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ON A GENERALIZED MODEL OF A LOT-SIZE INVENTORY SYSTEM

Abstract. This paper focuses not only on optimal policy in ordinary sense but also on all optimal policies of the first k orders in the sense of Wagner and Whitin in a multi-stage lot-size inventory problem which is called a generalized model of lot-size inventory system.

Since the theory and the computational tools presented by the author are not so well-known to scholars in the Western countries thus we shall briefly discuss them first.

key words. semi-field optimizing semi-field strongly optimizing semi-field N-THOPT

yin and yang elements modi-matrix jar-metric principle basic and generalized model of lot-size inventory system

1. Semi-field and modi-matrix

Definition 1. A semi-field is a triple $\{S,\oplus,\otimes\}$ where S is a set with two operations: modi-addition $\oplus$ and modi-multiplication $\otimes$ satisfying laws of commutativity, associativity and distributivity and there exists a zero element z in S.

Definition 2. A semi-field with identity e is called to be optimizing if there is no infinity element in S, and for a and b in S, we have

$a \oplus b = a$ or b.

In an optimizing semi-field, if $a \oplus b = a$, we say that a is no worse than b, denoted by $a \preccurlyeq b$. If $a \oplus b = a$ and $\neq b$, we say a is better than b or b is worse than a, denoted by $a \prec b$. If $a \prec e$, a is called a yin element, if $a \succ e$, a is called a yang element*, and e itself, the neutral element.

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*) Yin and yang are the alphabetic writings of two Chinese terms 陰 and 陽, borrowed from Chinese traditional Yin-yang analysis in a ancient book written by Laozi about more than two thousend years ago. Generally speaking, these two terms mean the two sides of any antitheses, such as positive and negative, good and bad, man and woman, sun and moon, and all such things.

Evidently, an optimizing semi-field is a totally ordered set.

Theorem 1. In an optimizing slemi-field $\{S, \oplus, \otimes\}$, we have

a) if $a \leqslant b$ and $b \leqslant a$, then $a = b$;

b) if $a \leqslant b$ and $b \leqslant c$, then $a \leqslant c$;

c) if $a \leqslant b$ and $c \leqslant d$, then $a \oplus c \leqslant b \oplus d$;

d) if $a \leqslant b$ and $c \leqslant d$, then $a \otimes c \leqslant b \otimes d$;

e) if $a \leqslant b$, then for any non-negative integer k, $a^k \leqslant b^k$;

f) $e^k = e$;

g) if a is a yang(yin, neutral) element, then for any positive k, a^k is a yang (yin, neutral) element.

Proof. By direct computation.

Definition 3. A semi-field is called to be strongly optimizing if it is optimizing and if $a \oplus b \neq b$ and $c \neq z$, we always have $a \otimes c \oplus b \otimes c \neq b \otimes c$.

Definition 4. A semi-field is called to be generalized optimizing if, for a and b in S. we always have

$$(a \oplus b) \oplus a = a \oplus b,$$
$$(a \oplus b) \oplus b = a \oplus b.$$

here $a \oplus b$ will not be necessarily equal to a or b.

It is evident that a generalized optimizing semi-field is a partial ordered set.

Now, let us define the concept of modi-matrix.

Let $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ be two given sets and a_{ij} $(i = 1,2,\ldots,m;\ j = 1,2,\ldots,n)$ are elements taken from a semi-field $\{S, \oplus, \otimes\}$.

An array A with m rows and n columns

$$A = \begin{array}{c} \\ x_1 \\ x_2 \\ \vdots \\ \\ x_m \\ y_j \end{array} \begin{array}{c} \begin{array}{cccc} y_1 & y_2 & \cdots\cdots & y_n \end{array} \\ \begin{bmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots\cdots & a_{2n} \\ \cdots & \cdots & \cdots\cdots & \cdots \\ a_{m1} & a_{m2} & \cdots\cdots & a_{mn} \end{bmatrix} \\ \\ \end{array}$$

or

$$A = x_i \begin{bmatrix} a_{ij} \end{bmatrix} \quad \text{or} \quad A = \begin{bmatrix} a_{ij} \end{bmatrix}$$

is called a m x n modi-matrix over the semi-field where $x_1, x_2, \ldots, x_m$ is called row margin, X, the rowe st, $y_1, y_2, \ldots, y_n$, the column margin and Y, the column set.

This array determines such a correspondence that from row x_i to column y_j there corresponds an element a_{ij} or there is a weight from x_i to y_j. Two modi-matrices A and B over the same semi-field are equal if they have the same row margin, the same column margin and the same correspondence.

We define modi-addition $\oplus$ and modi-multiplication $\otimes$ between modi-matrices in the same way as for ordinary matrices. It is easy to prove that commutative and associative laws of modi-addition, associative law of modi-multiplication and distributive law among modi-matrices hold true.

In paper [1], we develope the concept of modi-matrix in more general form, but it will not be used in this paper.

2. Jar-metric principle

Suppose we have a digraph, tis vertices set has an (n+1)-partition

$$V^{(0)}, V^{(1)}, \dots, V^{(n)}$$

where

$$V^{(i)} = \left\{ v_t^{(i)} \middle| t = 1,2,\dots,t_i \right\}$$

$$\left| V^{(i)} \right| = t_i, \quad i = 0,1,2,\dots,n$$

and each arc on the digraph has the property that if its tail is belonging to $V^{(i-1)}$, then its head must be in $V^{(i)}$. Such a digraph is called n (multi)-stage digraph G.

$V^{(i)}$ is called the i-th state of G and $v_t^{(i)}$ is called its vertex. The 0-th state is called an initial state, and the n-th state $V^{(n)}$ is called a terminus state. The bipartite digraph induced by $V^{(i-1)}$ and $V^{(i)}$ is called the i-th stage.

The most of multistage digraphs in our discussion are those with $t_0 = t_n = 1$. In such cases, we write $V^{(0)} = \left\{v_0^{(0)}\right\}$ and $V^{(n)} = \left\{v_0^{(n)}\right\}$.

