable ,then $Y^* = Y$;

iv) Y^* is unique;

v) $(Y^*)^* = Y^*$.

Proof. They come from the definition directly.

In SET ,two different elements may have the same non-worse element. In SET ,each element Y corresponds to an unique elementary element Y^*. All of those Y in SET which the same elementary element from a category . Then the set SET can be partitioned into several categories according to elementary element.

Theorem 6. To each Y in SET and any vector $\bar{y}$ in R^1, the necessary and sufficient condition for existing such a w in Y that $w \prec \bar{y}$ holds, is that there exists such a u in Y^* that $u \prec \bar{y}$ holds.

Proof. Necessarity: If there exists such a w in Y that $w \prec \bar{y}$ holds, then for w, Y^* contains a vector u which is either w itself or a vector, say u, better than w : $u \preceq w$. Therefore we have $u \prec \bar{y}$, i.e., there exists such a vector u in Y^* that $u \prec \bar{y}$ holds.

Sufficiency: It is evident, if we notice that $Y^* \subseteq Y$.

Corollary. For each Y in SET and any $\bar{y}$ in R^1, the necessary and sufficient condition for existing no element u in Y^* such that $u \preceq \bar{y}$, is that there exists no such element w in Y that $w \preceq \bar{y}$ holds.

Theorem 7. (Wu Cangpu)[1] For Y in SET and $\bar{y}$ in R^1, if

$$Y_{\bar{y}} = \{ y \mid y \preceq \bar{y} \quad , \quad y \in Y \} \quad , \tag{2}$$

then we have

$$(Y_{\bar{y}})^* = (Y^*)_{\bar{y}} \tag{3}$$

Proof. We discuss the situations first where some sets happen to be empty.

If $Y = \emptyset$, (3) is evidently true.

If $Y \neq \emptyset$ and $Y_{\bar{y}} = \emptyset$, then $(Y_{\bar{y}}^*) = \emptyset$. On the other hand, we have $Y^* \subseteq Y$, so $(Y^*)_{\bar{y}} \subseteq Y_{\bar{y}}$. Then $(Y^*)_{\bar{y}} = \emptyset$, and (3) is true.

If $Y^* \neq \emptyset$ and $(Y^*)_{\bar{y}} = \emptyset$, then there exists no element w in Y^* such that $w \preceq \bar{y}$. By the corollary of theorem 2, we have $Y_{\bar{y}} = \emptyset$. Thus $(Y_{\bar{y}})^* = \emptyset$, so (3) holds also.

In short, if any of Y, $Y_{\bar{y}}$ and $(Y^*)_{\bar{y}}$ is empty, (3) is always true
Now we suppose that the sets on both sides of (3) be nonempty.
We prove $(Y_{\bar{y}})^* \subseteq (Y^*)_{\bar{y}}$ first. Let q be in $(Y_{\bar{y}})^*$. Then we have

$$q \text{ in } Y_{\bar{y}} \tag{4}$$

and there exists no element w in $Y_{\bar{y}}$ such that $w \prec q$. We assert that q is in Y^*. If not, by definition, there exists v in Y^*, such that $v \prec q$. and by (2) we have $v \prec \bar{y}$. But it is impossible to have v in $Y_{\bar{y}}$ and q in $(Y_{\bar{y}})^*$ simultaneously. Thus q is in Y^*.
Noticing (4), we have q in $(Y^*)_{\bar{y}}$, then

$$(Y_{\bar{y}})^* \subseteq (Y^*)_{\bar{y}}$$

Next we prove $(Y^*)_{\bar{y}} \subseteq (Y_{\bar{y}})^*$. Let u be in $(Y^*)_{\bar{y}}$. We have $u \preceq \bar{y}$ and u in Y^*. Then there exists no such element t in Y that $t \prec u$. Since $Y_{\bar{y}} \subset Y$, therefore there exists no t in Y^* that $t \prec u$ holds. Hence we have

$$(Y_{\bar{y}})^* \supseteq (Y^*)_{\bar{y}}$$

Thus we have (3).

Theorem 8 (Wu Cangpu)[7].For each pair Y_1 and Y_2 in SET, we have

$$(Y_1 \cup Y_2)^* = (Y_1^* \cup Y_2^*)^* \qquad (5)$$

Proof. If any — of Y_1 and be Y_2 empty, by theorem 4 v), (5) is evidently true.

Now we suppose that Y_1 and Y_2 be nonempty.

We prove first $(Y_1 \cup Y_2)^* \subseteq (Y_1^* \cup Y_2^*)^*$ .Let $q \in (Y_1 \cup Y_2)^*$.

