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ON THE MACRO- AND MICRO-VIBRATIONS OF ELASTIC PLATES SUBJECTED TO PERIODICALLY DISTRIBUTED INERTIAL LOADINGS

Summary. The purpose of the work that follows is to present certain problems of dynamics of the linear elastic composite plates considered in the framework of the 2D Hencky-Reissner model. The starting point is the non-asymptotic, refined macrodynamics of microperiodic material structures proposed by prof. Cz. Woźniak. The paper presents the problem when microdynamics of the plate is connected with the periodical mass distribution.

MAKRO I MIKRO-DRGANIA PŁYT SPRĘŻYSTYCH WYWOŁANE PERIODYCZNYM ROZKŁADEM OBCIĄŻENIA INERCYJNEGO

Streszczenie. Przedmiotem tego opracowania są pewne zagadnienia płyt kompozytowych w ramach liniowej teorii sprężystości wg dwuwymiarowego (2D) modelu Hencky-Reissnera. Punktem wyjścia jest nieasymptotyczna, rafinowana makrodynamika mikroperiodycznych struktur materiałowych zaproponowana przez prof. Cz. Woźniaka. W artykule przedstawiono rozwiązanie zagadnienia, kiedy mikroperiodyczność płyty kompozytowej jest związana z rozkładem obciążających ją mas.

DIE MAKRO UND MIKRO-VIBRATION DER VERBUNDSTOFFPLATTEN VERURSACHT DURCH DIE PERIODISCHE VERTEILUNG DER INNERZIÄLEN BELASTUNG

Zusammenfassung. Die Bearbeitung hat gewisse Probleme der Dynamik der Verbundstoffplatten im Rahmen der linearen Elastizitätstheorie zum Gegenstand nach Hencky-Reissner Theorie. Den Ausgangspunkt gibt die von Professor Cz. Woźniak vorgeschlagene, nichtasymptotische, raffinierte Makrodynamik der mikro-periodischen Stoffgefüge ab. Im Artikel wurde die Lösung des Problems dargestellt, wenn die Mikroperiodizität der Verbundstoffplatte mit der Einteilung der belastenden Massen verbunden ist.

INTRODUCTION

A composite plate with a micro-periodic structure consists of many repeatable volumetric elements (with length dimensions much smaller than every characteristic dimension of a mid-plane of the plate) so called periodical cells. We assume simultaneously that each cell is a cuboid with a height equal to a thickness of the plate. Solving problems for such composite plates on the ground of the known equations of thin plates meets large analytical difficulties. It results of a fact that material functions and a distribution of mass are strongly oscillating and non-continuous. This makes it impossible to obtain solutions for many engineering problems. Thus, in the framework of micromechanics of the composite materials, simplified models are proposed in which features of the composite materials represent averaged (and constant) material stiffnesses [1-7,10]. These models, called homogenised, utilise an asymptotic approach and lead to differential equations with constant coefficients which considerably simplifies numerical procedures. However, such an approach neglects influence of a microstructure scale of the plate (i. e. dimensions of a periodic cell) on its dynamics. It makes research of vibrations impossible and waves of an order of the periodic dimensions cell length. As an example, for plates shown in the Fig. 1 which have the same dimensions L_1 and L_2, but different periodic scale, homogenised models yield identical effective stiffness and therefore identical effective results of dynamic problem solutions.

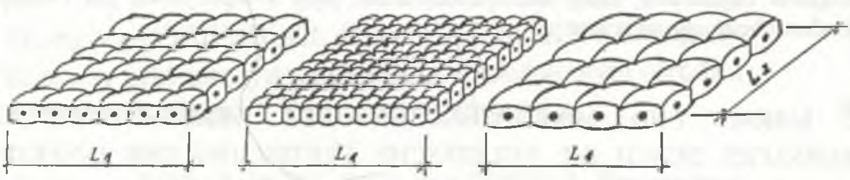

Fig. 1. Plates with different microstructure length parameters and identical values of the effective stiffnesses

Rys. 1. Płyty o różnej wartości parametru mikrostruktury i identycznych modułach efektywnych

Since in many engineering problems an influence of the periodic scale can not be neglected, then in [8, 9] there is proposed a non asymptotic, refined macrodynamics of microperiodical material structures. Bibliography of papers in this field can be found in [13]. In equations of this theory an influence of parameters describing dimensions of microstructure on macrodynamics behaviour of the body is taken into consideration. Moreover, the theory can be found a generalisation . of the asymptotic (effective stiffness) theory. In the case of the

Hencky-Reissner plate the refined theory was presented in [12] and for the Kirchhoff plate in [11].