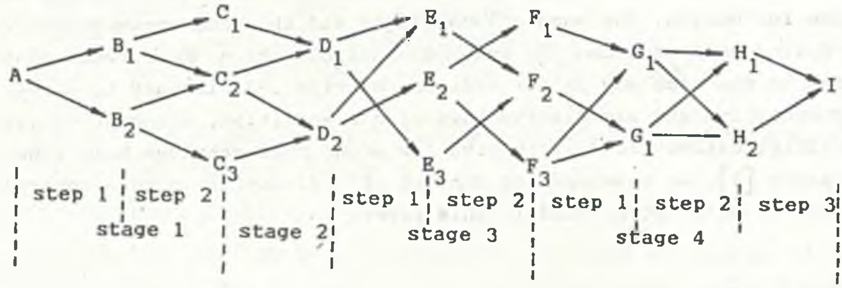
In a multistage digraph G, if $1 \leqslant h < k \leqslant n$, $1 \leqslant u \leqslant t_h$, $1 \leqslant v \leqslant t_K$, the subdigraph induced by all those vertices

$$\left\{v_u^{(h)}\right\} \cup V^{(h+1)} \cup \dots \cup V^{(k-1)} \cup \left\{v_v^{(k)}\right\}$$

is called an induced subdigraph from $v_u^{(h)}$ to $v_v^{(k)}$.

For practical reasons, we can sometimes make several stages form a new stage and call all those original stages to be steps. Such a digraph is called a complex multistage digraph.

Originally, the following figure is an 8-stage digraph, but we can say that this is a 4-stage(complex)digraph, the first and the third (new)stages are formed by two steps, the fourth(new)stage is formed by three steps, and the second state is formed by only one step.



Suppose we are given a semi-field. Now, with each link on G, we associate an element of the given semi-field. For explicity, the element associating with a link $v_\lambda^{(i-1)} v_\mu^{(i)}$ may be denoted by $J(v_\lambda^{(i-1)}, v_\mu^{(i)})$, called the jar-metric of the link*. The multistage digraph in which each link has a jar-metric is called the jared graph, denoted by G.

On the jared digraph, if there is no link from one vertex to the other, for example, from $v_s^{(h-1)}$ to $v_t^{(h)}$, we may imagine that it does have a link from $v_s^{(h-1)}$ to $v_t^{(h)}$, but its jar-metric $J(v_s^{(h-1)}, v_t^{(h)})$ equals zero element z of the semi-field.

The i-th stage can be represented by a $t_{i-1} \times t_i$ modi-matrix, denoted by STAGE$(v_\lambda^{(i-1)}, v_\mu^{(i)})$ or STAGE(i):

$$\mathrm{STAGE}(i) = \begin{array}{cc} & v_\mu^{(i)} \\ v_\lambda^{(i-1)} & \left[J(v_\lambda^{(i-1)}, v_\mu^{(i)}) \right] \end{array} \tag{1}$$

If $t_{i-1} = 1$, then (1) will be a row modi-vector, and if $t_i = 1$, then it is a column modivector. In the $t_{i-1} \times t_i$ modi-matrix, the $v_\lambda^{(i-1)}$ row is denoted by $(\mathrm{STAGE}(i))_\lambda$ and the $v_\mu^{(i)}$ column, by $(\mathrm{STAGE}(i))^\mu$.

We define the jar-metric from $v_\lambda^{(i-1)}$ to $v_\nu^{(i+1)}$ via $v_\mu^{(i)}$, denoted by $J(v_\lambda^{(i-1)}, v_\mu^{(i)}, v_\nu^{(i+1)})$, to be

$$J(v_\lambda^{(i-1)}, v_\mu^{(i)}, v_\nu^{(i+1)}) = J(v_\lambda^{(i-1)}, v_\mu^{(i)}) \otimes J(v_\mu^{(i)}, v_\nu^{(i+1)})$$

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*) Jar-metrix is a transliteration from the Chinese term 嘉量. The term originally means a kind of standard containers used in the Han-Dynasty about two thousend years ago. Emperors in history used to exhibit the jahr-metrics to signify the unification of his subjects. The reproducts are still exhibited in the Palace Museum in Beijing, China. We interpret it here as an abstract measure in our theory.

and the jar-metric from $v_\lambda^{(i-1)}$ to $v_\nu^{(i+1)}$ denoted by $J(v_\lambda^{(i-1)},v_\nu^{(i+1)})$ to be

$$J(v_\lambda^{(i-1)},v_\nu^{(i+1)}) = \sum_{\mu=1}^{t_i} J(v_\lambda^{(i-1)},v_\mu^{(i)},v_\nu^{(i+1)}) =$$

$$= \sum_{\mu=1}^{t_i} J(v_\lambda^{(i-1)},v_\mu^{(i)}) \otimes J(v_\mu^{(i)},v_\nu^{(i+1)})$$

$$= (\mathrm{STAGE}(i)) \otimes (\mathrm{STAGE}(i+1)) \tag{2}$$

Since we have

$$J(v^{(i-1)},v^{(i+1)},v^{(i+2)}) = \sum_{\nu=1}^{t_{i+1}} J(v_\lambda^{(i-1)},v_\nu^{(i+1)}) \otimes J(v_\nu^{(i+1)},v_\eta^{(i+2)})$$

$$= \sum_{\nu=1}^{t_{i+1}} \Big(\sum_{\mu=1}^{t_i} J(v_\lambda^{(i-1)},v_\mu^{(i)} \otimes J(v_\mu^{(i)},v_\nu^{(i+1)})\Big) \otimes J(v_\nu^{(i+1)},v_\eta^{(i+2)}) \tag{3}$$

$$J(v_\lambda^{(i-1)},v_\mu^{(i)},v_\eta^{(i+1)}) = \sum_{\mu=1}^{t_i} J(v_\lambda^{(i-1)},v_\mu^{(i)}) \otimes J(v_\mu^{(i)},v_\eta^{(i+2)}) =$$

$$= \sum_{\mu=1}^{t_i} J(v_\lambda^{(i-1)},v_\mu^{(i)}) \otimes \Big(\sum_{\nu=1}^{t_{i+1}} J(v_\mu^{(i)},v_\nu^{(i+1)}) \otimes J(v_\nu^{(i+1)},v_\eta^{(i+2)})\Big) \tag{4}$$

and by the operation laws on the semi-field, the right-hand sides of (3) and (4) are equal. We define the result to be the jar-metric from $v_\lambda^{(i-1)}$ to $v_\eta^{(i+2)}$:

$$J(v_\lambda^{(i-1)},v_\eta^{(i+2)}) = \sum_{\mu=1}^{t_i} J(v_\lambda^{(i-1)},v_\mu^{(i)},v_\eta^{(i+2)}) =$$

$$= \sum_{\nu=1}^{t_{i+1}} J(v_\lambda^{(i-1)},v_\nu^{(i+1)},v_\eta^{(i+2)}) =$$

$$= (\mathrm{STAGE}(i))_\lambda \otimes \mathrm{STAGE}(i+1) \otimes (\mathrm{STAGE}(i+2))^\eta \tag{5}$$

