CEB
27 th PLENARY SESSION
PARIS SEPTEMBER 1990

COMITÉ EURO-INTERNATIONAL DU BÉTON
CONTRIBUTION OF THE POLISH NATIONAL GROUP

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# ANALYSIS OF BOND AND CRACKS IN REINFORCED CONCRETE

## 1. MODEL OF THE SECONDARY BOND ZONE

A segment of the reinforced concrete element at the stage of cracks development is shown in Fig. 1. It has been assumed that the transmission of forces from reinforcement to concrete occurs on the length of the secondary bond zone only. In this zone the displacement of a bar with respect to the duct takes place.The bond stresses that accompany this displacement affect concrete around the bar and produce displacement of the duct as well. For each cross-section situated in secondary bond zone following relation can be written:

$$w_{sc}(x) = w_s(x) - w_c(x),$$

where: $w_{sc}(x)$ - local displacement of a bar with respect to a deformable duct,

$w_s(x)$ - local displacement of a bar with respect to undeformable duct,

$w_c(x)$ - local displacement of a deformable duct with respect to its initial position before loading.

The resultant displacement of a bar in a deformable duct $w(x)$ is equal to the sum of bar displacement with respect to the duct and of the duct displacement with respect to its initial position:

$$w(x) = w_{sc}(x) + w_c(x) = w_s(x).$$

Thus, the actual displacement of the bar at any chosen cross-section of a deformable duct equals $w_s(x)$ minus the displacement of that cross-section

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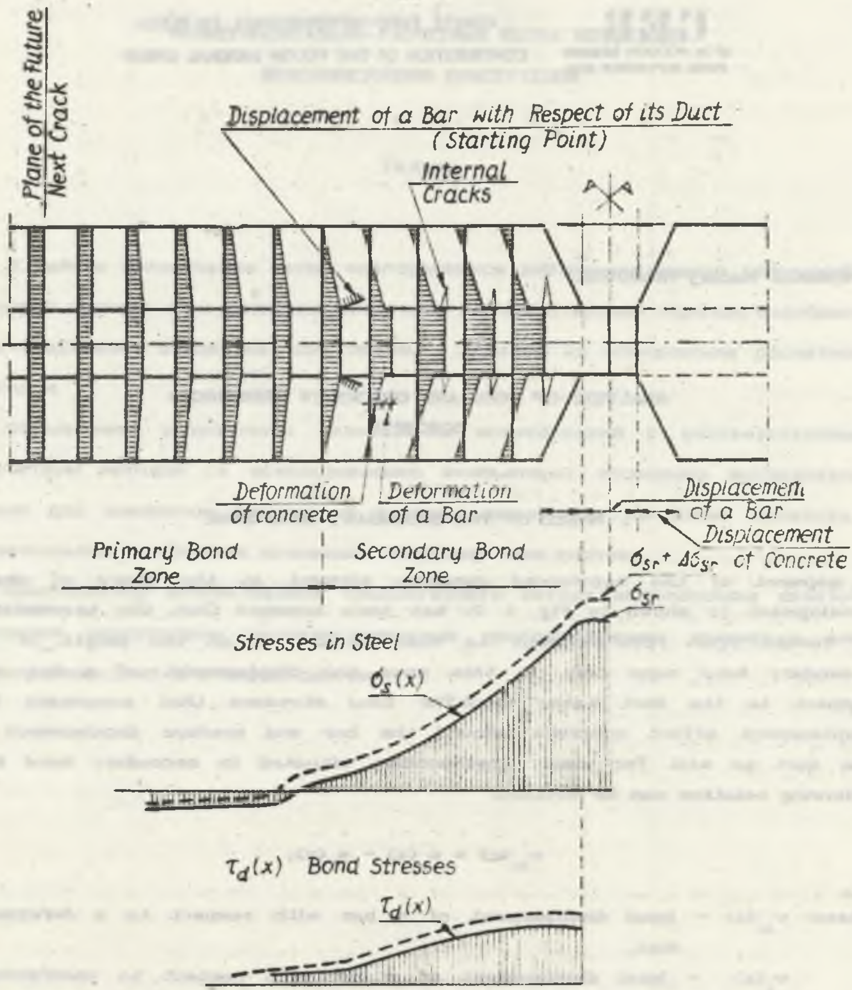


Fig.1 Reinforced concrete structure at the stage of cracks development.

with respect to its position in an undeformable duct.

