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A STRONG CONVERGENCE RESULT FOR SYSTEMS OF NONLINEAR OPERATOR EQUATIONS INVOLVING TOTAL ASYMPTOTICALLY NONEXPANSIVE MAPPINGS IN UNIFORMLY CONVEX BANACH SPACES

Abstract. We prove the strong convergence of an implicit iterative procedure to a solution of a system of nonlinear operator equations involving total asymptotically nonexpansive operators in uniformly convex Banach spaces.

1. Introduction

Let K be a nonempty subset of a real normed linear space E. A self mapping $T : K \to K$ is called

- nonexpansive if $\|Tx - Ty\| \leqslant \|x - y\|$ for every $x, y \in K$;
- asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that for every $n \geqslant 1$, $\|T^n x - T^n y\| \leqslant k_n \|x - y\|$ for all $x, y \in K$;

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- asymptotically quasi-nonexpansive if $F(T) = \{x \in K : Tx = x\} \neq \phi$ and there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that $\|T^n x - y\| \leqslant k_n \|x - y\|$ for all $x \in K$, $y \in F(T)$ and every $n \geqslant 1$;

- uniformly L-Lipschitzian if there exists a real number $L > 0$ such that $\|T^n x - T^n y\| \leqslant L\|x - y\|$ for all $x, y \in K$ and all $n \geqslant 1$.

The class of asymptotically nonexpansive mappings was introdued by Goebel and Kirk [9] and the class forms an important generalization of that of nonexpansive mappings. It was proved in [9] that if K is a nonempty closed convex subset of a real uniformly convex Banach space and T is an asymptotically nonexpansive self mapping on K, then T has a fixed point.

A mapping $T : K \to K$ is called asymptotically nonexpansive in the intermediate sense (see for example [2]) if it is continuous and the following inequality holds:

$$\limsup_{n\to\infty} \sup_{x,\, y\in K} \{\|T^n x - T^n y\| - \|x - y\|\} \leqslant 0.$$

If $F(T) \neq \emptyset$ and (2) holds for all $x \in K$, $y \in F(T)$, then T is called asymptotically quasi-nonexpansive in the intermediate sense. It is obvious that if

$$\sigma_n = \max\{ \sup_{x,\, y\in K} \{\|T^n x - T^n y\| - \|x - y\|\}, 0\},$$

then $\sigma_n \to 0$ as $n \to \infty$ and (2) reduces to

$$\|T^n x - T^n y\| \leqslant \|x - y\| + \sigma_n.$$

The class of mappings that are asymptotically nonexpansive in the intermediate sense was introduced by Bruck et al. [2] It is known from [11] that if K be a nonempty closed convex bounded subset of a uniformly convex Banach space E and T is a self mapping of K which is asymptotically nonexpansive in the intermediate sense, then T has a fixed point. It is worth mentioning that the class of mappings which are asymptotically nonexpansive in the intermediate sense contains properly the class of asymptotically nonexpansive mappings. The main tool for approximation of fixed points of generalizations of nonexpansive mappings remains iterative technique.

Iterative methods for approximating fixed points of nonexpansive mappings have been studied by many authors (see for example [3, 4, 7, 8, 10, 13, 16–19, 22] and the references therein).

In most of these papers, the well-known Mann iteration process [12],

$$x_1 \in K, \qquad x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \quad n \geqslant 1, \tag{1}$$

has been studied and the operator T has been assumed to map K into itself. The convexity of K then ensures that the sequence $\{x_n\}$ generated by (1) is well defined.

In 1991, Schu [21] introduced a modified iteration process to approximate fixed points of asymptotically nonexpansive self mappings in Hilbert space. More precisely, he proved the following theorem.

Theorem 1 ([21]). *Let H be a Hilbert space, K a nonempty closed convex and bounded subset of H. Let $T : K \to K$ be an asymptotically nonexpansive mapping with sequence $\{k_n\} \subset [1,\infty)$ for all $n \geqslant 1$, $\lim k_n = 1$ and $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$. Let $\{\alpha_n\}$ be a real sequence in [0,1] satisfying the condition $0 < a \leqslant \alpha_n \leqslant b < 1$, $n \geqslant 1$, for some constants a and b. Then the sequence $\{x_n\}$ generated from $x_1 \in K$ by*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n, \qquad n \geqslant 1,$$

converges strongly to some fixed point of T.

Since then, Schu's iteration process has been widely used to approximate fixed points of asymptotically nonexpansive self-mappings in Hilbert spaces or Banach spaces (see for example [14, 19, 20, 25]).

If, however, K is a proper subset of the real Banach space E and T maps K into E (as in the case in many applications), then the sequence given by (1) may not be well-defined. One method that has been used to overcome this in the case of a single operator T is to introduce a retraction $P : E \to K$ in the recursion formula (1) as follows:

$$x_1 \in K, \qquad x_{n+1} = (1-\alpha_n)x_n + \alpha_n PT x_n, \quad n \geqslant 1.$$

Recent results on approximation of fixed points of nonexpansive and asymptotically nonexpansive self and nonself single mappings can be found in [7, 8, 10, 13, 15, 22–24, 26, 28, 29] and the references therein.

The concept of nonself asymptotically nonexpansive mappings was introduced by Chidume et al. [7] as an important generalization of asymptotically nonexpansive self-mappings.

Definition 2 ([7]). *Let K be a nonempty subset of a real normed space E. Let $P : E \to K$ be a nonexpansive retraction of E onto K. A nonself mapping $T : K \to E$ is called asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that for every $n \geqslant 1$,*

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leqslant k_n\|x - y\| \quad \text{for all } x, y \in K.$$

T is said to be uniformly L-Lipschitzian if there exists a constant $L > 0$ such that for every $n \geqslant 1$,

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leqslant L\|x - y\| \qquad \text{for all } x, y \in K.$$

It is easy to see that a nonself asymptotically nonexpansive is uniformly L-Lipschitzian.

By studying the following iteration process

$$x_1 \in K, \qquad x_{n+1} = P((1 - \alpha_n)x_n + \alpha_n T(PT)^{n-1}x_n), \quad n \geqslant 1.$$

Chidume, Ofoedu and Zegeye [7] got some strong convergence theorems for nonself asymptotically nonexpansive mappings in uniformly convex Banach spaces.

Recently, Wang [28] proved the following strong convergence theorems for common fixed points of two nonself asymptotically nonexpansive mappings:

Theorem 3 ([28]). *Let K a nonempty closed convex subset of a uniformly convex Banach space E. Suppose that $T_1, T_2 : K \to E$ are two nonself asymptotically nonexpansive mappings with sequences $\{k_n\}, \{l_n\} \subset [1, \infty)$ such that $\sum\limits_{n=1}^{\infty}(k_n - 1) < \infty$, $\sum_{n=1}^{\infty}(l_n - 1) < \infty$. From arbitrary $x_1 \in K$, let $\{x_n\}$ be defined by*

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n T_1(PT_1)^{n-1}y_n \qquad n \geqslant 1, \\ y_n &= (1 - \beta_n)x_n + \beta_n T_2(PT_2)^{n-1}x_n, \end{aligned}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $[\epsilon, 1 - \epsilon]$ for some $\epsilon > 0$. If one of T_1 and T_2 is completely continuous and $F(T_1) \cap F(T_2) \neq \emptyset$ then $\{x_n\}$ converges strongly to a common fixed point of T_1 and T_2.