If $t_0 = t_n = 1$, the jar-metric from $v_0^{(0)}$ to $v_0^{(n)}$ can be defined in the similar way and be obtained by a following formula

$$J(v_0^{(0)},v_0^{(n)}) = \prod_{i=1}^{n} \text{STAGE}(i) \tag{6}$$

On the induced sub-digraph from $v_\alpha^{(i-1)}$ to $v_\beta^{(k)}$ $(1 \leqslant \alpha \leqslant t_{i-1},\ k \geqslant i+1,\ 1 \leqslant \beta \leqslant t_k)$, we have

$$J(v_\alpha^{(i-1)},v_\beta^{(k)}) = (\text{STAGE}(i))_\alpha \otimes \prod_{j=i+1}^{K-1} \text{STAGE}(j) \otimes (\text{STAGE}(k))^\beta \tag{7}$$

If we fix an integer s $(i-1 < s < k)$, by the associative law of modi-multiplication, we have

$$J(v_\lambda^{(i-1)} v_\tau^{(k)}) = \sum_{\xi=1}^{t_s} J(v_\lambda^{(i-1)},v_\xi^{(s)}) \otimes J(v_\xi^{(s)},v_\tau^{(k)}) \tag{8}$$

If $v_\lambda^{(i-1)}$ is called the start vertex of the induced digraph and $v_\tau^{(k)}$ the end vertex of it, we can formulate (8) in the following statement.

Jar-metric principle On a jared multistage digraph, the jar-metric from any start vertex to any end vertex equals the modi-sum of all modi-products of the jar-metrics from the start vertex to all those vertices of a middle state and that from those vertices of the middle state mentioned to the end vertex.

As special cases, the start vertex may be the initial vertex of the jared graph, the end vertex may be the final vertex, and the middle state may be just next to the state that the start vertex belongs to or just before the one that the end vertex belongs to.

Jar-metric principle is a very simple and intuitive one, it is just a kind of statement of the associative law of modi-multiplication of some modi-matrices.

If we develope the result on the right-hand side of (6), we have

$$J(v_0^{(0)},v_0^{(n)}) = \sum J(v_0^{(0)},v_{i_1}^{(1)}) \otimes J(v_{i_1}^{(1)},v_{i_2}^{(2)}) \otimes \cdots$$

$$\otimes J(v_{i_{h-1}}^{(h-1)},v_{i_h}^{(h)}) \otimes \cdots \otimes J(v_{i_{n-1}}^{(n-1)},v_0^{(n)}) \tag{9}$$

where under the modi-addition symbol $\sum$ we refer to all possible combinations $i_1,i_2,\ldots,i_{n-1}$ where $1 \leqslant i_j \leqslant t_j$ $(j = 1,2,\ldots,n-1)$. Geometrically, if we define the jar-metric of a path to be the modi-product of jar-metrics of all links on the path, the result on (9) equals the modi-sum of jar-metrics of all paths from initial vertex to final vertex. Of course, here, if there is no link from $v_r^{(k-1)}$ to $v_s^{(k)}$, that is to say, $J(v_r^{(k-1)},v_s^{(k)}) = z$, then the jar-metric of each path which passes through $v_r^{(k-1)}$ and $v_s^{(k)}$ will be a zero element.

If we want to find the shortest path and its length from the initial vertex to the final vertex on a multistage digraph on which with each link, there associated a real number, we must find the jar-metric from the initial vertex to the final one on the digraph over the strongly optimizing semi-field $\{\bar{R},\wedge,+\}$, where $\bar{R} = R \cup \{+\infty\}$. We can find it by the formula (9) which is equivalent to the formula (6). As to the shortest path itself, it will just be a by-product of the process of the computation which can be seen in the numerical example in the last section.

3. Semi-field N-THOPT and optimal path of the first N orders

Suppose we have a multistage digraph on which the jar-metric is taken from a strongly optimizing semi-field. There are several path from the initial vertex to the final one, denoted by $\mathcal{P}$. With each path p in $\mathcal{P}$ we associate a jar-metric, denoted by $\|p\|$. Then we have a subset of the strongly optimizing semi-field:

$$\{a \mid \|p\| = a,\ p \text{ in } \mathcal{P}\}$$

Since this subset is totally ordered, we can arrange all its elements in a monotonic-to-bad sequence

$$a_0 < a_1 < a_2 < \dots < a_t$$

The path with the jar-metric a_0 is called the optimal one of the zeroth order, that is, the optimal path in ordinary sense. The one with a_1 is called the optimal path of the first order, and the one with a_k is called the optimal path of the k-th order.

We have the following two theorems:

Optimal principle of the N-th order (Wu Xuemou) On a multistage digraph G over a strongly optimizing semi-field, if $L(0,n)$ is an optimal path of the N-th order and if the subpath $L(h,k)$ of $L(0,n)$ is the optimal path of the m-th order on the related induced subgraph, then we have

$$m \leq N$$

Theorem 2. (Qin Koukaung) On a multistage digraph G over a strongly optimizing semi-field, L(0,n), L(0,h) and L(h,n) are the optimal paths of N-th, m_1-th and m_2-th order respectively on the related (induced sub-) digraphs, then we have

$$m_1 + m_2 \leqslant N$$

<u>Corallary 1</u>. If $0 = h_0 < h_1 < h_2 < \ldots < h_{s-1} < n$, $L(0,n)$ and $L(h_{i-1},h_i)$ are the optimal paths of the N-th and m_i-th order on the related (induced sub-)graphs respectively, then we have

$$\sum_{i=1}^{1+\ell} m_i \leqslant N, \quad (0 \leqslant 1 < 1 + q \leqslant s)$$

Particularly, we have

$$\sum_{i=1}^{s} m_i \leqslant N$$

<u>Corollary 2</u>. If $0 < m_{i_0} < N$, then for all i, we have $m_i < n$. If $m_{i_0} = N$, then for all $i = i_0$, we have $m_i = 0$.

Now, we use these results to develope our theory.

On a strongly optimizing semi-field, we take N+1 yang elements or identity elements to form a sequence. If it satisfies the conditions

$$a_0 \prec a_1 \prec \ldots \prec a_{k-1} \prec a_k = z = \ldots = z$$

where $k \leqslant N$ and if we define that $z = z$ can be written as $z \prec z$, then we call this sequence with N+1 elements to be strictly monotonic to bad and write as

$$\{a_0, a_1, \ldots, a_k, z, \ldots, z\} \tag{1}$$

The family which contains all strictly monotonic-to-bad sequences like (1) is denoted by N-TH and the sequence will be called an element of the family.