The presented research utilises results of [12] and concerns the 2-D refined theory of the linear elastic thin composite plates with the micro-periodic structure. The aim of the work is to evaluate an influence of the plate microstructure length dimension on frequencies of proper vibrations under periodically distributed inertial loadings. There was proved that this influence plays a significant role in a research of the plates under periodical loadings of high frequencies.

Throughout the contribution summation convention holds. Subscripts $\alpha,\beta,\gamma...(i,j,k)$ run over $1,2$ $(1,2,3)$, indexes a,b,c and A,B,C takes the values $1,2,...,n$ and respectively $1,2,...,N$.

I. THEORETICAL FOUNDATIONS

Equations of the 2D refined theory of liner elastic composite plates with periodic structure were derived in [12]. For a sake of simplifying lecture of this contribution we quote main ideas of the above mentioned contribution.

Let $Ox_1x_2x_3$ be the orthogonal Cartesian co-ordinate system in the physical space. Let the considered plate occupies in a non-deformed configuration a region $\Omega = \Pi \times \left(-\frac{d}{2},\frac{d}{2}\right)$, where d is a thickness of the plate and Π is a regular region in plane Ox_1x_2. For orthogonal plates $\Pi = (0,L_1)\times(0,L_2)$. By $A=(0,l_1)\times(0,l_2)$ we describe periodic cell in the plane region Π, where $l=\sqrt{(l_1)^2+(l_2)^2}$ is called a microstructure parameter. We assume that $l \ll \min\{L_1,L_2\}$. For an arbitrary A-periodic integrable function $f(x_1,x_2)$, i.e. such that for each x_1,x_2 occurs $f(x_1,x_2)=f(x_1+l_1,x_2)=f(x_1,x_2+l_2)$ we define an averaging operator

$$\langle f\rangle \equiv \frac{1}{l_1 l_2}\int_A f(x_1,x_2)dx_1dx_2$$

where <f> is constant.

Coordinates of an arbitrary plate point before its deformation were described by x_1,x_2,z, where z is a distance of the point from the plane Ox_1x_2. Points of the mid-plane of the plate we denote by $\mathbf{x}=(x_1,x_2), \mathbf{x}\in\Pi$, by t-time coordinate $t>0$. By p^+,p^- are denoted loadings (along the axis x_3) on the top and bottom planes of the plate, respectively, by t_i loading on the surface $\partial\Pi \times \left(-\frac{d}{2},\frac{d}{2}\right)$, b is a constant body force (along the axis x_3) and ρ is a mass density. Further there was assumed that every plane z = const, $z \in \left(-\frac{d}{2},\frac{d}{2}\right)$, is a material symmetry plane and the material is linear elastic defined by a tensor of elasticity C_{ijkl}; hence $C_{3\alpha\beta\gamma} = C_{333\gamma} = 0$. According to the plane stress assumption we introduce modulae $D_{\alpha\beta\gamma\delta} = C_{\alpha\beta\gamma\delta} - C_{\alpha\beta33}C_{\gamma\delta33}(C_{3333})^{-1}$. Fields $C_{ijkl} = C_{ijkl}(\mathbf{x},z)$ and $\rho = \rho(\mathbf{x},z)$ are the A-periodic functions of an argument $\mathbf{x} = (x_1, x_2)$, and even functions of a variable z.