Theorem 4 ([28]). *Let K, E, T_1, T_2 and $\{x_n\}$ be as in Theorem 3. If one of T_1 and T_2 is demicompact then $\{x_n\}$ converges strongly to a common fixed point of T_1 and T_2.*

Definition 5 ([15]). *Let K be a nonempty subset of a real normed space E. Let $P : E \to K$ be a nonexpansive retraction of E onto K. A nonself mapping $T : K \to E$ is called asymptotically nonexpansive in the intermediate sense if T is uniformly continuous and*

$$\limsup_{n\to\infty} \sup_{x,\, y\in K} \{\|T(PT)^{n-1}x - T(PT)^{n-1}y\| - \|x-y\|\} \leqslant 0. \tag{2}$$

In 2007, Tian, Chang and Huang [27] introduced the following concept for nonself mappings:

Definition 6 ([27]). *Let E be a real Banach space, C a nonempty nonexpansive retract of E and P the nonexpansive retraction from E onto C. A mapping $T : C \to E$ is said to be a nonself asymptotically quasi-nonexpansive mapping if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim\limits_{n\to\infty} k_n = 1$ such that for every $n \geqslant 1$,*

$$\|T(PT)^{n-1}x - p\| \leqslant k_n\|x-p\| \qquad \text{for all } x \in K,\ p \in F(T).$$

Note that if $T : C \to E$ is a nonself asymptotically nonexpansive mapping, then T is a nonself asymptotically quasi-nonexpansive mapping.

Recently, Alber et al. [1] introduced a more general class of asymptotically nonexpansive mappings called total asymptotically nonexpansive mappings and studied methods of approximation of fixed points of mappings belonging to this class.

Very recently, Chidume and Ofoedu [6] introduced the following analogue for nonself mappings:

Definition 7. *Let K be a nonempty closed and convex subset of a real Banach space E. Let $P : E \to K$ be the nonexpansive retraction of E onto K. A nonself mapping $T : K \to E$ is said to be total asymptotically nonexpansive if there exist nonnegative real sequences $\{\mu_n\}_{n=1}^{\infty}$ and $\{l_n\}_{n=1}^{\infty}$ with $\mu_n,\, l_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\phi : R^+ \to R^+$ with $\phi(0) = 0$ such that for all $x,\, y \in K$,*

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leqslant \|x-y\| + \mu_n\phi(\|x-y\|) + l_n, \qquad n \geqslant 1. \tag{3}$$

Remark 8. If $\phi(\lambda) = \lambda$, then (3) reduces to

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leqslant (1+\mu_n)\|x-y\| + l_n, \qquad n \geqslant 1.$$

If, in addition, $l_n = 0$, $n \geqslant 1$, then nonself total asymptotically nonexpansive mappings coincide with nonself asymptotically nonexpansive mappings. If $\mu_n = 0$ and $l_n = 0$ for all $n \geqslant 1$, we obtain from (3) the class of mappings that includes the class of nonself nonexpansive mappings. If $\mu_n = 0$ and $l_n = \max\{\sup_{x,\, y\in K}\{\|T(PT)^{n-1}x - T(PT)^{n-1}y\| - \|x-y\|\}, 0\}$ for all $n \geqslant 1$, then (3) reduces to (2) which has been studied as nonself mappings asymptotically nonexpansive in the intermediate sense.

Definition 9. *A nonself mapping $T : K \to E$ is said to be total asymptotically quasi-nonexpansive if there exist nonnegative real sequences $\{\mu_n\}_{n=1}^{\infty}$ and $\{l_n\}_{n=1}^{\infty}$ with μ_n, $l_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\phi : R^+ \to R^+$ with $\phi(0) = 0$ such that for all $x \in K$, $x^* \in F(T)$,*

$$\|T(PT)^{n-1}x - x^*\| \leqslant \|x - x^*\| + \mu_n\phi(\|x-x^*\|) + l_n, \qquad n \geqslant 1. \tag{4}$$

If $\phi(\lambda) = \lambda$, then (4) reduces to

$$\|T(PT)^{n-1}x - x^*\| \leqslant (1+\mu_n)\|x-x^*\| + l_n, \qquad n \geqslant 1.$$

If, in addition, $l_n = 0$, $n \geqslant 1$, then nonself total asymptotically quasi-nonexpansive mappings coincide with nonself asymptotically quasi-nonexpansive mappings. If $\mu_n = 0$ and $l_n = 0$ for all $n \geqslant 1$, we obtain from (4) the class of mappings that includes the class of quasi-nonexpansive mappings.

If $\mu_n = 0$ and $l_n = \max\{\sup_{x\in K,\, x^*\in F(T)}\{\|T(PT)^{n-1}x - x^*\| - \|x-x^*\|\}, 0\}$ for all $n \geqslant 1$, then the mappings which has been studied as mappings asymptotically quasi-nonexpansive in the intermediate sense.

The idea behind Definitions 7 and 9 is to unify various definitions of classes of mappings associated with the class of nonself asymptotically nonexpansive mappings and which are extensions of nonexpansive mappings; and to prove general convergence theorems applicable to all these classes.

Very recently, Chidume and Bashir [5] constructed and used an implicit iterative sequence to approximate a common fixed point of a finite family of nonself asymptotically nonexpansive mappings. They proved the following theorem.

Theorem 10 ([5]). *Let E be a real uniformly convex Banach space and K a nonempty closed convex subset of E which is also a nonexpansive retract with retraction P. Let $T_1, T_2, \dots, T_m : K \to E$ be asymptotically nonexpansive mappings of K into E with sequences $\{k_{in}\} \subset [1,\infty)$ such that $\sum_{n=1}^{\infty}(k_{in}-1) < \infty$ and $\lim_{n\to\infty} k_{in} = 1$, $i = 1, 2, \dots, m$, respectively. Consider the following iterative scheme:*

$$\begin{cases} x_1 \in K, \\ x_{n+1} = P[(1-\alpha_{1n})x_n + \alpha_{1n}T_1(PT_1)^{n-1}y_{n+m-2}], \\ y_{n+m-2} = P[(1-\alpha_{2n})x_n + \alpha_{2n}T_2(PT_2)^{n-1}y_{n+m-3}], \\ \vdots \\ y_n = P[(1-\alpha_{mn})x_n + \alpha_{mn}T_m(PT_m)^{n-1}x_n], \qquad n \geqslant 1, \quad m \geqslant 2, \end{cases} \tag{5}$$

where $\{\alpha_{in}\}$ are sequences in $[\epsilon, 1-\epsilon]$ for some $\epsilon \in (0,1)$, $i = 1, 2, \dots, m$. Suppose that $\bigcap_{i=1}^{m} F(T_i) \neq \emptyset$ and that one of $\{T_i\}_{i=1}^m$ is either completely continuous or semicompact, then $\{x_n\}$ converges strongly to a common fixed point of $\{T_i\}_{i=1}^m$.

Motivated and inspired by the previous facts, we extend Theorem 10 to the class of total asymptotically nonexpansive nonself mappings.

2. Preliminaries

Let E be a real normed linear space. *The modulus of convexity* of E is the function $\delta_E : (0,2] \to [0,1]$ defined by

$$\delta_E(\epsilon) = \inf\{1 - \frac{1}{2}\|x+y\| : \|x\| = \|y\| = 1, \|x-y\| = \epsilon\}.$$

E is *uniformly convex* if and only if $\delta_E(\epsilon) > 0$ for every $\epsilon \in (0,2]$.

A subset K of E is said to be a *retract* of E if there exists a continuous map $P : E \to K$ such that $Px = x$, $x \in K$. Every closed convex subset of a uniformly convex Banach space is a retract. A map $P : E \to E$ is said to be a *retraction* if $P^2 = P$. It follows that if P is a retraction then $Py = y$ for all $y \in R(P)$, the range of P.

A mapping $T : K \to K$ is said to be *semicompact* if, for any bounded sequence $\{x_n\}$ in K such that $\|x_n - Tx_n\| \to 0$ as $n \to 0$, there exists a subsequence $\{x_{n_j}\}$, say, of $\{x_n\}$ such that $\{x_{n_j}\}$ converges strongly to some x^* in K. T is said to be *completely continuous* if, for any bounded sequence $\{x_n\}$, there exists a subsequence $\{Tx_{n_j}\}$, say, of $\{Tx_n\}$ such that $\{Tx_{n_j}\}$ converges strongly to some element of the range of the range of T.

We need the following lemmas in order to prove the main results of this paper.