Then q is in $Y_1 \cup Y_2$ and there exists no such — element u in $Y_1 \cup Y_2$ that $u \prec q$ .Without any loss of generality, we might say q being in Y_1, thus there is no such u in Y_1 that $u \prec q$. Therefore $q \in Y_1^* \subseteq Y_1^* \cup Y_2^*$. Since $Y_1^* \cup Y_2^* \subseteq Y_1 \cup Y_2$,and there is no such u in $Y_1^* \cup Y_2^*$ that $u \prec q$.

Therefore q is in $(Y_1^* \cup Y_2^*)^*$. Thus we have

$$(Y_1 \cup Y_2)^* \subseteq (Y_1^* \cup Y_2^*)^* .$$

Next we will prove $(Y_1^* \cup Y_2^*)^* \subseteq (Y_1 \cup Y_2)^*$. But this is evident since $Y_1^* \cup Y_2^* \subseteq Y_1 \cup Y_2$. Thus we have (5)

Corollary. For $Y_i \in$ SET ($i = 1,2, \ldots m$) ,we have

$$\left(\bigcup_{i=1}^{m} Y_i\right)^* = \left(\bigcup_{i=1}^{m} Y_i^*\right)^*$$

Theorem 9. For $Y \in$ SET and $y \in R$,if we write

$$Y^{\otimes y} = \{a \otimes y \mid a \in Y\} \qquad (6)$$

we have

$$(Y^{\otimes y})^* = ((Y^*)^{\otimes y})^* \qquad (7)$$

Without any loss of generality, we my write

$$Y = \{y^{(i)} \mid i = 1, 2, \ldots h\}$$

and let $Y^* = \{y^{(i)} \mid i = 1, 2, \ldots t\}$,

then

$$(Y^*)^{\otimes y} = \{y^{(i)} \otimes y \mid i = 1, 2, \ldots t\} \qquad (8)$$

If $Y = Y^*$, (7) is certainly true. If $Y \neq Y^*$,we have $t < h$ and $Y \setminus Y^* = \{y^{(i)} \mid i = t+1, \ldots h\}$. To each vector $y^{(s)}$ ($t < s \le h$) ,there always exists such a vector $y^{(k)}$ in Y^*($1 \le k < t$) that $y^{(k)} \prec y^{(s)}$. $y^{(k)}$ can be zero element . Then ,by our assumption ,the given semi-fields S_i are strongly optimum therefore we have

$$y^{(k)} \otimes y \prec y^{(s)} \otimes y$$

so in $Y^{\otimes y}$, $y^{(s)} \otimes y$ ($t < s \le h$) will not be in $(Y^{\otimes y})^*$.

Therefore we have (7) .

6.Semi-field PARETO and Multi -objective jar -metric principle

In the last section ,we denote the family of all finite subset in R^1 by SET , To each $Y \in$ SET ,there corresponds an elementary element Y^*. We denote the family formed by all of those elementary elements by PARETO. Now let us define modi-addition on it. For A,B,C in PARETO ,

$$A \oplus B = \{A \cup B\}^*$$

The law of commutativity evidently holds true :

$$A \oplus B = B \oplus A$$

Since we have

$$\begin{aligned}(A \oplus B) \oplus C &= (A \cup B)^* \oplus C = ((A \cup B)^* \cup C)^* \\ &= (((A \cup B)^*)^* \cup C^*)^* \quad \text{(theorem 8)} \\ &= ((A \cup B)^* \cup C^*)^* \quad \text{(theorem 5 v)} \\ &= ((A \cup B) \cup C)^* \quad \text{(theorem 8)} \\ &= (A \cup B \cup C)^*\end{aligned}$$

and similarly ,we have

$$A \oplus (B \oplus C) = (A \cup B \cup C)^*$$

Therefore the law of associativity holds :

$$(A \oplus B) \oplus C = A \oplus (B \oplus C)$$

We define modi-multiplication on PARETO as follows : for

$$A = \{a^{(i)} \mid i = 1, 2, \ldots h\}$$

$$B = \{b^{(j)} \mid j = 1, 2, \ldots k\} , \quad C = \{c^{(s)} \mid s = 1 ,2, \ldots m\}$$

and define

$$A \otimes B = \{a^{(i)} \otimes b^{(j)} \mid i = 1, \ldots h; \ j = 1, \ldots k\}^*$$