Given two elements A and B of N-TH, we rearrange all those 2N+2 terms monotonically to bad and take the first N+1 non-repeated (except zero) elements to form a new sequence which is unique and is an element of N-TH. We define this to be the modi-sum $A \oplus B$ of A and B.

To A and B, we rearrange $(N+1)^2$ modi-products $a_i \otimes b_j$ $(0 \leqslant i,j \leqslant N)$ monotonically to bad. Then taking the first N+1 non-repeated (except zero) elements to form a new sequence, we define this by $A \otimes B$.

For example, if we have

$$A = \{1,3,4,6\}, \quad B = \{1,2,4,z\}$$

then

$$A \oplus B = \{1,2,3,4\}, \quad A \otimes B = \{2,3,4,5\}$$

It is not so difficult to prove that the family N-TH with the operations $\oplus$, $\otimes$ is a generalized semi-field with zero element $Z = \{z,z,\ldots,z\}$ and identity element $E = \{e,z,\ldots,z\}$. We call it Shier Semi-field, denoted by N-THOPT.

When N=0, N-THOPT will reduce to the strongly optimizing semi-field itself.

If a sequence (1) contains some zero elements, we can omit those terms, for simplicity. For example, $\{a_0,a_1,a_2,z,\ldots,z\}$ may be written as $\{a_0,a_1,a_2\}$; $\{b_0,z,\ldots,z\}$ as $\{b_0\}$ or b_0. Of course, for $\{z,\ldots,z\}$, it would be better to write it as z.

Suppose that on a multistage digraph, the jar-metric of each link is a yang element or e taken from a strongly optimizing semi-field. If there are several links with different jar-metrics from v_i to v_j, we can arrange these jar-matrics in a monotonic-to-bad order. If there are more than N+1 terms, we take the first N+1 terms. If there are only $k(<N)$ terms, we can add N+1-k zero elements to them. Thus, in short, we can write the first N+1 jar-metrics, from v_i to v_j, as an element $A = \{a_0,a_1,\ldots,a_N\}$ of N-THOPT. We may say A being a jar-metric taken from N-THOPT. If there are two groups of links from v_i to v_j, their jar-metrics are A and B respectively. Then $A \oplus B$ will be the jar-metric of these two groups of links and geometrically, it represents the jar-metrics of the first N+1 non-repeated optimal links from these two groups of links. Similarly, if the jar-metric from v_i to v_j be A, and that from v_j to v_k be C, then the jar-metric from v_i to v_k via v_j will be $A \otimes C$.

For a n-stage digraph G, if each link corresponds to a jar-metric taken from N-THOPT, then the jar-metric from the initial vertex $v_0^{(0)}$ to the final vertex $v_0^{(n)}$ of the graph G can be calculated by jar-metric principle. In the process, we can find the optimal paths of the zeroth, first, and N-th orders.

We refer to this as a problem of finding optimal paths of the first N orders.

4. A generalized model of a lot-size inventory system

The following dynamic deterministic lot-size inventory system will be called a basic model.

Consider a company that sells a single product and would like to decide how many items to have in inventory for each of the next n time periods. Assume the company has an accurate forecast of the amount it requires to meet demand, d_i, a non-negative integer, for each of the n periods. At the beginning of each period i (or at the end of period i-1), the company reviews the inventory level v_i and decides how many items, z_i, to buy from its supplier. Assume that the initial and final inventory levels, v_0

and v_n, are equal to zero and that all demands must be satisfied on time, that is $v_i \geqslant 0$, for $i = 1,2,...,n$.

The objective of the company is to find an optimal policy, that is, to design such a buying schedule that minimizes the sum of the total buying cost and holding cost, subject to the restriction mentioned above.

Many people feel that this model may be the simplest but also the most important one in the dynamic inventory theory, because many complicated models can be reduced to this one, or can be solved with the help of the method for it.

This model can be solved by ordinary dynamic programming method. Many scholars have already got a lot of results about this basic model, among them, the following one seems the most important.

Theorem 3. (Wagner and Whitin) Suppose that the buying cost function $c_i(x_i)$ and the holding cost function $h_i(v_i)$ are concave. For an optimal policy, a purchase in period i is made if and only if the inventory level at the beginning of the period is zero.

A policy which has the property mentioned in the theorem 3 is called a policy in the sense of Wagner and Whitin.

Basing on this fact, the algorithm for solving the model will have be simplified.

Some generalizations of this basic model have been made in many aspects. Here we would like not only to discuss the optimal policy in ordinary sense, but also to find all optimal policies of the first k orders in the sense of Wagner and Whitin, where k is a preassigned positive integer. Such problem will be called a generalized model.

In this section, we would like to use the concepts and the method discussed in the last three sections to solve the generalized model.

Wagner and Whitin theorem permits us to focus our attention on the moments of time at which purchases are made, rather than on the quantities of items purchase. It may be more intuitive and more effective to use multi-stage digraph, instead of digraph, to depict the process for our model.

In our model, we have three sequences. The first one is the demand sequence D:

$$D = \{d_1, d_2, d_3, ..., d_n\} \tag{1}$$

where

$d_i > 0$, integer

the second one is the sequence V of inventory levels

$$V = \{v_0, v_1, v_2, ..., v_n\} \tag{2}$$

where

$$v_i \geqslant 0,\ i = 1,2,\dots,n-1$$

and

$$v_0 = v_n = 0$$

the last one is the purchase (buying) sequence X:

$$X = \{x_1, x_2, \dots, x_n\} \tag{3}$$

According to Wagner and Whitin theorem, we are now only interested in such a buying sequence X, that is, such a policy that satisfies the following condition. If the inventory level at the beginning of period i, v_{i-1} is positive, x_i must be zero; if v_{i-1} is zero, the possible values which x_i can take are one of d_i, $d_i + d_{i+1}, \dots, d_i + d_{i+1} + \dots + d_n$. If we define

$$d_{ij} = d_i + d_{i+1} + \dots + d_j$$

$$d_{ii} = d_i \quad \text{and} \quad d_{i(i-1)} = 0$$

then x_i can only take one of the values

$$d_{i(i-1)}, d_{ii}, d_{i(i+1)}, \dots, d_{in}$$

After purchase being made, the items in hand, here, d_{ij}, will meet the demand for the product d_i, so the ending inventory level will be $d_{(i+1)j}$. Now we use the symbol ij to represent d_{ij} and the symbol 'ij' to represent the inventory level d_{ij} at the beginning of period i or at the end of period i-1, the amount d_{ij} sufficiently meets the demands from period i [illegible] j.