Therefore, the problem of bond and cracks width can be reduced to an analysis of bar displacement in an undeformable duct. A test Compression-Pull-Out (C-POT) provides conditions approaching those of the suggested model. From these tests we obtain local bond stress as a function of local displacement of the bar in its duct and then the bond stress as a function of coordinate x.

Assuming that the normal stress in the bar in the boundary cross-section between the primary and secondary bond zones is small, a formula enabling to determine the normal stress in the bar in the secondary bond zone can be

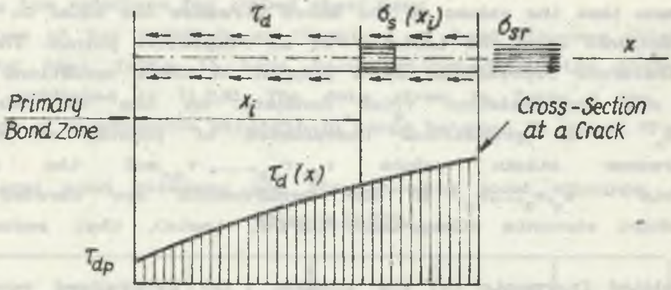


Fig.2 Equilibrium conditions of a bar in secondary bond zone.

obtained from equilibrium condition of projections of forces on the bar axis (Fig. 2):

$$\sigma(x) = \frac{u}{F_{s1}} \int \tau_d(x)dx . \tag{1}$$

where: u and $F_{s1}$ are the circumference and cross-sectional area of the bar, $\tau_d(x)$–bond stress.

Displacement of any cross-section denoted $x_i$ of the bar with respect of its initial position equals therefore:

$$w\Big|_{x_i} = \frac{1}{E_s} \frac{u}{F_{s1}} \int_0^{x_i} \Big[ \int \tau_d(x)dx \Big] dx. \tag{2}$$

In the case of $x_i = l_v$, where $l_v$ is the anchorage length of the bar (i.e. total length of secondary bond zone, corresponding to $\sigma_{sr}$ - the stress in steel at the crack cross-section), relation (2) represents $w_r$ - displacement of the bar with respect to the crack plane:

$$w_r(\sigma_{sr}) = \frac{1}{E_s} \frac{u}{F_{s1}} \int_c^{l_v} \Big[ \int \tau(x)dx \Big] dx. \tag{3}$$

## 2. BOND STRESS AS A FUNCTION OF GEOMETRICAL CO-ORDINATE

In order to determine the $\tau_d(x)$ relation, where x is the distance of a given cross-section from the boundary between primary and secondary zones, we divide the secondary bond zone into n portions. At extremal point of these segments the local displacement of the bar in its duct are denoted respectively:

$$0,\ w_1,\ w_2,\ w_3,\ \ldots\ldots,\ w_n,$$

and at these points the corresponding bond stresses apear:

$$\tau_{d,0},\ \tau_{d,1},\ \tau_{d,2},\ \tau_{d,3},\ldots\ldots,\ \tau_{d,n}.$$

We assume that the values of the above stresses are equal to mean bond stresses obtained from the tests C-POT at respective points. The relation $\tau_d(x)$ is therefore represented by a polygon. In these conditions the task to find out the relation $\tau_d(x)$ consists on the determination of $x_1, x_2, x_3, \ldots, x_n$ - the geometrical coordinates of points, at which the bond stresses attain values $\tau_{d,1}, \tau_{d,2}, \ldots, \tau_{d,n}$ and the respective displacements - $w_1, w_2, \ldots, w_n$. If the mesurements are carried out on suitably short elements (Compresion-Pull-Out tests), that assumption is correct.