Lemma 11 ([25]). *Let $\{a_n\}_{n=1}^{\infty}$, $\{c_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of nonnegative real numbers such that $a_{n+1} \leqslant (1+b_n)a_n + c_n$, $n \geqslant 1$. If $\sum\limits_{n=1}^{\infty} b_n < \infty$ and $\sum\limits_{n=1}^{\infty} c_n < \infty$ then $\lim\limits_{n\to\infty} a_n$ exists.*

Lemma 12 ([21]). *Let E be a real uniformly convex Banach space and $0 < \alpha \leqslant t_n \leqslant \beta < 1$ for all positive integers $n \geqslant 1$. Suppose that $\{x_n\}$ and $\{y_n\}$ are two sequences of E such that*

$$\limsup_{n\to\infty} \|x_n\| \leqslant r, \qquad \limsup_{n\to\infty} \|y_n\| \leqslant r \quad \text{and} \qquad \limsup_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = r$$

hold for some $r \geqslant 0$, then $\lim\limits_{n\to\infty} \|x_n - y_n\| = 0$.

3. Main results

In this section we state and prove the main results of this paper. We start with giving the following proposition. The purpose of presenting this proposition is to give an idea of our method of proof.

Proposition 13. *Let E be a real normed linear space and K a nonempty convex subset of E which is also a nonexpansive retract of E with the nonexpansive retraction $P : E \to K$. Let T_1, T_2, T_3 and T_4 be total asymptotically nonexpansive mappings from K into E with a nonempty common fixed point set such that*

$$\|T_i^n x - T_i^n y\| \leqslant \|x-y\| + \mu_{in}\phi_i(\|x-y\|) + l_{in}, \qquad n \geqslant 1, \quad i \in \{1,2,3,4\},$$

where $\{\mu_{in}\}_{n=1}^{\infty}$ and $\{l_{in}\}_{n=1}^{\infty}$, $i \in \{1,2,3,4\}$ are nonnegative real sequences with $\sum\limits_{n=1}^{\infty} \mu_{in} < \infty$, $\sum\limits_{n=1}^{\infty} l_{in} < \infty$, $i \in \{1,2,3,4\}$ and $\phi_i : R^+ \to R^+$, $i \in \{1,2,3,4\}$ are strictly increasing continuous functions with $\phi_i(0) = 0$, $i \in \{1,2,3,4\}$. Suppose that there exist constants M_i, $M_i^* > 0$ such that $\phi_i(\lambda) \leqslant M_i^*\lambda$ forall $\lambda \geqslant M_i$,

$i \in \{1,2,3,4\}$. Let $\{x_n\}$ be the sequence defined by the following iterative scheme total asymptotically:

$$\begin{cases} x_1 \in K, \\ x_{n+1} = P[(1-\alpha_{1n})x_n + \alpha_{1n}T_1(PT_1)^{n-1}y_{n+2}], \\ y_{n+2} = P[(1-\alpha_{2n})x_n + \alpha_{2n}T_2(PT_2)^{n-1}y_{n+1}], \\ y_{n+1} = P[(1-\alpha_{3n})x_n + \alpha_{3n}T_3(PT_3)^{n-1}y_n], \\ y_n = P[(1-\alpha_{4n})x_n + \alpha_{4n}T_4(PT_4)^{n-1}x_n], \qquad n \geqslant 1, \end{cases}$$

where $\{\alpha_{in}\}$ are sequences in $[\epsilon, 1-\epsilon]$ for some $\epsilon \in (0,1)$, $i \in \{1,2,3,4\}$. Then $\{x_n\}$ is bounded and $\lim_{n\to\infty} \|x_n - q\|$ exists for each $q \in \bigcap_{i=1}^{4} F(T_i) \neq \emptyset$.

Proof. For any $q \in \bigcap_{i=1}^{4} F(T_i) \neq \emptyset$, we have the following estimates:

$$\begin{aligned} \|x_{n+1} - q\| &= \|P[(1-\alpha_{1n})x_n + \alpha_{1n}T_1(PT_1)^{n-1}y_{n+2}] - q\| \leqslant \\ &\leqslant \|(1-\alpha_{1n})(x_n - q) + \alpha_{1n}(T_1(PT_1)^{n-1}y_{n+2} - q)\| \leqslant \\ &\leqslant (1-\alpha_{1n})\|x_n - q\| + \alpha_{1n}\|T_1(PT_1)^{n-1}y_{n+2} - q\| \leqslant \\ &\leqslant (1-\alpha_{1n})\|x_n - q\| + \alpha_{1n}[\|y_{n+2} - q\| + \mu_{1n}\phi_1(\|y_{n+2} - q\|) + l_{1n}]. \end{aligned}$$

By our assumption, there exist constants $M_1, M_1^* > 0$ such that

$$\phi_1(\|y_{n+2} - q\|) \leqslant M_1^* \|y_{n+2} - q\|$$

whenever

$$\|y_{n+2} - q\| \geqslant M_1.$$

Moreover, since ϕ_1 is a strictly increasing function, then $\phi_1(\|y_{n+2} - q\|) \leqslant \phi_1(M_1)$ if $\|y_{n+2} - q\| \leqslant M_1$. In either case we have

$$\phi_1(\|y_{n+2} - q\|) \leqslant \phi_1(M_1) + M_1^* \|y_{n+2} - q\|.$$

Therefore, continuing in the same way, we get

$$\begin{aligned} \|x_{n+1} - q\| \leqslant (1-\alpha_{1n})\|x_n - q\| &+ \alpha_{1n}[\|y_{n+2} - q\| + \mu_{1n}(\phi_1(M_1) + \\ &+ M_1^*\|y_{n+2} - q\|) + l_{1n}] = \\ &= (1-\alpha_{1n})\|x_n - q\| + \alpha_{1n}(1 + M_1^*\mu_{1n})\|y_{n+2} - q\| + \\ &+ \alpha_{1n}[\phi_1(M_1)\mu_{1n} + l_{1n}] \leqslant \end{aligned}$$