Then at p[illegible]d 1, the initial inventory $v_0 = 0$, x_i must take one of the values $d_{11}, d_{12}, \dots, d_{1n}$. After meeting the demand d_1, we have the ending inventory level $d_{21}, d_{22}, \dots, d_{2n}$. We depict all those facts in the first stage (with two steps) in the following figure. Again, if the inventory level v_1 is at the beginning of period 2, that is, at vertex [21], we must purchase x_2 items, which may be one of $d_{22}, d_{23}, \dots, d_{2n}$ and if the inventory level is d_{2j} $(j \geqslant 2)$ that is from vertex [2j], we must not purchase any more items in period 2. And so on.

Thus the process of the model can be depicted as a multistage digraph on fig. 1.

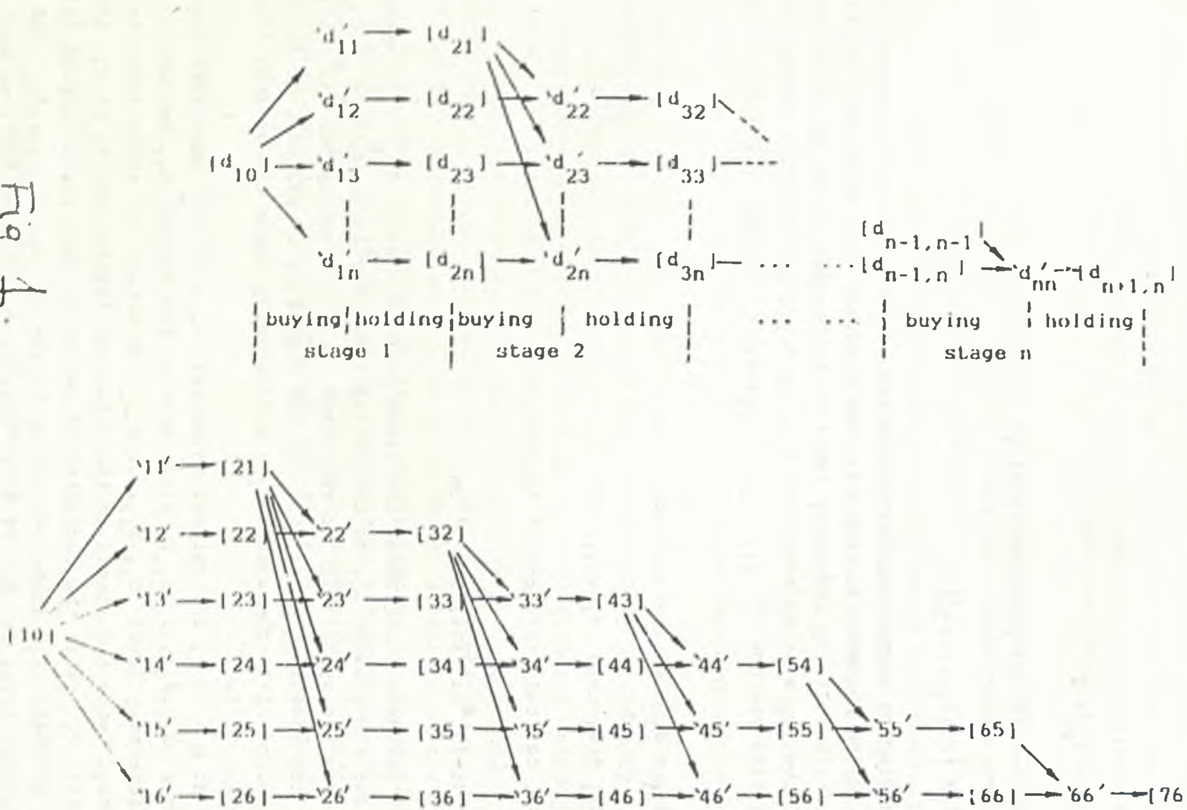

Fig. 1. A multistage digraph of the process

Rys. 1. Wieloetapowy graf skierowany procesu

Thus a solution of the generalized model is equivalent to finding the optimal paths and their length of the first k orders from $[10]$ to the vertex $[(n+1)n]$ on the multistage digraph. These can be obtained by formula (6) in section 2.

5. Numerical example

Solve a generalized model where we have

$$n = 0,\ v_1 = v_6 = 0$$

$$D = \{4,\ 2,\ 3,\ 1,\ 5,\ 2\}$$

The buying cost function is

$$c_i(x_i) = c(x) = \begin{cases} 0 & x = 0 \\ 20 + 10x, & x = 1,2,\dots \end{cases}$$

the holding cost function is

$$h_i(v_i) = h(v) = 2v, \quad v = 0,1,2,\dots$$

.and $k = 2$.
That is to say: find the optimal buying decision of the first two orders, i.e., find such policies that the total costs are minimal second minimal and the third minimal.

Solution: According to Wagner and Whitin theorem, we must first find d_{ij} and $c(d_{ij})$. Thus $d_{12} = d_1 + d_2 = 4 + 2 = 6$, and $c(d_{12}) = c(6) = 80$, $d_{25} = d_2 + d_2 + d_3 + d_4 + d_5 = 2 + 3 + 1 + 5 = 11$ and $c(d_{25}) = c(11) = 130$, and so on. So we have

$$[d_{ij}] = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 4 & 6 & 9 & 10 & 15 & 17 \\ & 2 & 5 & 6 & 11 & 13 \\ & & 3 & 4 & 9 & 11 \\ & & & 1 & 6 & 8 \\ & & & & 5 & 7 \\ & & & & & 2 \end{bmatrix} \end{array}$$

$$
\left[c(d_{ij})\right] = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}
\begin{array}{c}
\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\
\begin{bmatrix}
60 & 80 & 110 & 120 & 170 & 190 \\
 & 40 & 70 & 80 & 130 & 150 \\
 & & 50 & 60 & 110 & 130 \\
 & & & 30 & 80 & 100 \\
 & & & & 70 & 90 \\
 & & & & & 40
\end{bmatrix}
\end{array}
$$