For isolated fragments of the polygon $\tau_d(x)$ generalized equations of bond stresses can be written:

$$\tau_d(x) = \tau_{d,o} + \frac{\tau_{d,1} + \tau_{d,o}}{x_1} \qquad \text{for } 0 \le x \le x_1,$$

$$\tau_d(x) = \tau_{d,1} + (\tau_{d,2} - \tau_{d,1}) \frac{x - x_1}{x_2 - x_1} \qquad \text{for } x_1 \le x \le x_2, \qquad (4)$$

$$\tau_d(x) = \tau_{d,n} + (\tau_{d,n+1} - \tau_{d,n}) \frac{x - x_n}{x_{n+1} - x_n} \qquad \text{for } x_n \le x \le x_{n+1}.$$

Substituting the right-hand sides of equation (4) into right-hand sides of equation (2) and writing on its left-hand side suitable components of displacements generated at particular segments of secondary bond zone, we obtain the following equations:

$$w_1 = \frac{u}{E_s F_{s1}} \int_0^{x_1} \left[ \int (\tau_{d,o} + \frac{\tau_{d,1} - \tau_{d,o}}{x_1} x)dx \right] dx,$$

$$w_2 - w_1 = \frac{u}{E_s F_{s1}} \int_{x_1}^{x_2} \left\{ \int \left[ \tau_{d,1} + (\tau_{d,2} - \tau_{d,1}) \frac{x - x_1}{x_2 - x_1} \right] dx \right\} dx, \qquad (5)$$

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$$w_{n+1} - w_n = \frac{u}{E_s F_{s1}} \int_{x_n}^{x_{n+1}} \left\{ \int \left[ \tau_{d,n} + (\tau_{d,n+1} - \tau_{d,n}) \frac{x - x_n}{x_{n+1} - x_n} \right] dx \right\} dx.$$

Solving the set of equations (5) we can find the coordinates: $x_1$, $x_2 \ldots x_n$ of points, at which stresses $\tau_{d,1}, \tau_{d,2}, \ldots, \tau_{d,n}$ appear.

### 3. DETAILED FORM OF THE RELATION $\tau_d(x)$ AND RELATIONS DERIVED

The suggested method of approaching the problem can be applied to all sorts of reinforcing bars. In this paper the subject has been limited to establishing the relations for ribbed steel bars.

The values of bar slippings as functions of bond stresses obtained from C-POT tests (see Tabte 1) have been estimated on the basis of the researches published in [1,2,3]. The data given in Table 1 are applicable to concretes of compressive strength in range between 20 and 40 MPa.

Table 1. Local bond stresses and corresponding local slippings, calculated data in [1,2,3].

| | | | | |
|---|---|---|---|---|
| $\tau_d(x)$ [MPa] | 0.0334 | 0.1360 | 0.2270 | 0.3860 |
| w [mm] | 0 | 0.01 | 0.10 | 1.00 |

In the considered case the set of equations (5) includes 3 equations. Results of calculations carried out, are shown in Table 2.

Table 2. Geometrical co-ordinates of the bond stresses.

| $\tau_d(x)$ [MPa] | x [mm] |
|---|---|
| $\tau_{d,0}$= 0.84 | $x_0$= 0 |
| $\tau_{d,1}$= 3.40 | $x_1$= $2.5\sqrt{\phi}$ |
| $\tau_{d,2}$= 5.68 | $x_2$= $60\sqrt{\phi}$ |
| $\tau_{d,3}$= 9.65 | $x_3$= $146\sqrt{\phi}$ |

A graph plotted on the basis of data from Table 2 is presented in Fig. 3. It represents the situation at which the maximum bond forces develop and this graph is referred to as a "complete graph". On this graph geometrical coordinate $x_3$ of the maximum length of bar anchorage zone is $l_{vmax}=146\sqrt{\phi}$, here $\phi$ is the bar diameter. The complete graph presents a polygon which in Fig.3 has been substituted by a curve described by the following formula:

$$\tau_d(x)= \frac{0.794}{\sqrt{\phi}}\, x^{0.6} \quad \text{[MPa]} \tag{6}$$

where: x-dimentionless coefficient that has to be assumed equal to the distance (in milimeters) between the cross-section considered and the end of primary bond zone,

$\phi$-dimensionless coefficient equal to the diameter of a bar (in milimeters).