$$\leqslant (1-\alpha_{1n})\|x_n-q\| + \alpha_{1n}(1-\alpha_{2n})(1+M_1^*\mu_{1n})\|x_n-q\| +$$
$$+ \alpha_{1n}\alpha_{2n}(1+M_1^*\mu_{1n})\|T_2(PT_2)^{n-1}y_{n+1}-q\| +$$
$$+ \alpha_{1n}[\phi_1(M_1)\mu_{1n}+l_{1n}] \leqslant$$
$$\leqslant (1-\alpha_{1n})\|x_n-q\| + \alpha_{1n}(1-\alpha_{2n})(1+M_1^*\mu_{1n})\|x_n-q\| +$$
$$+ \alpha_{1n}\alpha_{2n}(1+M_1^*\mu_{1n})[(1+M_2^*\mu_{2n})\|y_{n+1}-q\| +$$
$$+ (\phi_2(M_2)\mu_{2n}+l_{2n})] + \alpha_{1n}[\phi_1(M_1)\mu_{1n}+l_{1n}] =$$
$$= (1-\alpha_{1n})\|x_n-q\| + \alpha_{1n}(1-\alpha_{2n})(1+M_1^*\mu_{1n})\|x_n-q\| +$$
$$+ \alpha_{1n}\alpha_{2n}(1+M_1^*\mu_{1n})(1+M_2^*\mu_{2n})\|y_{n+1}-q\| +$$
$$+ \alpha_{1n}\alpha_{2n}(1+M_1^*\mu_{1n})[\phi_2(M_2)\mu_{2n}+l_{2n}] + \alpha_{1n}[\phi_1(M_1)\mu_{1n}+l_{1n}] \leqslant$$
$$\leqslant (1-\alpha_{1n})\|x_n-q\| + \alpha_{1n}(1-\alpha_{2n})(1+M_1^*\mu_{1n})\|x_n-q\|+$$
$$+ \alpha_{1n}\alpha_{2n}(1-\alpha_{3n})(1+M_1^*\mu_{1n})(1+M_2^*\mu_{2n})\|x_n-q\|+$$
$$+ \alpha_{1n}\alpha_{2n}\alpha_{3n}(1+M_1^*\mu_{1n})(1+M_2^*\mu_{2n})(1+M_3^*\mu_{3n})\|y_n-q\|+$$
$$+ \alpha_{1n}\alpha_{2n}\alpha_{3n}(1+M_1^*\mu_{1n})(1+M_2^*\mu_{2n})[\phi_3(M_3)\mu_{3n}+l_{3n}]+$$
$$+ \alpha_{1n}\alpha_{2n}(1+M_1^*\mu_{1n})[\phi_2(M_2)\mu_{2n}+l_{2n}] + \alpha_{1n}[\phi_1(M_1)\mu_{1n}+l_{1n}] \leqslant$$
$$\leqslant (1-\alpha_{1n})\|x_n-q\| + \alpha_{1n}(1-\alpha_{2n})(1+M_1^*\mu_{1n})\|x_n-q\|+$$
$$+ \alpha_{1n}\alpha_{2n}(1-\alpha_{3n})(1+M_1^*\mu_{1n})(1+M_2^*\mu_{2n})\|x_n-q\|+$$
$$+ \alpha_{1n}\alpha_{2n}\alpha_{3n}(1-\alpha_{4n})(1+M_1^*\mu_{1n})(1+M_2^*\mu_{2n})(1+M_3^*\mu_{3n})\|x_n-q\|+$$
$$+ \alpha_{1n}\alpha_{2n}\alpha_{3n}\alpha_{4n}(1+M_1^*\mu_{1n})(1+M_2^*\mu_{2n})(1+M_3^*\mu_{3n})(1+M_4^*\mu_{4n})\|x_n-q\|+$$
$$+ \alpha_{1n}\alpha_{2n}\alpha_{3n}\alpha_{4n}(1+M_1^*\mu_{1n})(1+M_2^*\mu_{2n})(1+M_3^*\mu_{3n})[\phi_4(M_4)\mu_{4n}+l_{4n}]+$$
$$+ \alpha_{1n}\alpha_{2n}\alpha_{3n}(1+M_1^*\mu_{1n})(1+M_2^*\mu_{2n})[\phi_3(M_3)\mu_{3n}+l_{3n}]+$$
$$+ \alpha_{1n}\alpha_{2n}(1+M_1^*\mu_{1n})[\phi_2(M_2)\mu_{2n}+l_{2n}] + \alpha_{1n}[\phi_1(M_1)\mu_{1n}+l_{1n}] =$$
$$= [1-\alpha_{1n}+\alpha_{1n}(1-\alpha_{2n})(1+M_1^*\mu_{1n})+$$
$$+ \alpha_{1n}\alpha_{2n}(1-\alpha_{3n})(1+M_1^*\mu_{1n})(1+M_2^*\mu_{2n})+$$
$$+ \alpha_{1n}\alpha_{2n}\alpha_{3n}(1-\alpha_{4n})(1+M_1^*\mu_{1n})(1+M_2^*\mu_{2n})(1+M_3^*\mu_{3n})+$$
$$+ \alpha_{1n}\alpha_{2n}\alpha_{3n}\alpha_{4n}(1+M_1^*\mu_{1n})(1+M_2^*\mu_{2n})(1+M_3^*\mu_{3n})(1+M_4^*\mu_{4n})]\|x_n-q\|+$$
$$+ \alpha_{1n}\alpha_{2n}\alpha_{3n}\alpha_{4n}(1+M_1^*\mu_{1n})(1+M_2^*\mu_{2n})(1+M_3^*\mu_{3n})[\phi_4(M_4)\mu_{4n}+l_{4n}]+$$
$$+ \alpha_{1n}\alpha_{2n}\alpha_{3n}(1+M_1^*\mu_{1n})(1+M_2^*\mu_{2n})[\phi_3(M_3)\mu_{3n}+l_{3n}]+$$
$$+ \alpha_{1n}\alpha_{2n}(1+M_1^*\mu_{1n})[\phi_2(M_2)\mu_{2n}+l_{2n}] + \alpha_{1n}[\phi_1(M_1)\mu_{1n}+l_{1n}] =$$

$$
\begin{aligned}
&= [1 + \alpha_{1n} M_1^* \mu_{1n} + \alpha_{1n}\alpha_{2n}(1 + M_1^*\mu_{1n})M_2^*\mu_{2n} + \\
&+ \alpha_{1n}\alpha_{2n}\alpha_{3n}(1 + M_1^*\mu_{1n})(1 + M_2^*\mu_{2n})M_3^*\mu_{3n} + \\
&+ \alpha_{1n}\alpha_{2n}\alpha_{3n}\alpha_{4n}(1 + M_1^*\mu_{1n})(1 + M_2^*\mu_{2n})(1 + M_3^*\mu_{3n})M_4^*\mu_{4n}]\|x_n - q\| + \\
&+ \alpha_{1n}\alpha_{2n}\alpha_{3n}\alpha_{4n}(1 + M_1^*\mu_{1n})(1 + M_2^*\mu_{2n})(1 + M_3^*\mu_{3n})[\phi_4(M_4)\mu_{4n} + l_{4n}] + \\
&+ \alpha_{1n}\alpha_{2n}\alpha_{3n}(1 + M_1^*\mu_{1n})(1 + M_2^*\mu_{2n})[\phi_3(M_3)\mu_{3n} + l_{3n}] + \\
&+ \alpha_{1n}\alpha_{2n}(1 + M_1^*\mu_{1n})[\phi_2(M_2)\mu_{2n} + l_{2n}] + \alpha_{1n}[\phi_1(M_1)\mu_{1n} + l_{1n}] \leqslant \\
&\leqslant [1 + M_1^*\mu_{1n} + (1 + M_1^*\mu_{1n})M_2^*\mu_{2n} + (1 + M_1^*\mu_{1n})(1 + M_2^*\mu_{2n})M_3^*\mu_{3n} + \\
&+ (1 + M_1^*\mu_{1n})(1 + M_2^*\mu_{2n})(1 + M_3^*\mu_{3n})M_4^*\mu_{4n}]\|x_n - q\| + \\
&+ (1 + M_1^*\mu_{1n})(1 + M_2^*\mu_{2n})(1 + M_3^*\mu_{3n})[\phi_4(M_4)\mu_{4n} + l_{4n}] + \\
&+ (1 + M_1^*\mu_{1n})(1 + M_2^*\mu_{2n})[\phi_3(M_3)\mu_{3n} + l_{3n}] + \\
&+ (1 + M_1^*\mu_{1n})[\phi_2(M_2)\mu_{2n} + l_{2n}] + [\phi_1(M_1)\mu_{1n} + l_{1n}] = \\
&= [1 + M_1^*\mu_{1n} + M_2^*\mu_{2n} + M_3^*\mu_{3n} + M_4^*\mu_{4n} + M_1^*\mu_{1n}M_2^*\mu_{2n} + M_1^*\mu_{1n}M_3^*\mu_{3n} + \\
&+ M_1^*\mu_{1n}M_4^*\mu_{4n} + M_2^*\mu_{2n}M_3^*\mu_{3n} + M_2^*\mu_{2n}M_4^*\mu_{4n} + M_3^*\mu_{3n}M_4^*\mu_{4n} + \\
&+ M_1^*\mu_{1n}M_2^*\mu_{2n}M_3^*\mu_{3n} + M_1^*\mu_{1n}M_2^*\mu_{2n}M_4^*\mu_{4n} + \\
&+ M_1^*\mu_{1n}M_3^*\mu_{3n}M_4^*\mu_{4n} + M_2^*\mu_{2n}M_3^*\mu_{3n}M_4^*\mu_{4n} + \\
&+ M_1^*\mu_{1n}M_2^*\mu_{2n}M_3^*\mu_{3n}M_4^*\mu_{4n}]\|x_n - q\| + \\
&+ (1 + M_1^*\mu_{1n})(1 + M_2^*\mu_{2n})(1 + M_3^*\mu_{3n})[\phi_4(M_4)\mu_{4n} + l_{4n}] + \\
&+ (1 + M_1^*\mu_{1n})(1 + M_2^*\mu_{2n})[\phi_3(M_3)\mu_{3n} + l_{3n}] + (1 + M_1^*\mu_{1n})[\phi_2(M_2)\mu_{2n} + l_{2n}] + \\
&+ [\phi_1(M_1)\mu_{1n} + l_{1n}].
\end{aligned}
$$

Hence, there exists $Q > 0$ such that

$$\|x_{n+1} - q\| \leqslant [1 + 4\nu_n + 6\nu_n^2 + 4\nu_n^3 + \nu_n^4]\|x_n - q\| + Q\sum_{i=1}^{4}[\phi_i(M_i)\mu_{in} + l_{in}],$$

where $\nu_n = \max\{M_k^*\mu_{kn}, k = 1, 2, 3, 4\}$, $n \geqslant 1$.

