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STOCHASTIC CONTROLLABILITY OF A CLASS OF DISCRETE TIME SYSTEMS WITH MULTIPLICATIVE NOISE*)

Summary. The stochastic ε-controllability of discrete time systems is considered. Using a stochastic Liapunov-like approach, sufficient conditions for stochastic ε-controllability of discrete time systems are formulated. An illustrative example is given.

1. Introduction

Using a stochastic Liapunov-like approach, the problem of stochastic controllability for continuous- and discrete-stochastic dynamical systems has been recently studied (see [1], [2], [3], [4]). In these papers the main condition that the conditional mean value of Liapunov function along trajectory obtained by the solution of the system is not increasing, was used. Because of this condition, the stochastic controllability in these papers, some time, may be regarded as a kind of stochastic stability. This paper is devoted to a detailed study of the stochastic relative ε-controllability in probability φ of discrete time dynamical systems perturbed by multiplicative noise, with conditions that are more general than the ones, obtained previously for discrete time systems.

2. System description and basic definitions

We shall consider the folowing discrete systems

$$x_{k+1} = (A_k + V_k)x_k + (B_k + W_k)U_k + D_k\eta_k \tag{2.1}$$

the coefficient A_k and the stochastic disturbance V_k are n×n-matrices, B_k and the stochastic disturbance W_k are n×m matrices D_k is n×p matrix, η_k is random p-vector, the control imput U_k is an m vector for $k \in \{0,1,2,\dots,N\} = [0,N]$, $x_k \in R^n$ is a state of the system.

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We impose the following assumptions on the system (2.1)

i) The elements of V_k may be correlated random variables

ii) The elements of W_k may be correlated random variables.

iii) V_k is uncorrelated or correlated with W_k for each k

iv) V_k, W_k, and η_k are "white matrix sequences" and "white vector sequences", respectively.

v) The random variables of V_k have distribution F_k and those of W_k have distribution G_k.

vi) η_k is uncorrelated with V_k and W_k for each k.

vii) $E\eta_k'\eta_j = I\delta_{ij}$, where I an identity matrix. $\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$

viii) The initial state and the processes are uncorrelated at any time.

ix) All random variables have zero means.

In the material which follows, condition ix) is in no way restrictive but is used for mathematical convinience.

The following definitions will be used in sequence, [1], [2], [3], [4].

Definition 2.1. An initial state $x(o) = x_o \in R^n$ of the system (2.1) is said to be stochastically ε-controllable in probability φ in square norm sense, with respect to a specified target domain with the norm $\sqrt{\varepsilon}$, in the time interval $[0,N]$, if there exists a control $U(k,x)$ such that

$$P\left\{\|x(N)\|^2 \geqslant \varepsilon \mid x(o) = x_o\right\} \leqslant 1 - \varphi \tag{1}$$

where $0 < \varphi < 1$, and the norm $\|\cdot\|$ is expressed as $\sqrt{x'x}$ and symbol $'$ denotes the transpose of a vector or matrix.

Definition 2.2. The system (2.1) is said to be completely ε-controllable in probability φ in the square norm sense with respect to a specified target domain with the norm $\sqrt{\varepsilon}$, in the time interval $[0,N]$. If the inequality (1) holds for every initial state $x_o \in R^n$.

Let us denote: $\Delta_k V(k.x_k) = E\left[V(k+1,x_{k+1} \mid x_k) - V(k,x)\right]$ where $V(\cdot,\cdot)$ is a function form $[0,N] \times R^n$ into R, x_k is a solution of the system (2.1), the conditional mean value is denoted by $E\left[\cdot \mid x_k\right]$.

3. Main results

In this section we present sufficient conditions for stochastic controllability of the system (2.1). Let us start with the main theorem.