For brevity ,we write

$$A \otimes B = \{\bigcup_{i,j=1}^{h,k} a^{(i)} \otimes b^{(j)}\}^*$$

For brevity ,we write

$$A \otimes B = A \otimes B$$

Since we have

$$\begin{aligned}(A \otimes B) \otimes C &= \{\bigcup_{s=1}^{m} (A \otimes B) \otimes c^{(s)}\}^* \\ &= \{\bigcup_{s=1}^{m} ((A \otimes B) \otimes c^{(s)})^*\}^* \quad \text{(theorem 8)}\end{aligned}$$

$$= \{\bigcup_{s=1}^{m} \{ (\{\bigcup_{i,j=1}^{h,k} a^{(i)} \otimes b^{(j)}\}^*) \otimes c^{(k)}\}^* \}^*$$

$$= \{\bigcup_{s=1}^{m} \{ (\bigcup_{i,j=1}^{h,k} a^{(i)} \otimes b^{(j)}) \otimes c^{(k)} \}^* \}^*$$

$$= \{\bigcup_{s=1}^{m} \{\bigcup_{i,j=1}^{h,k} a^{(i)} \otimes b^{(j)} \otimes c^{(s)}\}^* \}^*$$

$$= \{ \bigcup_{s=1}^{m} (\bigcup_{i,j=1}^{h,k} a^{(i)} \otimes b^{(j)} \otimes c^{(s)}) \}^*$$

$$= \{ \bigcup_{i,j,s=1}^{h,k,m} a^{(i)} \otimes b^{(j)} \otimes c^{(s)} \}^*$$

We can check easily that $A \otimes (B \otimes C)$ has the same result . Therefore the law of associativity holds :

$$(A \otimes B) \otimes C = A \otimes (B \otimes C)$$

The law of distrutivity holds true also :

$$(A \oplus B) \otimes C = (A \cup B)^* \otimes C$$

$$= \{ \bigcup_{s=1}^{m} ((A \cup B)^*) \otimes c^{(s)} \}^*$$

$$= \{ \bigcup_{s=1}^{m} (((A \cup B)^* \otimes c^{(s)})^* \}^*$$

$$= \{ \bigcup_{s=1}^{m} ((A \cup B) \otimes c^{(s)})^* \}^*$$

$$= \{ \bigcup_{s=1}^{m} (A \cup B) \otimes c^{(s)} \}^*$$

$$= \{ \bigcup_{s=1}^{m} A \otimes c^{(s)} \cup \bigcup_{s=1}^{m} B \otimes c^{(s)}) \}^*$$

$$= \{ (\bigcup_{s=1}^{m} A \otimes c^{(s)})^* \cup (\bigcup_{s=1}^{m} B \otimes c^{(s)})^* \}^*$$

$$= \{ (A \otimes C) \cup (B \otimes C) \}^*$$

$$= A \otimes C \oplus B \otimes C$$

and finally , $E = \{e\} = \{[e_1, e_2 , \dots e_l]\}$ and $Z = \{z\} = \{[z_1, z_2, \dots z_l]\}$ are identity and zero element in PARETO.

What's more, we have

$$(A \oplus B) \oplus A = A \oplus B$$
$$(A \oplus B) \oplus B = A \oplus B$$

Thus we have

Theorem 10 .For A, B, C being in PARETO ,we define

$$A \oplus B = \{A \cup B\}^*$$

$$A \otimes B = \{ \bigcup_{i,j=1}^{h,k} a^{(i)} \otimes b^{(j)} \}^*$$

then PARETO is generalized optimizing semi-field with identity.

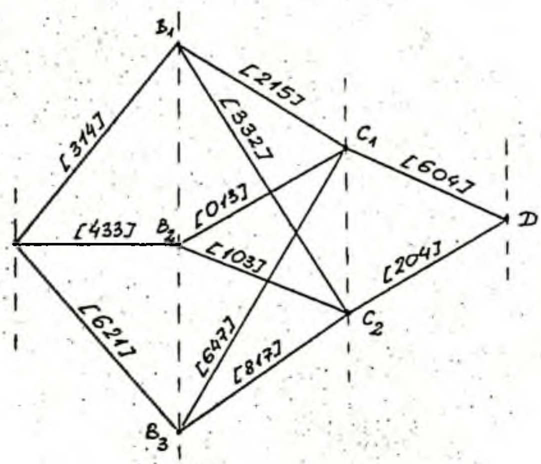
And we also have

Multi-objective jar-metric principle. On a multistage directed graph, to each link ,there corresponds a jar -metric an element taken from the semi-field PARETO.