Then we can draw the 6-stage digraph on which each stage has two steps-buying and holding. Then we write all modi-matrices and calculate the results sucessfully

$$
\text{BUY(1)} = \begin{array}{c|cccccc}
 & \text{`11'} & \text{`12'} & \text{`13'} & \text{`14'} & \text{`15'} & \text{`16'} \\
\hline
[10] & 60 & 80 & 110 & 120 & 170 & 190
\end{array}
$$

$$
\text{HOLD(1)} = \begin{array}{c|cccccc}
 & [21] & [22] & [23] & [24] & [25] & [26] \\
\hline
\text{`11'} & 0 & & & & & \\
\text{`12'} & & 4 & & & & \\
\text{`13'} & & & 10 & & & \\
\text{`14'} & & & & 12 & & \\
\text{`15'} & & & & & 22 & \\
\text{`16'} & & & & & & 26
\end{array}
$$

$$
\text{BUY(2)} = \begin{array}{c|ccccc}
 & \text{`22'} & \text{`23'} & \text{`24'} & \text{`25'} & \text{`26'} \\
\hline
[21] & 40 & 70 & 80 & 130 & 150 \\
[22] & 0 & & & & \\
[23] & & 0 & & & \\
[24] & & & 0 & & \\
[25] & & & & 0 & \\
[26] & & & & & 0
\end{array}
$$

$$
\text{HOLD(2)} = \begin{array}{c|ccccc}
 & [32] & [33] & [34] & [35] & [36] \\
\hline
\text{`22'} & 0 & & & & \\
\text{`23'} & & 6 & & & \\
\text{`24'} & & & 8 & & \\
\text{`25'} & & & & 18 & \\
\text{`26'} & & & & & 22
\end{array}
$$

$$
\text{BUY(3)} = \begin{array}{c|cccc}
 & \text{`33'} & \text{`34'} & \text{`35'} & \text{`36'} \\
\hline
[32] & 50 & 60 & 110 & 130 \\
[33] & 0 & & & \\
[34] & & 0 & & \\
[35] & & & 0 & \\
[36] & & & & 0
\end{array}
$$

$$
\text{HOLD}(3) = \begin{array}{c} \\ `33' \\ `34' \\ `35' \\ `36' \end{array}
\begin{array}{cccc} [43] & [44] & [45] & [46] \\ \hline \end{array}
$$

$$
\text{HOLD}(3) = \begin{array}{cc}
 & \begin{array}{cccc} [43] & [44] & [45] & [46] \end{array} \\
\begin{array}{c} `33' \\ `34' \\ `35' \\ `36' \end{array} & \begin{bmatrix} 0 & & & \\ & 2 & & \\ & & 12 & \\ & & & 16 \end{bmatrix}
\end{array}
$$

$$
\text{BUY}(4) = \begin{array}{cc}
 & \begin{array}{ccc} `44' & `45' & `46' \end{array} \\
\begin{array}{c} [43] \\ [44] \\ [45] \\ [46] \end{array} & \begin{bmatrix} 30 & 80 & 100 \\ 0 & & \\ & 0 & \\ & & 0 \end{bmatrix}
\end{array}
\qquad
\text{HOLD}(4) = \begin{array}{cc}
 & \begin{array}{ccc} [54] & [55] & [56] \end{array} \\
\begin{array}{c} `44' \\ `45' \\ `46' \end{array} & \begin{bmatrix} 0 & & \\ & 10 & \\ & & 14 \end{bmatrix}
\end{array}
$$

$$
\text{BUY}(5) = \begin{array}{cc}
 & \begin{array}{cc} `55' & `56' \end{array} \\
\begin{array}{c} [54] \\ [55] \\ [56] \end{array} & \begin{bmatrix} 70 & 90 \\ 0 & \\ & 0 \end{bmatrix}
\end{array}
\qquad
\text{HOLD}(5) = \begin{array}{cc}
 & \begin{array}{cc} [65] & [66] \end{array} \\
\begin{array}{c} `55' \\ `56' \end{array} & \begin{bmatrix} 0 & \\ & 4 \end{bmatrix}
\end{array}
$$

$$
\text{BUY}(6) = \begin{array}{cc}
 & `66' \\
\begin{array}{c} [65] \\ [66] \end{array} & \begin{bmatrix} 40 \\ 0 \end{bmatrix}
\end{array}
\qquad
\text{HOLD}(6) = \begin{array}{cc}
 & [76] \\
`66' & [\ 0\ ]
\end{array}
$$

Note that all elements which we ought to write but not write out are z' s.
Now, let us calculate the modi-product

$$
\prod_{i=1}^{6} \text{BUY}(i) \otimes \text{HOLD}(i)
$$

over the strongly optimizing semi-field $\{\bar{R}, \wedge, +\}$.

$$
\text{BUY}(1) \otimes \text{HOLD}(1) =
$$

$$
= [10] \begin{array}{c} \begin{array}{cccccc} [21] & [22] & [23] & [24] & [25] & [26] \end{array} \\ \begin{bmatrix} 60+0 & 80+4 & 110+10 & 120+12 & 170+22 & 190+26 \end{bmatrix} \end{array} \quad (1)
$$

$$
= [10] \begin{array}{c} \begin{array}{cccccc} [21] & [22] & [23] & [24] & [25] & [26] \end{array} \\ \begin{bmatrix} \frac{60}{`11'} & \frac{84}{`12'} & \frac{120}{`13'} & \frac{132}{`14'} & \frac{192}{`15'} & \frac{216}{`16'} \end{bmatrix} \end{array}
$$

$$
\text{BUY}(1) \otimes \ldots \otimes \text{BUY}(2) =
$$

$$
= [10] \begin{array}{c} \begin{array}{ccccc} `22' & `23' & `24' & `25' & `26' \end{array} \\ \begin{bmatrix} 100 \oplus 84 & 130 \oplus 120 & 140 \oplus 132 & 190 \oplus 192 & 210 \oplus 216 \end{bmatrix} \end{array} \quad (2)
$$

$$
= [10] \begin{array}{c} \begin{array}{ccccc} `22' & `23' & `24' & `25' & `26' \end{array} \\ \begin{bmatrix} \left(\frac{84}{[22]} \frac{100}{[21]}\right) \left(\frac{120}{[23]} \frac{130}{[21]}\right) \left(\frac{132}{[24]} \frac{140}{[21]}\right) \left(\frac{190}{[21]} \frac{192}{[25]}\right) \left(\frac{210}{[21]} \frac{216}{[26]}\right) \end{bmatrix} \end{array} \quad (3)
$$

Expression (2) can be written in the following from for our coming use.