Displacements $u_i(\mathbf{x},z,t)$, $\mathbf{x} \in \Pi$, $i = 1,2,3$ at an arbitrary point of the microperiodic plate in the framework of [12] are defined by :

$$
\begin{aligned}
u_\alpha(\mathbf{x},z,t) &= z\left[\vartheta_\alpha(\mathbf{x},t) + h^a(\mathbf{x})\vartheta^a_\alpha(\mathbf{x},t)\right],\\
u_3(\mathbf{x},z,t) &= w(\mathbf{x},t) + g^A(\mathbf{x})w^A(\mathbf{x},t),
\end{aligned}
\tag{1}
$$

where new unknown functions w, ϑ_α were called macrodeflections and macrorotations, respectively , w^A, ϑ^a_α were called correctors for displacements and rotations, respectively. These unknown fields are so-called macrofunctions i. e. together with their derivatives are approximately constant in every element with dimension $l_1 \times l_2$. Functions $h^a(\mathbf{x}), g^A(\mathbf{x})$, $a = 1,2,...,n$, $A = 1,2,...,N$, are assumed to be A-periodic microshape functions, defining qualitatively disturbances of a displacement state evoked by a microperiodic structure or a loading of the plate. They satisfy the following conditions: $(i)\langle g^A_\alpha\rangle = 0$, $(ii)\, g^A(\mathbf{x}) \in O(l)$, $(iii)\, g^A_\alpha(\mathbf{x}) \in O(1)$, i. e. maximum values of g^A_α are independent of l. The same condition must satisfy functions h^a. Microshape functions take values of an order of the microstructure parameter l, i.e. their values are very small as compared to dimensions of the plate L_1, L_2, and their influence reveals in stresses and strains: maximal values of $h^a{}_{,\alpha}$ and $g^A{}_{,\alpha}$ are not small. As it was mentioned the microshape functions are assumed a priori, and their selection depends on a kind of the analysed problem, for example on the character of expected microvibrations of the microperiodic plate.

By the principle of virtual work, utilising presented in [12] hypotheses and defining constant coefficients:

$$
\mu \equiv \int_{-\frac{d}{2}}^{\frac{d}{2}} \rho\, dz,\quad J \equiv \int_{-\frac{d}{2}}^{\frac{d}{2}} z^2 \rho\, dz,\quad G_{\alpha\beta\gamma\delta} \equiv \int_{-\frac{d}{2}}^{\frac{d}{2}} z^2 D_{\alpha\beta\gamma\delta}\, dz,\quad C_{\alpha\beta} \equiv \int_{-\frac{d}{2}}^{\frac{d}{2}} C_{\alpha 3\beta 3}\, dz,\quad f \equiv p^+ + p^- + \langle\mu\rangle b.
$$

$$
J^a \equiv \langle Jh^a\rangle l^{-1},\quad J^{ab} \equiv \langle Jh^a h^b\rangle l^{-2},
$$

$$
\mu^A \equiv \langle \mu g^A\rangle l^{-1},\quad \mu^{AB} \equiv \langle \mu g^A g^B\rangle l^{-2}.
$$

There were obtained the following 2-D motion equations of the refined theory of a thin linear elastic composite plate with microperiodic structure:

$$
\begin{aligned}
&M_{\alpha\beta,\beta} - Q_\alpha - \langle J\rangle \ddot{\vartheta}_\alpha - lJ^a \ddot{\vartheta}^a_\alpha = 0, \\
&Q_{\alpha,\alpha} - \langle \mu\rangle \ddot{w} - l\mu^A \ddot{w}^A + f = 0, \\
&M^a_\alpha + l^2 J^{ab} \ddot{\vartheta}^b_\alpha + lJ^a \ddot{\vartheta}_\alpha = 0, \\
&Q^A + l^2 \mu^{AB} \ddot{w}^B + l\mu^A \ddot{w} = 0
\end{aligned} \tag{2}
$$

as well as the constitutive equations:

$$
\begin{aligned}
&M_{\alpha\beta} = \langle G_{\alpha\beta\gamma\delta}\rangle \vartheta_{(\gamma,\delta)} + \langle G_{\alpha\beta\gamma\delta}\, h^a_{,\gamma}\rangle \vartheta^a_\delta, \\
&Q_\alpha = \langle C_{\alpha\beta}\rangle \left(\vartheta_\beta + w_{,\beta}\right) + \langle C_{\alpha\beta}\, g^A_{,\beta}\rangle w^A, \\
&M^a_\alpha = \langle G_{\alpha\beta\gamma\delta}\, h^a_{,\beta}\rangle \vartheta_{(\gamma,\delta)} + \langle G_{\alpha\beta\gamma\delta}\, h^a_{,\beta} h^b_{,\gamma}\rangle \vartheta^b_\delta, \\
&Q^A = \langle C_{\alpha\beta}\, g^A_{,\alpha}\rangle \left(\vartheta_\beta + w_{,\beta}\right) + \langle C_{\alpha\beta}\, g^A_{,\alpha} g^B_{,\beta}\rangle w^B.
\end{aligned} \tag{3}
$$