Then the jar-metric from a start vertex to an end vertex in the graph will be the modi-sum of all modi-product of those jar-metrics from the start vertex to all vertices on a certain middle state and these from the vertices mentioned to the end vertex. And this modi-sum is independent of all those states before the start vertex and after the end vertex.

If Y is in SET and q is in Y , then there exists no such a vector w that $w \prec q$. Now ,we say q is a Pareto solution of Y . Then ,on a multistage directed graph,the modi-sum of jar-metrics of all the path from initial vertex to final vertex is a set of all Pareto solutions from initial vertex to final vertex of the graph.And we can calculate them by the multi-objective jar-metric principle.

Example 5. The multistage directed graph is the following one, Suppose that $S_1 = S_2 = S_3 = \{ \bar{R} , \wedge , + \}$. The jar-metric of each link be denoted on the graph. Find the set of Pareto solution from A to D.



Solution. We can consider the jar-metric on each link being taken from the semi-field PARETO of 3 dimension. Then we have

$$\text{STAGE}(A, B) = \begin{array}{c|ccc} \text{from}\backslash\text{to} & B_1 & B_2 & B_3 \\ A & [[3\ 1\ 4] & [4\ 3\ 3] & [6\ 2\ 1]] \end{array}$$

$$\text{STAGE}(B, C) = \begin{array}{c|cc} \text{from}\backslash\text{to} & C_1 & C_2 \\ B_1 & [2\ 1\ 5] & [3\ 3\ 2] \\ B_2 & [0\ 1\ 3] & [1\ 0\ 3] \\ B_3 & [6\ 4\ 7] & [8\ 1\ 7] \end{array}$$

$$\text{STAGE}(C, D) = \begin{array}{c|c} \text{from}\backslash\text{to} & D \\ C_1 & [6\ 0\ 4] \\ C_2 & [2\ 0\ 4] \end{array}$$

Then we do

$$\text{STAGE}(A, C) = \text{STAGE}(A, B) \otimes \text{STAGE}(B, C)$$

Since the jar-metric from A to C_1 equals

$$\begin{aligned}
&[3\ 1\ 4] \otimes [2\ 1\ 5] \oplus [4\ 3\ 3] \otimes [0\ 1\ 3] \\
&\quad \oplus [6\ 2\ 1] \otimes [6\ 4\ 7] \\
&= [5\ 2\ 9] \oplus [4\ 4\ 6] \oplus [12\ 6\ 8] \\
&= \{[5\ 2\ 9]/B_1, [4\ 4\ 6]/B_2\}
\end{aligned}$$

and that from A to C_2 is

$$\begin{aligned}
&[3\ 1\ 4] \otimes [3\ 3\ 2] \oplus [4\ 3\ 3] \otimes [1\ 0\ 3] \\
&\quad \oplus [6\ 2\ 1] \otimes [8\ 1\ 7] \\
&= [6\ 4\ 6] \oplus [5\ 3\ 6] \oplus [14\ 3\ 8] \\
&= \{[5\ 3\ 6]/B_2\}
\end{aligned}$$

$$\text{so STAGE}(A, C) = \begin{array}{c|cc} \text{from}\backslash\text{to} & C_1 & C_2 \\ A & [\{[5\ 2\ 9]/B_1, [4\ 4\ 6]/B_2\} & \{[5\ 3\ 6]/B_2\}] \end{array}$$

$$\text{STAGE}(A, D) = \text{STAGE}(A, C) \otimes \text{STAGE}(C, D)$$

$$= \begin{array}{c|c} \text{from}\backslash\text{to} & \\ A & [[5\ 2\ 9], [4\ 4\ 6]\ [5\ 3\ 6]] \end{array} \otimes \begin{array}{c|c} \text{from}\backslash\text{to} & D \\ C_1 & [6\ 0\ 4] \\ C_2 & [2\ 0\ 4] \end{array}$$

$$= \begin{array}{c|c} \text{from}\backslash\text{to} & D \\ A & [\{[11\ 2\ 13]/C_1, [10\ 4\ 10]/C_1\ [7\ 3\ 10]/C_2\}^*] \end{array}$$

$$= \begin{array}{c|c} \text{from}\backslash\text{to} & D \\ A & [\{[11\ 2\ 13]/C_1, [7\ 3\ 10]/C_2\}] \end{array}$$

Therefore the Pareto solutions are

$$y^{(1)} = [11\ 2\ 13] \qquad y^{(2)} = [7\ 3\ 10]$$

and the related paths are AB_1C_1D and AB_2C_2D.