Since $\sum\limits_{n=1}^{\infty} \mu_{in} < \infty$ and $\sum\limits_{n=1}^{\infty} l_{in} < \infty$ for all $i \in \{1, 2, 3, 4\}$, then applying Lemma 11, we obtain that $\lim\limits_{n\to\infty} \|x_n - q\|$ exists for any $q \in \bigcap\limits_{i=1}^{4} F(T_i)$ and hence $\{x_n\}$ is bounded. □

Now, we can prove the following lemma for finitely many maps. In the sequel, we denote the set $\{1, 2, \ldots, m\}$ by I.

Lemma 14. *Let E be a real normed linear space and K a nonempty convex subset of E which is also a nonexpansive retract of E with the nonexpansive retraction $P: E \to K$. Let $\{T_i, i \in I\}$, be a finite family of m total asymptotically nonexpansive mappings from K into E with a nonempty common fixed point set, F, such that*

$$\|T_i^n x - T_i^n y\| \leqslant \|x - y\| + \mu_{in}\phi_i(\|x - y\|) + l_{in}, \qquad n \geqslant 1, \quad i \in I,$$

where $\{\mu_{in}\}_{n=1}^{\infty}$ and $\{l_{in}\}_{n=1}^{\infty}$, $i \in I$ are nonnegative real sequences with $\sum\limits_{n=1}^{\infty} \mu_{in} < \infty$, $\sum\limits_{n=1}^{\infty} l_{in} < \infty$, $i \in I$ and $\phi_i : R^+ \to R^+$, $i \in I$ are strictly increasing continuous functions with $\phi_i(0) = 0$, $i \in I$. Suppose that there exist constants M_i, $M_i^* > 0$ such that $\phi_i(\lambda) \leqslant M_i^* \lambda$ for all $\lambda \geqslant M_i$, $i \in I$. Let $\{x_n\}$ be the sequence defined by the iterative scheme (5), where $\{\alpha_{in}\}$ are sequences in $[\epsilon, 1-\epsilon]$ for some $\epsilon \in (0,1)$, $i \in I$. Then $\{x_n\}$ is bounded and $\lim\limits_{n\to\infty} \|x_n - q\|$ exists for each $q \in F \neq \emptyset$.

Proof. For an arbitrary $q \in F \neq \emptyset$, we have the following estimates:

$$\begin{aligned}
\|x_{n+1} - q\| &= \|P[(1-\alpha_{1n})x_n + \alpha_{1n}T_1(PT_1)^{n-1}y_{n+m-2}] - q\| \leqslant \\
&\leqslant \|(1-\alpha_{1n})(x_n - q) + \alpha_{1n}(T_1(PT_1)^{n-1}y_{n+m-2} - q)\| \leqslant \\
&\leqslant (1-\alpha_{1n})\|x_n - q\| + \alpha_{1n}\|T_1(PT_1)^{n-1}y_{n+m-2} - q\| \leqslant \\
&\leqslant (1-\alpha_{1n})\|x_n - q\| + \alpha_{1n}[\|y_{n+m-2} - q\| + \mu_{1n}\phi_1(\|y_{n+m-2} - q\|) + l_{1n}].
\end{aligned}$$

By our assumption, there exist constants $M_1, M_1^* > 0$ such that $\phi_1(\|y_{n+m-2} - q\|) \leqslant M_1^*\|y_{n+m-2} - q\|$ whenever $\|y_{n+m-2} - q\| \geqslant M_1$. Moreover, since ϕ_1 is a strictly increasing function, then $\phi_1(\|y_{n+m-2} - q\|) \leqslant \phi_1(M_1)$ if $\|y_{n+m-2} - q\| \leqslant M_1$. In either case we have

$$\phi_1(\|y_{n+m-2} - q\|) \leqslant \phi_1(M_1) + M_1^*\|y_{n+m-2} - q\|.$$

Therefore, proceeding in this way, we get

$$\begin{aligned}
&\|x_{n+1} - q\| \leqslant (1-\alpha_{1n})\|x_n - q\| + \alpha_{1n}[\|y_{n+m-2} - q\| + \mu_{1n}(\phi_1(M_1) + \\
&\quad + M_1^*\|y_{n+m-2} - q\|) + l_{1n}] = \\
&\quad = (1-\alpha_{1n})\|x_n - q\| + \alpha_{1n}(1 + M_1^*\mu_{1n})\|y_{n+m-2} - q\| + \alpha_{1n}[\phi_1(M_1)\mu_{1n} + l_{1n}] \leqslant \\
&\quad \leqslant (1-\alpha_{1n})\|x_n - q\| + \alpha_{1n}(1-\alpha_{2n})(1 + M_1^*\mu_{1n})\|x_n - q\| + \\
&\quad + \alpha_{1n}\alpha_{2n}(1 + M_1^*\mu_{1n})\|T_2(PT_2)^{n-1}y_{n+m-3} - q\| + \alpha_{1n}[\phi_1(M_1)\mu_{1n} + l_{1n}] \leqslant
\end{aligned}$$