Theorem 3.1. The initial state x_o of the system (2.1) is stochastically ε-controllable in probability φ in the square norm sense with respect to the terminal target domain with norm $\sqrt{\varepsilon}$ in the time interval $[0,N]$, if the following conditions are satisfied:

i) In the time interval $[0,N]$, a scalar nonegative function $V(k,x)$ is defined with respect to k and x.

ii) $V(N,x_N) \geq \frac{1}{\mathcal{L}} x'_N x_N$

where $\mathcal{L}$ is a positive number such that $\mathcal{L} << \varepsilon$.

iii) The given initial state x_o satisfies $V(0,x_o) \leq (1-\varphi) \frac{\varepsilon}{\mathcal{L}}$

iv) A control $U(k.x)$ exists such that along trajectory obtained by the solution of the equation (2.1) in $[0,N]$, the following inequalities hold

$$\Delta_k V(k,x_k) \leq \varphi(k) \qquad \text{(a.s)} \quad \text{for} \quad k[0,N-1]$$

where $\varphi(k)$ are scalar functions such that

$$\sum_{i=0}^{N-1} \varphi(i) \leq 0 \qquad \text{(a.s)}$$

Proof. Let us write $V(N,x_N)$ in the form

$$V(N,x_N) = V(0,x_o) + \sum_{k=0}^{N-1} (V(k+1,x(k+1)) - V(k,x_k)) \tag{2}$$

Because of the property iv) and from (2) we have

$$E(V(N,x_N)|x(o) = x_o) \leq V(0,x_o) \tag{3}$$

On the other side

$$\frac{\varepsilon}{\mathcal{L}} P(\|x(N)\|^2 \geq \varepsilon | x(o) = x) \leq E(V(N,x(N))|x(o) = x_o) \tag{4}$$

From (3), (4) and ii) we obtained

$$P(\|x(N)\|^2 \geq \varepsilon | x(o) = x_o) \leq (1-\varphi)$$

Hence the theorem is prowed.

Our next result deals specifically with the class of systems defined in equation (2.1).

Theorem 3.2. An initial state $x(o)$ of system (2.1) is stochastically ε-controllable in probability φ, in the square norm sense, with respect to a terminal state in the time interval $[0,N]$ if the following conditions are satisfied.

a) There exists a sequence of matrices P_k which are bounded, symmetric, positive definite in $0,N$ and which satisfy the following recursive difference equation

$$P_k = (A_k - B_kL_k)' P_{k+1}(A_k - B_kL_k) + E(V_k - W_kL_k)' P_{k+1}(V_k - W_kL_k) +$$

$$+ 2\,E(V_k - W_kL_k)' P_{k+1}(A_k - B_kL_k) + L_kRL_k + Q \tag{5}$$

with terminal condition $P_N = I/\mathcal{L}$ where R, Q are symmetric $n \times n$-dimensional, non-negative definite matrices.

b) $E((B_k + W_k)' P_{k+1}(B_k + W_k) + R)$ is positive definite for each $k \in [0,N-1]$

c) $x_o'P_ox_o + \sum_{k=o}^{N-1} \operatorname{tr} D_k'P_{k+1}D_k \leqslant (1-\varphi)\frac{\varepsilon}{\mathcal{L}}$

where the symbol t_r denotes trace of matrix.

Proof. Let a scalar function $V(k,x)$ be given by the following formula

$$V(o,x_o) = x_o'P_ox_o + h_o$$

$$V(k,x) = x'P_kx \qquad k = 1,2,\dots,N \tag{6}$$

where $h_o = \sum_{k=o}^{N-1} \operatorname{tr} D_k'P_{k+1}D_k$.

define the following control function $U(k,x) = -L_kx$ where L_k is defined by

$$L_k = (E(R + (B_k + W_k)' P_{k+1}(B_k + W_k)))^{-1}E((B_k + W_k)' P_{k+1}(A_k + V_k))$$

In order to establish condition iv) of the theorem 3.1, using equation (2.1) we can state

$$\Delta_kV(k,x_k) = x_k'((A_k - B_kL_k)' P_{k+1}(A_k - B_kL_k) +$$

$$+ E(V_k - W_kL_k)' P_{k+1}(A_k - B_kL_k) +$$

$$+ 2\,E(V_k - W_kL_k)' P_{k+1}(V_k - W_kL_k) - P_k)x_k + \operatorname{tr} D_k'P_{k+1}D_k$$

for $k \geqslant 1$. It implies that

$$\Delta_kV(k,x_k) \leqslant \operatorname{tr} D_k'P_{k+1}D_k \quad \text{for} \quad k \geqslant 1. \tag{7}$$