References

[1] Dreyfus, S.E. and Law ,A.M. , The Art and Theory of Dynamic Programming, Academik Press,1977.

[2] Qin Yuyuan ,On Jar -metric Principle,Science Exploration,(I) 1(1981) 59-76;(II) 3(1981) 101-108; (III) 1(1984) 91-102 ; (IV) 1 (1985) 53-64 (Chinese).

[3] Qin Yuyuan ,Modi-matrix and Shortest Path Problem,Jou.of Wuhan Iron and Steel University, 2(1979) 1-21 (Chinese).

[4] Wu Xuemou ,Pansystems Metodology : Concepts,Theorems and Applications (IV) , Science Exploratin, 1 (1984) 107-116.

[5] Qin Guoguang, Some Applications of Pangraph and Pansystem Operation Projection Principle to Markov Process and Optimization Problems ,Science Exploratin , 3(1981) 55 -62 (Chinese).

[6] Shier.D.R., Iterative Methods for Determining the k Shortest Path in a Network,Networks , 6 (1976) 205-230.

[7] Wu Canpu ,On Multiobjective Dynamic Programming ,Scientia Sinica,4(1980) 388- 395 (Chinese).

[8] Qin Yuyuan ,Algoritmus on Optimal Path Problems,Science Exploration, 4(1981) 61-94 (Chinese).

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ПРИНЦИПЫ МЕТРИКИ ТИПА ДЖАР. ПРИМЕНЕНИЕ К ОПРЕДЕЛЕНИЮ ОПТИМАЛЬНОГО ПУТИ В НАПРАВЛЕННОМ ГРАФЕ

Р е з ю м е

В работе предотавлен новый оригинальный метод определения оптимального пути в направленном графе , который может быть использован для оптимизации многоэтапных дискретных процессов . Метод этот основан на сформулированном в данной статье " принципе типа джар". Слово джар на китайском языке обозначает сосуд и было в употреблении более 2000 лет назад. В данной работе слово это употребляется в качестве абстракционной меры , связанной с отдельными рёбрами направленного графа. Для определения метрики типа джар автором вводится ряд понятий из абстракционной алгебры таких как: семи-поля, моди-действия а также моди-матрицы.

Принцип метрики типа джар используется в данной работе для определения оптимальной дороги между двумя избранными вершинами направленного графа а также для определения оптимальной дороги для всего графа. Детально оговорены взаимозависимости между известным из литературы принципом Беллмана а введённым автором принципом метрики типа джар. Указаны основные различия между ними а также случай, когда оба принципа равнозначны.

Необходимо подчеркнуть, что представленный в работе метод , основанный на принципе метрики типа джар, даёт возможность обобщить и унифицировать подход к разным проблемам оптимизации многоэтапных дискретных процессов. В работе приводятся многочисленные примеры хорошо иллюстрирующие представленные теоремы и определения. Примеры эти способствуют правильному пониманию трудных и абстрактных понятий представляемых в работе.

O ZASADACH METRYKI "JAR" - ZUNIFIKOWANE PODEJŚCIE DO ROZWIĄZYWANIA PROBLEMÓW POSZUKIWANIA OPTYMALNEJ ŚCIEŻKI NA WIELOETAPOWYCH GRAFACH SKIEROWANYCH

S t r e s z c z e n i e

W pracy przedstawiono nową, oryginalną metodę wyznaczania optymalnej drogi w grafie skierowanym, która może być wykorzystywana do optymalizacji wieloetapowych procesów dyskretnych. Metoda ta oparta jest na sformułowanej w artykule "zasadzie metryki typu jar". Termin "jar" oznacza w języku chińskim pojemnik stosowany powszechnie w Chinach około 2000 lat temu. W pracy termin ten stosowany jest jako abstrakcyjna miara związana z poszczególnymi krawędziami grafu skierowanego. W celu zdefiniowania metryki typu jar, wprowadzono wiele pojęć z dziedziny algebry abstrakcyjnej takich jak: semi-pola, modi-działania oraz modi-macierze.

Zasada metryki typu jar wykorzystywana jest w pracy do wyznaczania optymalnej drogi pomiędzy dwoma wybranymi wierzchołkami grafu skierowanego, jak również do wyznaczania optymalnej drogi w całym grafie. Szeroko dyskutowane są wzajemne relacje pomiędzy znaną z literatury zasadą optymalności Bellmana a wprowadzoną przez Autora zasadą metryki typu jar. Przedstawiono zasadnicze różnice pomiędzy nimi a także przypadki, w których obie te zasady są sobie równoważne.