$$
\begin{aligned}
&\leqslant (1-\alpha_{1n})\|x_n-q\| + \alpha_{1n}(1-\alpha_{2n})(1+M_1^*\mu_{1n})\|x_n-q\|+\\
&+\alpha_{1n}\alpha_{2n}(1+M_1^*\mu_{1n})[(1+M_2^*\mu_{2n})\|y_{n+m-3}-q\| + (\phi_2(M_2)\mu_{2n}+l_{2n})]+\\
&+\alpha_{1n}[\phi_1(M_1)\mu_{1n}+l_{1n}] =\\
&= (1-\alpha_{1n})\|x_n-q\| + \alpha_{1n}(1-\alpha_{2n})(1+M_1^*\mu_{1n})\|x_n-q\|+\\
&+\alpha_{1n}\alpha_{2n}(1+M_1^*\mu_{1n})(1+M_2^*\mu_{2n})\|y_{n+m-3}-q\|+\\
&+\alpha_{1n}\alpha_{2n}(1+M_1^*\mu_{1n})[\phi_2(M_2)\mu_{2n}+l_{2n}] + \alpha_{1n}[\phi_1(M_1)\mu_{1n}+l_{1n}] \leqslant\\
&\leqslant (1-\alpha_{1n})\|x_n-q\| + \alpha_{1n}(1-\alpha_{2n})(1+M_1^*\mu_{1n})\|x_n-q\|+\\
&+\alpha_{1n}\alpha_{2n}(1-\alpha_{3n})(1+M_1^*\mu_{1n})(1+M_2^*\mu_{2n})\|x_n-q\|+\\
&+\alpha_{1n}\alpha_{2n}\alpha_{3n}(1+M_1^*\mu_{1n})(1+M_2^*\mu_{2n})(1+M_3^*\mu_{3n})\|y_{n+m-4}-q\|+\\
&+\alpha_{1n}\alpha_{2n}\alpha_{3n}(1+M_1^*\mu_{1n})(1+M_2^*\mu_{2n})[\phi_3(M_3)\mu_{3n}+l_{3n}]+\\
&+\alpha_{1n}\alpha_{2n}(1+M_1^*\mu_{1n})[\phi_2(M_2)\mu_{2n}+l_{2n}] + \alpha_{1n}[\phi_1(M_1)\mu_{1n}+l_{1n}] \leqslant\\
&\leqslant \ \ldots \ \leqslant\\
&\leqslant \{1-\alpha_{1n}+\alpha_{1n}(1-\alpha_{2n})(1+M_1^*\mu_{1n})+\\
&+\alpha_{1n}\alpha_{2n}(1-\alpha_{3n})(1+M_1^*\mu_{1n})(1+M_2^*\mu_{2n})+\ldots+\\
&+\alpha_{1n}\ldots\alpha_{(m-2)n}(1-\alpha_{(m-1)n})(1+M_1^*\mu_{1n})\ldots(1+M_{m-2}^*\mu_{(m-2)n})\}\|x_n-q\|+\\
&+\{\alpha_{1n}\ldots\alpha_{(m-1)n}(1+M_1^*\mu_{1n})\ldots(1+M_{m-1}^*\mu_{(m-1)n})\}\|y_n-q\|+\\
&+\alpha_{1n}\ldots\alpha_{(m-1)n}(1+M_1^*\mu_{1n})\ldots(1+M_{m-2}^*\mu_{(m-2)n})\cdot\\
&\cdot[\phi_{m-1}(M_{m-1})\mu_{(m-1)n}+l_{(m-1)n}]+\\
&+\alpha_{1n}\alpha_{2n}\ldots\alpha_{(m-2)n}(1+M_1^*\mu_{1n})(1+M_2^*\mu_{2n})\ldots\\
&\ldots(1+M_{m-3}^*\mu_{(m-3)n})[\phi_{m-2}(M_{m-2})\mu_{(m-2)n}+l_{(m-2)n}]+\ldots+\\
&+\alpha_{1n}[\phi_1(M_1)\mu_{1n}+l_{1n}] \leqslant\\
&\leqslant \{1-\alpha_{1n}+\alpha_{1n}(1-\alpha_{2n})(1+M_1^*\mu_{1n})+\\
&+\alpha_{1n}\alpha_{2n}(1-\alpha_{3n})(1+M_1^*\mu_{1n})(1+M_2^*\mu_{2n})+\ldots+\\
&+\alpha_{1n}\alpha_{2n}\ldots\alpha_{mn}(1+M_1^*\mu_{1n})(1+M_2^*\mu_{2n})\ldots(1+M_m^*\mu_{mn})\}\|x_n-q\|+\\
&+\alpha_{1n}[\phi_1(M_1)\mu_{1n}+l_{1n}] + \alpha_{1n}\alpha_{2n}(1+M_1^*\mu_{1n})[\phi_2(M_2)\mu_{2n}+l_{2n}]+\\
&+\alpha_{1n}\alpha_{2n}\alpha_{3n}(1+M_1^*\mu_{1n})(1+M_2^*\mu_{2n})[\phi_2(M_2)\mu_{2n}+l_{2n}]+\ldots+\\
&+\alpha_{1n}\ldots\alpha_{mn}(1+M_1^*\mu_{1n})\ldots(1+M_{(m-1)}^*\mu_{(m-1)n})[\phi_m(M_m)\mu_{mn}+l_{mn}].
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\|x_{n+1}-q\| \leqslant \{1+M_1^*\mu_{1n}+M_2^*\mu_{2n}+\ldots+M_m^*\mu_{mn}+M_1^*\mu_{1n}M_2^*\mu_{2n}+\\
&+M_1^*\mu_{1n}M_3^*\mu_{3n}+\ldots+M_1^*\mu_{1n}M_m^*\mu_{mn}+M_2^*\mu_{2n}M_3^*\mu_{3n}+\\
&+M_2^*\mu_{2n}M_4^*\mu_{4n}+\ldots+M_2^*\mu_{2n}M_m^*\mu_{mn}+\ldots+M_{(m-1)}^*\mu_{(m-1)n}M_m^*\mu_{mn}+\ldots+\\
&+M_1^*\mu_{1n}M_2^*\mu_{2n}\ldots M_m^*\mu_{mn}\}\|x_n-q\|+\\
&+\alpha_{1n}[\phi_1(M_1)\mu_{1n}+l_{1n}]+\alpha_{1n}\alpha_{2n}(1+M_1^*\mu_{1n})[\phi_2(M_2)\mu_{2n}+l_{2n}]+\\
&+\alpha_{1n}\alpha_{2n}\alpha_{3n}(1+M_1^*\mu_{1n})(1+M_2^*\mu_{2n})[\phi_2(M_2)\mu_{2n}+l_{2n}]+\ldots+\\
&+\alpha_{1n}\alpha_{2n}\ldots\alpha_{mn}(1+M_1^*\mu_{1n})(1+M_2^*\mu_{2n})\ldots(1+M_{(m-1)}^*\mu_{(m-1)n})\cdot\\
&\cdot[\phi_m(M_m)\mu_{mn}+l_{mn}]\leqslant\\
&\leqslant\{1+\tbinom{m}{1}\nu_n+\tbinom{m}{2}\nu_n^2+\ldots+\tbinom{m}{m-1}\nu_n^{m-1}+\nu_n^m\}\|x_n-q\|+\\
&+\alpha_{1n}[\phi_1(M_1)\mu_{1n}+l_{1n}]+\alpha_{1n}\alpha_{2n}(1+M_1^*\mu_{1n})[\phi_2(M_2)\mu_{2n}+l_{2n}]+\\
&+\alpha_{1n}\alpha_{2n}\alpha_{3n}(1+M_1^*\mu_{1n})(1+M_2^*\mu_{2n})[\phi_2(M_2)\mu_{2n}+l_{2n}]+\ldots+\\
&+\alpha_{1n}\alpha_{2n}\ldots\alpha_{mn}(1+M_1^*\mu_{1n})(1+M_2^*\mu_{2n})\ldots(1+M_{(m-1)}^*\mu_{(m-1)n})\cdot\\
&\cdot[\phi_m(M_m)\mu_{mn}+l_{mn}]\leqslant\\
&\leqslant(1+\nu_n)^m\|x_n-q\|+[\phi_1(M_1)\mu_{1n}+l_{1n}]+(1+M_1^*\mu_{1n})[\phi_2(M_2)\mu_{2n}+l_{2n}]+\\
&+(1+M_1^*\mu_{1n})(1+M_2^*\mu_{2n})[\phi_2(M_2)\mu_{2n}+l_{2n}]+\ldots+\\
&+(1+M_1^*\mu_{1n})(1+M_2^*\mu_{2n})\ldots(1+M_{(m-1)}^*\mu_{(m-1)n})[\phi_m(M_m)\mu_{mn}+l_{mn}].
\end{aligned}
$$

Hence, there exists $Q^*>0$ such that

$$\|x_{n+1}-q\| \leqslant (1+\nu_n)^m\|x_n-q\|+Q^*\sum_{i=1}^{m}[\phi_i(M_i)\mu_{in}+l_{in}],$$

where $\nu_n=\max\{M_k^*\mu_{kn}, k\in I\}$, $n\geqslant 1$.