$$\Delta_o V(0,x_o) = x_o'((A_o - B_oL_o)' P_1 (A_o - B_oL_o) +$$

$$+ E(V(0) - W(0)L_o)' P_1 (A_o - B_oL_o)$$

$$+ E(V(0) - W(o)L_o)' P_i (V(o) - W(o)L_o) - P_o)x_o - \sum_{k=1}^{N=1} \operatorname{tr} D_k' P_{k+1} D_k$$

for $k = 0$. Hence

$$\Delta_o V(0,x_o) \leqslant - \sum_{k=1}^{N-1} \operatorname{tr} D_k' P_{k+1} D_k \tag{8}$$

Now we put

$$\varphi(0) = - \sum_{k=1}^{N-1} \operatorname{tr} D_k' P_{k+1} D_k$$

$$\varphi(k) = \operatorname{tr} D_k' P_{k+1} D_k \qquad 1 \leqslant k \leqslant N-1$$

From (7), (8) we have

$$\sum_{k=0}^{N-1} \varphi(k) \leqslant 0$$

It is easy to see that the function $V(k,x)$ and $\varphi(k)$ satisfy the conditions of the theorem 3.1. Hence the theorem is proved.

Remark. The condition b) is satisfied if R is a positive matrix. The condition a) is satisfied for some specific systems see [5], [6], [7]).

Example. Let us consider as an illustrative example the one dimensional stationary discrete dynamical system with the following parameters

$$A_k = A, \quad B_k = 1, \quad V_k = W_k = 0, \quad D_k = D, \quad Q = 0, \quad R = 1$$

where A, D are constant. Hence the equation (2.1) is of the form

$$x_{k+1} = Ax_k + U_k + D\eta_k$$

Then the equation (5) has a simple form

$$P_N = 1/\mathcal{L}$$

$$P_k = A^2(P_{k+1}/(P_{k+1} + 1) = A^{2(N-k)}/(\sum_{i=0}^{N-k-1} A^{2i} + \mathcal{L})$$

We put

$$h_o = \sum_{k=0}^{N-1} D^2 P_{k+1} = D^2(\sum_{k=0}^{N-k} (A^{2(N-k)}/(\sum_{i=0}^{N-k-1} A^{2i} + \mathcal{L})))$$

Defining φ as follows

$$\varphi = P(\|x(N)\|^2 < \varepsilon \,|\, x(0) = x_o)$$

and using the assumption c) of the theorem 3.2, we can derive the following formula for φ_{max}

$$\varphi_{max} = 1 - \frac{\mathcal{L}}{\varepsilon}(x_o' P_o x_o + h_o).$$

4. Conclusion

In this paper, using a Liapunov-like approach, the problem of stochastic ε-controllability of discrete time system has been considered. Sufficient conditions for stochastic ε-controllability of linear discrete dynamical system have been obtained. The theorems given in this paper extend the results of the paper [4], because the conditions of the theorems and the class of discrete systems are more general than those in [4].

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СТОХАСТИЧЕСКАЯ УПРАВЛЯЕМОСТЬ НЕКОТОРОГО КЛАССА ДИСКРЕТНЫХ СИСТЕМ С МУЛЬТИПЛИКАТИВНЫМ ШУМОМ

Р е з ю м е

В статье рассматривается стохастическая ε-управляемость динамических дискретных систем. Применяя подход, основан на функции Ляпунова дана формулировка достаточности условий стохастической ε-управляемости для дискретных динамических систем. Представленная теория иллюстрируется примером.

STOCHASTYCZNA STEROWALNOŚĆ PEWNEJ KLASY DYSKRETNYCH UKŁADÓW Z MULTIPLIKATYWNYM SZUMEM

S t r e s z c z e n i e

W artykule rozpatrywana jest stochastyczna ε-sterowalność dyskretnych układów dynamicznych. Stosując podejście bazujące na funkcji Lapunowa, sformułowano warunki wystarczające stochastycznej ε-sterowalności dla dyskretnych układów dynamicznych. Podano również przykład ilustrujący przedstawioną teorię.