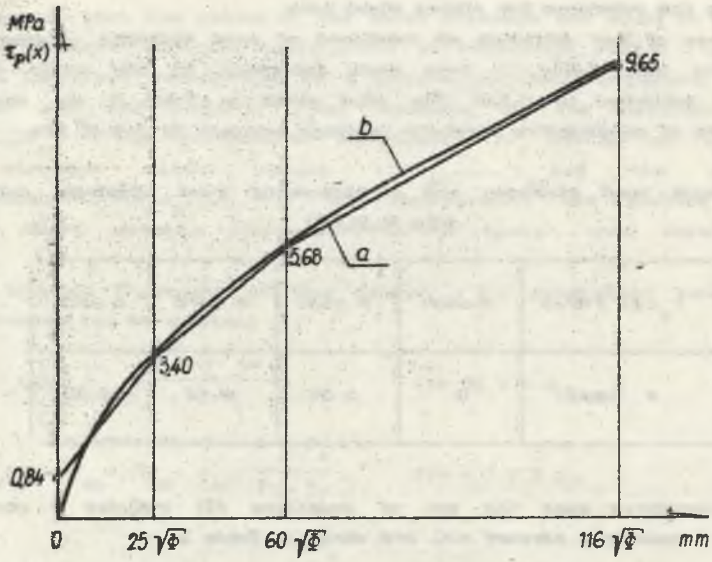


Fig.3 Bond stresses - Complete diagram $\tau_p(x)$:
a) as a polygon, b) as an approximate curve.

From the relation (6), all relations concerning the secondary bond zone can be obtained. Thus, substituting relation (6) into equation (1) we obtain a formula enabling calculation of stresses $\sigma_s(x)$ in reinforcement at any cross-section of the anchorage zone, situated at distance $x_i$ from the end of the primary bond zone:

$$\sigma_s\Big|_{x_i} = 1.985\,\frac{x_i^{1.6}}{\phi^{1.5}} \quad \text{[MPa]}, \qquad (7)$$

In a particular case for calculation of stress $\sigma_{sr}$ in reinforcement at the crack cross-section, we obtain:

$$\sigma_{sr} = 1.985\,\frac{l_v^{1.6}}{\phi^{1.5}} \quad \text{[MPa]}, \qquad (8)$$

where: $l_v$-length of bar anchorage zone.

Now, transforming the relation (8) we present the anchorage length of a bar $l_v$ as a function of stresses $\sigma_{sr}$ in reinforcement at the crack cross-section:

$$l_v = 0.651\, \sigma_{sr}^{0.625}\, \phi^{0.9375} \quad [mm]. \qquad (9)$$

The relation for evaluation of bar displacement at any point of secondary bond zone may be obtained from equation (3):

$$w\Big|_{x_i} = \frac{0.763}{E_s \phi^{1.5}}\, x_i^{2.6} \quad [mm].$$

Substituting $x = l_v$ we obtain the bar displacement with respect to the plane of a crack:

$$w_r = \frac{0.2499}{E_s}\, \sigma_{sr}^{1.625}\, \phi^{0.9375}\, c, \qquad (10)$$

where: c is a coefficient, equal $\frac{mm^{0.0625}}{MPa^{0.625}}$.

The mean deformation of reinforcement on the length of secondary bond zone may be determined by dividing the displacement $w_r$ of the bar with respect to the crack by the length $l_v$ of the anchorage zone, where $w_r$ is obtained from (10) and $l_v$ - from (9):

$$\varepsilon_{s,m} = \frac{w_r}{l_v} = 0.3839\, \frac{\sigma_{sr}}{E_s}. \qquad (11)$$

## 4. LIMIT CRACKS SPACINGS

The limit local spacing of cracks may be determined from the equation (9). We obtain:

- for the elements subjected to axial tension:

$$s_r = 0.651\, \left( \frac{m\, f_{ct,m}}{\mu} \right)^{0.625} \phi^{0.938}, \qquad (12)$$

- for the rectangular elements in bending:

$$s_r = 0.251\, \left( \frac{m\, f_{ct,m}}{\mu} \right)^{0.625} \phi^{0.938}, \qquad (13)$$

where: m, $f_{ct,m}$ -local tensile strength of concrete (at the crack) expressed as a product of m and the mean strength ( $f_{ct,m}$ ),

m -numerical coefficient

$\mu$ -reinforcement, $\mu = A_s / A_c$, (here $A_s$ is the area of reinforcement and $A_c$ -total area of concrete).

When m=1, relations (12) and (13) allow to evaluate $s_r$-mean limit cracks spacing.