Again, applying Lemma 11, we obtain that $\lim\limits_{n\to\infty}\|x_n-q\|$ exists for any $q\in F$ and hence $\{x_n\}$ is a bounded sequence. □

Lemma 15. *Let E be a real uniformly convex Banach space and K a nonempty closed convex subset of E which is also a nonexpansive retract of E with the nonexpansive retraction $P:E\to K$. Let $\{T_i, i\in I\}$, be a finite family of m uniformly L-Lipschitzian total asymptotically nonexpansive mappings from K into E with a nonempty common fixed point set, F, such that*

$$\|T_i^n x-T_i^n y\| \leqslant \|x-y\|+\mu_{in}\phi_i(\|x-y\|)+l_{in}, \qquad n\geqslant 1, \quad i\in I,$$

where $\{\mu_{in}\}_{n=1}^{\infty}$ *and* $\{l_{in}\}_{n=1}^{\infty}$, $i \in I$ *are nonnegative real sequences with* $\sum_{n=1}^{\infty} \mu_{in} < \infty$, $\sum_{n=1}^{\infty} l_{in} < \infty$, $i \in I$ *and* $\phi_i : R^+ \to R^+$, $i \in I$ *are strictly increasing continuous functions with* $\phi_i(0) = 0$, $i \in I$. *Suppose that there exist constants* M_i, $M_i^* > 0$ *such that* $\phi_i(\lambda) \leqslant M_i^* \lambda$ *for all* $\lambda \geqslant M_i$, $i \in I$. *Let* $\{x_n\}$ *be the sequence defined by the iterative scheme (5), where* $\{\alpha_{in}\}$ *are sequences in* $[\epsilon, 1-\epsilon]$ *for some* $\epsilon \in (0,1)$, $i \in I$. *Then* $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$, $i \in I$.

Proof. Lemma 14 asserts that $\lim_{n\to\infty} \|x_n - q\|$ exists for any $q \in F \neq \emptyset$. Assume that

$$\lim_{n\to\infty} \|x_n - q\| = c. \tag{6}$$

Observe that for any positive integer j, $2 \leqslant j < m$, we have the following estimates

$$\begin{aligned} \|y_{n+m-j} - q\| \leqslant {} & [1 + M_j^* \mu_{jn} + (1 + M_j^* \mu_{jn}) M_{j+1}^* \mu_{(j+1)n} + \\ & + (1 + M_j^* \mu_{jn})(1 + M_{j+1}^* \mu_{(j+1)n}) M_{j+2}^* \mu_{(j+2)n} + \ldots + \\ & + (1 + M_j^* \mu_{jn})(1 + M_{j+1}^* \mu_{(j+1)n}) \ldots (1 + M_{m-1}^* \mu_{(m-1)n}) M_m^* \mu_{mn}] \|x_n - q\| + \\ & + \phi_j(M_j) \mu_{jn} + l_{jn} + (1 + M_j^* \mu_{jn}) [\phi_{(j+1)}(M_{(j+1)}) \mu_{(j+1)n} + l_{(j+1)n}] + \\ & + (1 + M_j^* \mu_{jn})(1 + M_{(j+1)}^* \mu_{(j+1)n}) [\phi_{(j+2)}(M_{(j+2)}) \mu_{(j+2)n} + l_{(j+2)n}] + \ldots + \\ & + (1 + M_j^* \mu_{jn})(1 + M_{(j+1)}^* \mu_{(j+1)n}) \ldots \\ & \ldots (1 + M_{(m-1)}^* \mu_{(m-1)n}) [\phi_m(M_m) \mu_{mn} + l_{mn}] \end{aligned} \tag{7}$$

and for any positive integer j, $1 \leqslant j < m$, we have

$$\|T_j(PT_j)^{n-1} y_{n+m-j-1} - q\| \leqslant (1 + M_j^* \mu_{jn}) \|y_{n+m-j-1} - q\| + \phi_j(M_j) \mu_{jn} + l_{jn}. \tag{8}$$

In the light of our assumptions, it follows from (7) and (8) that for each positive integer j, $1 \leqslant j < m$:

$$\limsup_{n\to\infty} \|T_j(PT_j)^{n-1} y_{n+m-j-1} - q\| \leqslant c. \tag{9}$$

It follows from (6) that

$$\begin{aligned} & \lim_{n\to\infty} \|x_{n+1} - q\| = \\ & \quad = \lim_{n\to\infty} \|(1 - \alpha_{1n})(x_n - q) + \alpha_{1n}(T_1(PT_1)^{n-1} y_{n+m-2} - q)\| = c. \end{aligned} \tag{10}$$

Using (6), (9), (10) and Lemma 12, we obtain

$$\lim_{n\to\infty} \|x_n - T_1(PT_1)^{n-1} y_{n+m-2}\| = 0. \tag{11}$$

Since

$$\begin{aligned}\|x_n - q\| &\leqslant \|x_n - T_1(PT_1)^{n-1} y_{n+m-2}\| + \|T_1(PT_1)^{n-1} y_{n+m-2} - q\| \leqslant \\ &\leqslant \|x_n - T_1(PT_1)^{n-1} y_{n+m-2}\| + (1 + M_1^* \mu_{1n})\|y_{n+m-2} - q\| + \\ &+ \phi_1(M_1)\mu_{1n} + l_{1n},\end{aligned}$$

using (11) and taking lim inf on both sides of the above inequality, we get

$$c \leqslant \liminf_{n\to\infty} \|y_{n+m-2} - q\|. \tag{12}$$

Inequality (7) implies

$$\limsup_{n\to\infty} \|y_{n+m-2} - q\| \leqslant c. \tag{13}$$

Now, (12) together with (13) yield

$$\lim_{n\to\infty} \|y_{n+m-2} - q\| = c,$$

which implies that

$$\lim_{n\to\infty} \|(1 - \alpha_{2n})(x_n - q) + \alpha_{2n}(T_2(PT_2)^{n-1} y_{n+m-3} - q)\| = c,$$

and similarly by (6), (9) and Lemma 12, we get

$$\lim_{n\to\infty} \|x_n - T_2(PT_2)^{n-1} y_{n+m-3}\| = 0.$$

Continuing in the same fashion we obtain

$$\lim_{n\to\infty} \|x_n - T_j(PT_j)^{n-1} y_{n+m-j-1}\| = 0 \qquad \text{for each } \ j \in \{1, 2, \ldots, m-1\} \tag{14}$$

and

$$\lim_{n\to\infty} \|y_{n+m-j} - q\| = 0 \qquad \text{for each } \ j \in \{2, 3, \ldots, m-1\}.$$

Now,

$$\begin{aligned}
&\|x_n - T_j(PT_j)^{n-1}x_n\| \leqslant \\
&\quad \leqslant \|x_n - T_j(PT_j)^{n-1}y_{n+m-j-1}\| + \|T_j(PT_j)^{n-1}y_{n+m-j-1} - T_j(PT_j)^{n-1}x_n\| \leqslant \\
&\quad \leqslant \|x_n - T_j(PT_j)^{n-1}y_{n+m-j-1}\| + \mu_{jn}\phi_j(\|y_{n+m-j-1} - x_n\|) + l_{jn} \leqslant \\
&\quad \leqslant \|x_n - T_j(PT_j)^{n-1}y_{n+m-j-1}\| + (1 + M_j^*\mu_{jn})\|y_{n+m-j-1} - x_n\| + \\
&\quad + \phi_j(M_j)\mu_{jn} + l_{jn} = \\
&\quad = \|x_n - T_j(PT_j)^{n-1}y_{n+m-j-1}\| + \phi_j(M_j)\mu_{jn} + l_{jn} + \\
&\quad + (1 + M_j^*\mu_{jn})\|(1-\alpha_{(j+1)n})x_n + \alpha_{(j+1)n}T_{j+1}(PT_{j+1})^{n-1}y_{n+m-j-2} - x_n\| \leqslant \\
&\quad \leqslant \|x_n - T_j(PT_j)^{n-1}y_{n+m-j-1}\| + \phi_j(M_j)\mu_{jn} + l_{jn} + \\
&\quad + (1 + M_j^*\mu_{jn})\alpha_{(j+1)n}\|T_{j+1}(PT_{j+1})^{n-1}y_{n+m-j-2}\|.
\end{aligned}$$