$$
\begin{aligned}
&\mathrm{BUY}(1) \otimes \ldots \otimes \mathrm{BUY}(2) = \\
&= [10]\begin{array}{c} [21] \\ \left[\; 60 \;\right] \end{array} \otimes [10]\begin{array}{ccccc} `22' & `23' & `24' & `25' & `26' \\ \multicolumn{5}{c}{\left[\begin{array}{ccccc} 40 & 70 & 80 & 130 & 150 \end{array}\right]} \end{array} \\
&\otimes [10]\begin{array}{c} \begin{array}{ccccc} `22' & `23' & `24' & `25' & `26' \end{array} \\ \left[\begin{array}{ccccc} \frac{84}{[22]} & \frac{120}{[23]} & \frac{132}{[24]} & \frac{192}{[25]} & \frac{216}{[26]} \end{array}\right] \end{array}
\end{aligned}
\tag{4}
$$

Note that the vertex (or vertices) under a number divided by a short line is the one where the optimal path passes through. Such vertices will not enter for the computation henceafter. For example, from the second summond of (3), we can read that the paths from $[10]$ to $`22'$ via $[22]$ is the optimal one of the zero order with length 84 and via $[21]$ is the optim[illegible] one of the first order with length 100.

$$
\begin{aligned}
&\mathrm{BUY}(1) \otimes \ldots \otimes \mathrm{HOLD}(2) = \\
&= [10]\begin{array}{c} \begin{array}{ccccc} [32] & [33] & [34] & [35] & [36] \end{array} \\ \left[\left(\frac{84}{`22'}\ \frac{100}{`22'}\right) \left(\frac{126}{`23'}\ \frac{136}{`23'}\right) \left(\frac{140}{`24'}\ \frac{148}{`24'}\right) \left(\frac{208}{`25'}\ \frac{210}{`25'}\right) \left(\frac{232}{`26'}\ \frac{238}{`26'}\right) \right] \end{array}
\end{aligned}
$$

$$
\begin{aligned}
&\mathrm{B}[illegible]) \otimes \ldots \mathrm{BUY}(3) = \\
&= [10]\begin{array}{c} [32] \\ \left[(84\ \ 100)\right] \end{array} \otimes [32]\begin{array}{c} \begin{array}{cccc} `33' & `34' & `35' & `36' \end{array} \\ \left[\begin{array}{cccc} 50 & 60 & 110 & 130 \end{array}\right] \end{array} \\
&\otimes [10]\begin{array}{c} \begin{array}{cccc} `33' & `34' & `35' & `36' \end{array} \\ \left[\left(\frac{126}{[33]}\ \frac{136}{[33]}\right) \left(\frac{140}{[34]}\ \frac{148}{[34]}\right) \left(\frac{208}{[35]}\ \frac{210}{[35]}\right) \left(\frac{232}{[36]}\ \frac{238}{[36]}\right) \right] \end{array} \\
&= [10]\begin{array}{c} \begin{array}{cccc} `33' & `34' & `35' & `36' \end{array} \\ \left[\left(\frac{134}{[32]_1}\ \frac{150}{[32]_2}\right) \left(\frac{144}{[32]_1}\ \frac{160}{[32]_2}\right) \left(\frac{194}{[32]_1}\ \frac{210}{[32]_2}\right) \left(\frac{214}{[32]_1}\ \frac{230}{[32]_2}\right) \right] \end{array} \\
&\otimes [10]\begin{array}{c} \begin{array}{cccc} `33' & `34' & `35' & `36' \end{array} \\ \left[\left(\frac{126}{[33]_1}\ \frac{136}{[33]_2}\right) \left(\frac{140}{[34]_1}\ \frac{148}{[34]_2}\right) \left(\frac{208}{[35]_1}\ \frac{210}{[35]_2}\right) \left(\frac{232}{[36]_1}\ \frac{238}{[36]_2}\right) \right] \end{array} \qquad (5) \\
&= [10]\begin{array}{c} \begin{array}{cccc} `33' & `34' & `35' & `36' \end{array} \\ \left[\left(\frac{126}{[33]_1}\ \frac{134}{[32]_1}\ \frac{136}{[33]_2}\right) \left(\frac{140}{[34]_1}\ \frac{144}{[32]_1}\ \frac{148}{[34]_2}\right) \left(\frac{192}{[32]_1}\ \frac{208}{[35]_1}\ \frac{210}{[32]_2}\right) \left(\frac{214}{[32]_1}\ \frac{230}{[32]_2}\ \frac{232}{[36]_1}\right) \right] \end{array}
\end{aligned}
$$

where, for example, we can read that the path from $[10]$ to $`33'$ via $[33]$ is the optimal one of the zeroth order with length 126 and the path from $[10]$ to $[33]$ must be the optimal one of the zeroth order and the subscript "1" means to take the first term of the entry at row $[10]$ and column $[33]$.

Similarly, we have

BUY(1) $\otimes$ .... $\otimes$ HOLD(3) =

$$= [10]\left[\left(\frac{126}{[33]_1}\ \overset{[43]}{\frac{134}{[33]_2}}\ \frac{136}{33_3}\right)\left(\frac{142}{[34]_1}\ \overset{[44]}{\frac{146}{[34]_2}}\ \frac{150}{[34]_3}\right)\left(\frac{206}{[35]_1}\ \overset{[45]}{\frac{220}{[35]_2}}\ \frac{222}{[35]_3}\right)\left(\frac{230}{[36]_1}\ \overset{[46]}{\frac{246}{[36]_2}}\ \frac{248}{[36]_3}\right)\right]$$

BUY(1) $\otimes$ ... $\otimes$ BUY(4) =

$$= [10]\left[\left(\frac{142}{[44]_1}\ \overset{[44]}{\frac{146}{[44]_2}}\ \frac{150}{[££]_3}\right)\left(\frac{206}{[45]_1}\ \overset{[45]}{\frac{214}{[45]_1[42]_2}}\ \frac{216}{[43]_3}\right)\left(\frac{226}{[43]_1}\ \overset{[46]}{\frac{230}{[46]_1}}\ \frac{234}{[43]_2}\right)\right]$$

BUY(1) $\otimes$ ... $\otimes$ HOLD(4) =

$$= [10]\left[\left(\frac{142}{[44]_1}\ \overset{[54]}{\frac{146}{[44]_2}}\ \frac{150}{[44]_3}\right)\left(\frac{216}{[45]_1}\ \overset{[55]}{\frac{224}{[45]_2}}\ \frac{226}{[45]_3}\right)\left(\frac{240}{[46]_1}\ \overset{[56]}{\frac{244}{[46]_2}}\ \frac{248}{[46]_3}\right)\right]$$

BUY(1) $\otimes$ ... $\otimes$ BUY(5) =

$$= [10]\left[\left(\frac{212}{[54]_1}\ \overset{[55]}{\frac{216}{[54]_2[55]_1}}\ \frac{220}{[54]_3}\right)\left(\frac{232}{[54]_1}\ \overset{[56]}{\frac{236}{[54]_2}}\ \frac{240}{[54]_3[56]_1}\right)\right]$$

BUY(1) $\otimes$ ... $\otimes$ HOLD(5) $= [10]\left[\left(\frac{212}{[55]_1}\ \overset{[65]}{\frac{216}{[55]_2}}\ \frac{220}{[55]_3}\right)\left(\frac{236}{[56]_1}\ \overset{[66]}{\frac{240}{[56]_2}}\ \frac{244}{[56]_3}\right)\right]$.