## 5. WIDTH OF CRACKS AT THE STATE OF THE LIMIT CRACK SPACING

The general formula is:

$$w_r = 2\,\varepsilon_{a,m} s_r, \tag{14}$$

where: $\varepsilon_{a,m}$ - mean strain of reinforcement calculated from relation (11),

$s_r$ - local limit crack spacing.

After suitable substitutions we obtain relations for calculation of local cracks widths:

- in elements subjected to axial tension:

$$w_r = 0.500 \frac{(m\, f_{ct,m})^{0.625}\, \phi^{0.938}}{\mu^{1.625}\, E_s}\, \sigma_b, \tag{15}$$

- in rectangular elements in bending:

$$w_r = 0.042 \frac{(m\, f_{ct,m})^{0.625}\, \phi^{0.938}}{\mu^{1.625}\, E_s}\, \sigma_{bd}, \tag{16}$$

where: $\sigma_b$ -apparent stresses in elements in axial tension with cracks, calculated as in uncracked element,

$\sigma_{bd}$ -apparent boundary stresses in elements in bending with cracks.

In a case of m=1 relations (15) and (16) enable to calculate mean width of cracks $w_{r,m}$.

## 6. WIDTH OF THE EXTREME CRACKS

The extreme cracks can attain wider opening than the intermediate cracks. This phenomenon is due to the lack of possibilities of enforcing the crack development in the space beyond the extreme crack. Thus, after the extreme crack has developed, further increase of loading causes in the zone of anchorage of the bar a considerably growth of bond stresses; anchorage zone length increases and large displacement appear (Fig.4.). This can occur in the elements subjected to axial tension and in these cross-sections of element of a frame which are adjacent to the frame joints. The equation to asses the width of the extreme crack in the elements in axial tension takes the following form:

$$w_{re} = w_r \left[ 1 + \left( \frac{\sigma_b}{R_r} \right)^{0.625} \right]. \tag{17}$$

In the average load conditios, the value of apparent stresses is $\sigma = (5 \div 6) f_{ct,m}$ and the width of the extreme cracks can be twice as large as that of the intermediate ones.

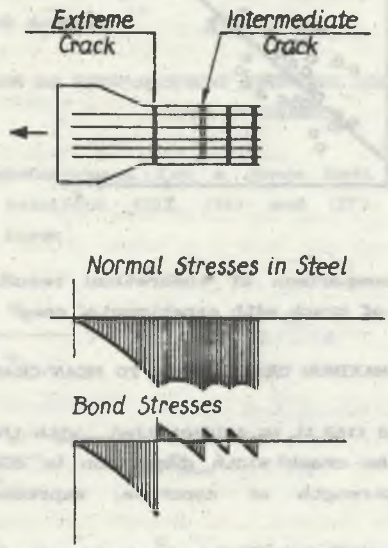


Fig.4 Comparison of conditions accompanying the developement of outermost and intermediate cracks.

## 7. COMPARISON OF THEORETICAL ESTIMATION CONCERNING MEAN WIDTH OF CRACKS WITH EXPERIMENTAL RESULTS

For the above purpose the results of tests carried out in the Institute of Building Technology in Warsaw on 49 rectangular beams has been used. Each experimental result is a mean value of approximately 10 crack width measurement, which were under observation in the span of pure flexion action in each beam. In the test the variable parameters were: the percentage of reinforcement, bar diameters and apparent edge stress $\sigma_{bd} = 6M/bh^2$.

Statistical analysis of the foregoing data have proved that the mean ratio of theoretical estimation to experimental results (Fig.5.) equals 1.04 and the value of correlation coefficient is equal to 0.8. This shows a very close agreement between the mean theoretical estimation and experimental results.

The comparative analysis proves that the mean cracks width evaluated on basis of the method suggested by author confirmed the mean cracks width calculated on basis of CEB formula [5].

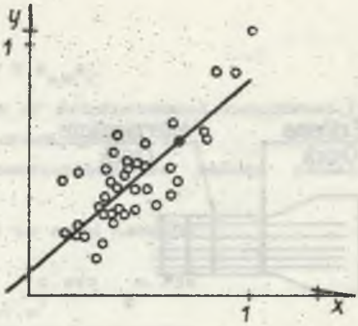


Fig.5 Regression line - comparison of theoretical results concerning mean width of crack with experimental ones.