Therefore, it follows from (14) that

$$\lim_{n\to\infty} \|x_n - T_j(PT_j)^{n-1}x_n\| = 0 \qquad \text{for each } \ j \in \{1, 2, \ldots, m-1\}. \tag{15}$$

Since T_j is uniformly L-Lipschitzian for each $j \in I - \{m\}$, then noting that $x_n = Px_n$, $n \geqslant 1$, and that P is nonexpansive, we have the following estimate:

$$\begin{aligned}
&\|x_n - T_jx_n\| \leqslant \|x_n - T_j(PT_j)^{n-1}x_n\| + \|T_j(PT_j)^{n-1}x_n - T_j(PT_j)^{n-1}y_{n+m-j-1}\| + \\
&\quad + \|T_j(PT_j)^{n-1}y_{n+m-j-1} - T_jx_n\| \leqslant \\
&\quad \leqslant \|x_n - T_j(PT_j)^{n-1}x_n\| + \|x_n - T_j(PT_j)^{n-1}y_{n+m-j-1}\| + \\
&\quad + \|T_j(PT_j)^{n-1}x_n - x_n\| + L\|T_j(PT_j)^{n-2}y_{n+m-j-1} - x_n\|.
\end{aligned}$$

Hence, using (14) and (15), we get that

$$\lim_{n\to\infty} \|x_n - T_jx_n\| = 0 \qquad \text{for each } \ j \in I - \{m\}. \tag{16}$$

For $j = m$, we have

$$\begin{aligned}
&\|y_n - q\| = \|(1 - \alpha_{mn})(x_n - q) + \alpha_{mn}(T_m(PT_m)^{n-1}x_n - q)\| \leqslant \\
&\quad \leqslant (1 - \alpha_{mn})\|x_n - q\| + \alpha_{mn}\|T_m(PT_m)^{n-1}x_n - q\| \leqslant \\
&\quad \leqslant (1 - \alpha_{mn})\|x_n - q\| + \alpha_{mn}[(1 + M_m^*\mu_{mn})\|x_n - q\| + \phi_m(M_m)\mu_{mn} + l_{mn}] \leqslant \\
&\quad \leqslant (1 + M_m^*\mu_{mn})\|x_n - q\| + \phi_m(M_m)\mu_{mn} + l_{mn}.
\end{aligned}$$

Thus

$$\limsup_{n\to\infty} \|y_n - q\| \leqslant c. \tag{17}$$

On the other hand,

$$\begin{aligned}\|x_n - q\| &\leqslant \|x_n - T_{m-1}(PT_{m-1})^{n-1}y_n\| + \|T_{m-1}(PT_{m-1})^{n-1}y_n - q\| \leqslant \\ &\leqslant \|x_n - T_{m-1}(PT_{m-1})^{n-1}y_n\| + ((1 + M^*_{m-1}\mu_{(m-1)n}))\|y_n - q\| + \\ &+ \phi_{m-1}(M_{m-1})\mu_{(m-1)n} + l_{(m-1)n}.\end{aligned}$$

Again, (14) implies

$$\liminf_{n\to\infty} \|y_n - q\| \geqslant c. \tag{18}$$

It, therefore, follows from (17) and (18) that

$$\lim_{n\to\infty} \|y_n - q\| = c,$$

which is equivalent to

$$\lim_{n\to\infty} \|(1-\alpha_{mn})(x_n - q) + \alpha_{mn}(T_m(PT_m)^{n-1}x_n - q)\| = c. \tag{19}$$

Also,

$$\|T_m(PT_m)^{n-1}x_n - q\| \leqslant (1 + M^*_m\mu_{mn})\|x_n - q\| + \phi_m(M_m)\mu_{mn} + l_{mn},$$

which implies

$$\limsup_{n\to\infty} \|T_m(PT_m)^{n-1}x_n - q\| \leqslant c. \tag{20}$$

Finally, applying Lemma 12 in presence of (6), (19) and (20), we get that

$$\lim_{n\to\infty} \|x_n - T_m(PT_m)^{n-1}x_n\| = 0.$$

Moreover,

$$\begin{aligned}\|x_n - T_mx_n\| &\leqslant \|x_n - T_m(PT_m)^{n-1}x_n\| + \|T_m(PT_m)^{n-1}x_n - T_mx_n\| \leqslant \\ &\leqslant \|x_n - T_m(PT_m)^{n-1}x_n\| + L\|T_m(PT_m)^{n-2}x_n - x_n\|,\end{aligned}$$

so that

$$\lim_{n\to\infty} \|x_n - T_mx_n\| = 0.$$

This completes the proof. □

The following theorem extends Theorem 10 to the case of the class of total asymptotically nonexpansive mappings on uniformly convex Banach spaces.

Theorem 16. *Let E be a real uniformly convex Banach space and K a nonempty closed convex subset of E which is also a nonexpansive retract of E with the nonexpansive retraction $P : E \to K$. Let $\{T_i, i \in I\}$, be a finite family of m uniformly L-Lipschitzian total asymptotically nonexpansive mappings from K into E with a nonempty common fixed point set, F, such that*

$$\|T_i^n x - T_i^n y\| \leqslant \|x - y\| + \mu_{in}\phi_i(\|x - y\|) + l_{in}, \qquad n \geqslant 1, \quad i \in I,$$

where $\{\mu_{in}\}_{n=1}^{\infty}$ and $\{l_{in}\}_{n=1}^{\infty}$, $i \in I$ are nonnegative real sequences with $\sum_{n=1}^{\infty} \mu_{in} < \infty$, $\sum_{n=1}^{\infty} l_{in} < \infty$, $i \in I$ and $\phi_i : R^+ \to R^+$, $i \in I$ are strictly increasing continuous functions with $\phi_i(0) = 0$, $i \in I$. Suppose that there exist constants M_i, $M_i^* > 0$ such that $\phi_i(\lambda) \leqslant M_i^* \lambda$ for all $\lambda \geqslant M_i$, $i \in I$. Let $\{x_n\}$ be the sequence defined by the iterative scheme (4), where $\{\alpha_{in}\}$ are sequences in $[\epsilon, 1 - \epsilon]$ for some $\epsilon \in (0, 1)$, $i \in I$. If one of the mappings $\{T_i\}_{i=1}^{m}$ is either completely continuous or semicompact then $\{x_n\}$ converges strongly to a common fixed point of $\{T_i\}_{i=1}^{m}$.

Proof. The proof follows from Lemma 15 and the proof of Theorem 3.4 in [5]. □

The following corollary follows from Theorem 16.

Corollary 17. *Let E, K, $\{T_i, i \in I\}$ be as defined in Theorem 16. Let $\{x_n\}$ be the sequence defined by the following iterative scheme*

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1 - \alpha_{1n})x_n + \alpha_{1n}T_1^n y_{n+m-2}, \\ y_{n+m-2} = (1 - \alpha_{2n})x_n + \alpha_{2n}T_2^n y_{n+m-3}, \\ \vdots \\ y_n = (1 - \alpha_{mn})x_n + \alpha_{mn}T_m^n x_n, \qquad n \geqslant 1, \quad m \geqslant 2, \end{cases} \tag{21}$$

where $\{\alpha_{in}\}$ are sequences in $[\epsilon, 1 - \epsilon]$ for some $\epsilon \in (0, 1)$, $i \in I$. If one of the mappings $\{T_i\}_{i=1}^{m}$ is either completely continuous or semicompact then $\{x_n\}$ converges strongly to a common fixed point of $\{T_i\}_{i=1}^{m}$.

Remark 18.

(i) The main results of this paper are obviously valid for the case where $\{T_i : K \to E, i \in I\}$ is a finite family of N uniformly continuous asymptotically nonexpansive and uniformly continuous asymptotically nonexpansive in the intermediate sense nonself mappings with a nonempty common fixed point set.

(ii) Theorem 16 of this paper and its corollary trivially carry over to the class of total asymptotically quasi-nonexpansive mappings defined above with little or no modification.

(iii) Our results mainly generalize and extend those obtained by Chidume and Bashir [5].

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