BUY(1) $\otimes$ ... $\otimes$ BUY(6) = $[10]\left[\left(\frac{236}{[66]_1}\ \overset{[66]}{\frac{240}{[66]_2}}\ \frac{244}{[66]_3}\right)\right]$

and finaly, we have

BUY(1) $\otimes$ ... $\otimes$ HOLD(6) = $[10]\left[\left(\frac{236}{66_1}\ \overset{[76]}{\frac{240}{66_2}}\ \frac{244}{66_3}\right)\right]$

Therefore the total cost of the optimal policy of the zeroth order, that is, the length of the shortest path is 236. And its path is

1 2 3 4 5 6	I	II	III	IV	V	VI
	60 $\otimes$ 0	80 4	110 10	120 12	170 22	190 26
40 70 80 130 150	$\otimes$ 60	84 $\oplus$ 100	120 130	132 140	192 190	216 210
		$(\frac{84}{f}\ \frac{100}{s})$ $\otimes$ 0	$(\frac{120}{f}\ \frac{130}{s})$ 6	$(\frac{132}{f}\ \frac{140}{s})$ 8	$(\frac{190}{s}\ \frac{192}{f})$ 18	$(\frac{210}{s}\ \frac{216}{f})$ 22
50 60 110 130	$\otimes$	(84 100)	(126 136) $\oplus$ (134 150)	(140 148 (144 160)	(208 210) (194 210)	(232 238) (214 230)
			$(\frac{126}{f_1}\ \frac{134}{s_1}\ \frac{136}{f_2})$ $\otimes$ 0	$(\frac{140}{f_1}\ \frac{144}{s_1}\ \frac{148}{f_2})$ 2	$(\frac{194}{s_1}\ \frac{208}{f_1}\ \frac{210}{f_2 s_2})$ 12	$(\frac{214}{s_1}\ \frac{230}{s_2}\ \frac{232}{f_1})$ 16
30 80 100		$\otimes$	(126 134 136)	(142 146 150) (156 164 166)	(206 220 222) (206 214 216)	(230 246 248) (226 234 236)
				$(\frac{142}{f_1}\ \frac{146}{f_2}\ \frac{150}{f_3})$ 0	$(\frac{206}{f_1 s_1}\ \frac{214}{s_2}\ \frac{216}{s_3})$ 10	$(\frac{226}{s_1}\ \frac{230}{f_1}\ \frac{234}{s_2})$ 14
70 90			$\otimes$	(142 146 150)	(216 224 226) (212 216 220)	(240 244 248) (232 236 240)
					$(\frac{212}{s_1}\ \frac{216}{f_2}\ \frac{220}{s_3})$ 0	$(\frac{232}{s_1}\ \frac{236}{s_2}\ \frac{240}{f_1 s_3})$ 4
40				$\otimes$	(212 216 220)	(236 240 244) 252 256 210)
						$(\frac{236}{f_1}\ \frac{240}{f_2}\ \frac{244}{f_3})$ 0
						(236 240 241)

f: first line

f_1: the i-th term on the first line

s: second line

s_1: the i-th term on the second line

$$[10] \to `14' \to [24] \to `24' \to [34] \to `34' \to [44] \to `44' \to [54] \to `56' \to [66] \to `66' \to [76]$$

stage 1 | stage 2 | stage 3 | stage 4 | stage 5 | stage 6

to buy the amount of items to meet the following 4 periods

to buy the amount of items to meet the following two periods

The buying sequence X_0^* for the optimal policy is

$$X_0^* = \{10,\ 0,\ 0,\ 0,\ 7,\ 0\}$$

with the total cost 236.

The total cost of the optimal policy of the first order, that is, the second shortest path is 240. And its path is

$$[10] \to `12' \to [22] \to `22' \to [32] \to `34'_3 \to [44]_2 \to `44'_2 \to [54]_2 \to `56'_2 \to [66]_2 \to `66'_2 \to [76]$$

to buy the amount of items for two periods

to buy the amount of items for two periods

to buy the amount of items for two periods

The buying sequence X_1^* for the optimal policy of the first order is

$$X_1^* = \{6,\ 0,\ 4,\ 0,\ 7,\ 0\}$$

with the total cost 240.

Similarly, we can easily find the optimal policy of the second order:

$$X_2^* = \{4,\ 6,\ 0,\ 0,\ 7,\ 0\}$$

with the total cost 244.

Before concluding this section, let us pay attention to formulas from (1) to (4): we can give a compact form of tabulating the information and the calculation for solving the numerical example as follows.

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ОБОБЩЁННАЯ МОДЕЛЬ СИСТЕМЫ ВЫСОКОСТЕЛЛАЖНОГО СКЛАДИРОВАНИЯ

Р е з ю м е

В статье затрагиваются вопросы поиска оптимальной политики, не только в обычном смысле, но также всех оптимальных и первых политик в смысле Вагнера и Уайттина в многоэтапной проблеме высокостеллажного складирования. Так как теория и расчётные возможности, представлены автором, мало известны специалистам в западных странах, то в начале статьи эти проблемы будут кратко рассмотрены.

UOGOLNIONY MODEL SYSTEMU WYSOKIEGO SKŁADOWANIA

S t r e s z c z e n i e

Artykuł dotyczy poszukiwania optymalnej polityki nie tylko w zwykłym sensie, ale także wszystkich optymalnych k pierwszych polityk w sensie Wagnera i Whitina w wieloetapowym problemie wysokiego składowania, który jest nazywany uogólnionym modelem systemu wysokiego składowania. Ponieważ teoria i narzędzia obliczeniowe przedstawione przez autora są mało znane specjalistom w krajach zachodnich na początku artykułu zostaną one w skrócie przedyskutowane.