## 8. RELATION OF MAXIMUM CRACK WIDTH TO MEAN CRACK WIDTH

From the relations (15) and (16) it is evident that with the condition of the same reinforcement the crack width dispertion is due to the variation in the local tensile strength of concrete, expressed numerically by coefficient m.

When the tensile stress in the bottom edge varies along the element length, the analysis of crack width should be carried out in the most stressed cross-section. In this case probable value of the tensil strength of concrete is the mean one, (thus m=1).

In the elements uniformly stressed however, m can attain much greater values. Values of m which have to be applied in the in the calculations may be assessed then in the following manner:

1.A distribution of local tensile strength values has to be fixed. Let it be e.g an normal distribution, represented by normalised function $\varphi(u)$.
2.An allowable width of crack opening has to be assumed ($w_{per}$) and the number of cracks (n) of that opening in a structure, at the stage of working loading must be evaluated.
3.It is assumed that in the cross-section corresponding to the last crack, the concrete tensile strength equals $m f_{ct,m}$, and that there exist only one such a cross-section in the whole element, thus:

$$\int_{-\infty}^{u_R} \varphi(u)du = 1 - \frac{1}{n} \qquad (18)$$

4.After calculation of the integral from the above formula the value of of variable $u_R$ may be found out from the tables of normal distributions factors. Next, the value of a coefficient m may be calculated from the the formula:

$$m = 1 \mp u_R \sigma, \qquad (19)$$

where $\delta$ is a mean standard deviation.

Therefore, the maximum tensile strength in the concrete is a function of the number of cracks. The greater that number, the higher strength should be assumed in crack analyses.

## 9. DESIGN OF REINFORCEMENT FROM THE CRITERIUM OF LIMIT WIDTH OF CRACKS

The amount of reinforcement for a given limit width of crack , can be determined from relations (15), (16) and (17), which after modification take the following form:

$$\mu = 0.653 \left( \frac{\sigma_b}{E_s w_{lim}} \right)^{0.615} \left( m f_{ct,m} \right)^{0.385} \phi^{0.577} , \qquad (20)$$

$$\mu = 0.142 \left( \frac{\sigma_a}{E_a w_{lim}} \right)^{0.615} \left( m f_{ct,m} \right)^{0.385} \phi^{0.577} , \qquad (21)$$

$$\mu = 0.425 \left\{ \frac{\sigma_b}{E_a w_{lim}} \left[ 1 + \left( \frac{\sigma_b}{m f_{ct,m}} \right)^{0.625} \right] \right\}^{0.615} \left( m f_{ct,m} \right)^{0.385} \qquad (22)$$

## 10. EXEMPLE OF ANALYSIS

Design of the reinforcement for a r.c. tie 25 m long and subjected to axial tension for the limit crack width equal to $w_{per}$ = 0.20 mm. Apparent stresses at the stage of working loading $\sigma_b$ = 10 MPa. As reinforcement 25 mm dia ribbed steel bars should be used of design strength 310 MPa. Concrete of mean tensile strength 2 MPa. Mean standart deviation of strength of concrete $\delta$ = 0.25.

Preliminary design is carried on with an assumption that concrete tensile strength is 2 MPa. By applying equation (20), we obtain:

$$\mu = 0.653 \left( \frac{10}{213000 \times 0.2} \right)^{0.615} 2^{0.385}\, 25^{0.577} = 0.032 \text{ i.e. } 3.2\%$$

Mean ultimate spacing of cracks can be calculated from relation (12):

$$s_{r,m} = 0.651 \left( \frac{2}{0.032} \right)^{0.625} 25^{0.938} = 177 \text{ mm.}$$

Number of cracks:

$$n = \frac{25000}{177} = 141.$$

The value of the integral of function $\varphi(u)$:

$$\int_{-\infty}^{u_n} \varphi(u)du = 1 - \frac{1}{141} = 0.992024.$$

From tables of normal distribution function we find:

$$u_n = 2.41.$$

Hence, the coefficient:

$$m = 1 + 2.41 \times 0.25 = 1.60.$$

Maximum tensile strength of concrete at the crack:

$$m\, f_{ct,m} = 1.60 \times 2 = 3.2 \text{ MPa.}$$

The final design of reinforcement should be carried on taking into account that the number of imposed cracks is equal to 141, the highest probable tensile strength of concrete at a crack is equal to 3.2 MPa. Therefore:

$$\mu = 0.653 \left(\frac{10}{213000 \times 0.2}\right)^{0.645} 3.2^{0.985}\, 25^{0.577} = 0.0383 \text{ i.e. } 3.83\% .$$

Checking the stresses in the reinforcement:

$$\sigma_a = \frac{10}{0.0383} = 261 \text{ MPa} < 310 \text{ MPa.}$$

In this case the area of reinforcement is determined by the criterion of a limit crack width and not of a bearing capacity.

We shall calculate now the area of reinforcement required at the extreme crack. Considering that the number of extreme cracks is very small (e.g. two cracks), and beside their position is stricly determined, we assume, that the most probable value of concrete tensile strength at the extreme crack is the mean strength i.e. 2 MPa. Applying equation (22), we obtain:

$$\mu\rho = 0.425 \times 2^{0.985}\, 25^{0.577} \left\{\frac{10}{213000 \times 0.2}\left[1 + \left(\frac{10}{2}\right)^{0.625}\right]\right\}^{0.645}$$

$$\mu\rho = 0.0472 \text{ i.e. } 4.7\% .$$

The control of the crack width may be also made on the basis of the CEB formulae ( Bulletin d'Information No 176 pp 15.1-15.4 ):

$$\sigma_{sm} = \frac{10}{0.0472} = 210 \text{ MPa;} \quad \sigma_{sr} = \frac{2}{0.0472} = 42 \text{ MPa;}$$

$$s_{r,m} = 50 + 0.25 \times 0.8 \frac{25}{0.0472} = 156 \text{ mm},$$

$$\varepsilon_{sm} = \frac{212}{200000} \left( 1 - 0.3 \times \frac{42}{212} \right) = 0.000991,$$

$$w_m = 156 \times 0.000991 = 0.154 \text{ mm}.$$

The characteristic crack width $w_k$ is equal:

for small elements $w_k$=1.3x0.165=0.200 mm

for big elements $w_k$=1.7x0.165=0.262 mm.

The conclusion can be drawn out that the proposed method yields smaller reinforcement than that calculated from CEB formulae.

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## ANALYSIS OF BOND AND CRACKS IN REINFORCED CONCRETE

Summary

The results of the author's investigations [1] concerning the model of cracked reinforced-concrete elements are presented. The technical bond theory based on experimental relation $\tau_p$-s, taking into account the Compresion-Pull-Out test results has been formulated. It enables to determine mean crack width. It has been suggested that dispersion of test results concerning crack width is caused by the local differences in tensile strength.

## METODA ANALIZY PRZYCZEPNOSCI I RYS W ŻELBECIE

( Streszczenie )

Opracowana przez autora techniczna teoria przyczepności, w zakresie związków fizykalnych opiera się na zależności naprężenie przyczepności-poślizg pręta w kanale, uzyskanej w trybie badań typu Compression-Pull-Out. Na podstawie tej teorii wyprowadzono zależności pozwalające na określenie średniej szerokości rozwarcia rysy w stanie granicznego rozstawu rys.

Maksymalną prawdopodobną szerokość rozwarcia rysy w konstrukcjach równomiernie naprężonych oszacowano, wychodząc z założenia, że rozrzut szerokości rozwarcia rysy jest w takich konstrukcjach wynikiem zróżnicowania lokalnej wytrzymałości betonu na rozciąganie.

## МЕТОД АНАЛИЗА СЦЕПЛЕНИЯ АРМАТУРЫ И ШИРИНЫ РАСКРЫТИЯ ТРЕЩИН В ЖЕЛЕЗОБЕТОНЕ

/ Резюме /

Разработанная автором, техническая теория сцепления арматуры с бетоном в области физических связей базирует на экспериментальной зависимости напряжение сцепления - скольжение стержня в канале, полученной из опытов типа C-Pull-Out.

В связи с этой теорией построен метод расчета средней ширины раскрытия трещин в состоянии предельного расстояния трещин.

Оценка максимальной вероятной ширины раскрытия трещин /в равномерно напряженных конструкциях/ осуществляется исходя из предпосылки, что в таких же конструкциях разбросы ширины раскрытия трещин являются результатом дифференцирования местной прочности бетона на растяжение.