Eqs. (2) and (3) have to be satisfied for every $\mathbf{x} \in \Pi$ and $t > 0$. The natural boundary conditions have the form: $M_{\alpha\beta} n_\beta = m_\alpha$, $Q_\alpha n_\alpha = q$ for $\mathbf{x} \in \partial\Pi$, $t > 0$, where n_α are is the unit normal outward to $\partial\Pi$, and m_α and q are defined by

$$
m_\alpha \equiv \int_{-\frac{d}{2}}^{\frac{d}{2}} z\, t_\alpha\, dz, \quad q \equiv \int_{-\frac{d}{2}}^{\frac{d}{2}} t_3\, dz .
$$

Eqs. (2) and (3) lead to the system of differential equations for unknown macrofunctions $w(\mathbf{x},t)$, $\vartheta_\alpha(\mathbf{x},t)$, $w^A(\mathbf{x},t)$, $\vartheta^a_\alpha(\mathbf{x},t)$. A discussion of the above equations can be found in [12].

II. HOMOGENOUS PLATES SUBJECTED TO MICRO-PERIODIC INERTIAL LOADINGS

Let us consider the problem, where the micro-periodic structure of the composite plate is due exclusively to a periodic distribution of masses. Under this assumption for every microshape function we obtain

$$
\begin{aligned}
&\left\langle G_{\alpha\beta\gamma\delta} h^a_{,\alpha}\right\rangle = G_{\alpha\beta\gamma\delta}\left\langle h^a_{,\alpha}\right\rangle = 0, \\
&\left\langle C_{\alpha\beta} g^A_{,\alpha}\right\rangle = C_{\alpha\beta}\left\langle g^A_{,\alpha}\right\rangle = 0.
\end{aligned}
$$

Then the constitutive equations (3) take the form

$$M_{\alpha\beta} = \langle G_{\alpha\beta\gamma\delta} \rangle \vartheta_{(\gamma,\delta)}, \quad M^{a}_{\alpha} = \langle G_{\alpha\beta\gamma\delta} h^{a}_{,\beta} h^{b}_{,\gamma} \rangle \vartheta^{b}_{\delta},$$
$$Q_{\alpha} = \langle C_{\alpha\beta} \rangle (\vartheta_{\beta} + w_{,\beta}), \quad Q^{A} = \langle C_{\alpha\beta} g^{A}_{,\alpha} g^{B}_{,\beta} \rangle w^{B} \tag{4}$$

and the equations of motion (2) are

$$\begin{aligned} &Q_{\alpha,\alpha} - \langle \mu \rangle \ddot{w} - l\mu^{A} \ddot{w}^{A} + f = 0, \\ &M_{\alpha\beta,\beta} - Q_{\alpha} - \langle J \rangle \ddot{\vartheta}_{\alpha} - lJ^{a} \ddot{\vartheta}^{a}_{\alpha} = 0, \\ &M^{a}_{\alpha} + l^{2} J^{ab} \ddot{\vartheta}^{b}_{\alpha} + lJ^{a} \ddot{\vartheta}_{\alpha} = 0, \\ &Q^{A} + l^{2} \mu^{AB} \ddot{w}^{B} + l\mu^{A} \ddot{w} = 0. \end{aligned} \tag{5}$$

By neglecting in Eqs. (4) and (5) terms depending on rotational inertia terms i.e. under formal assumption that $J=0$, $J^{a} = J^{ab} = 0$, correctors of rotations are equal to zero, $\vartheta^{a}_{\alpha} = 0$, and the equations of motion take the form:

$$\begin{aligned} &Q_{\alpha,\alpha} - \langle \mu \rangle \ddot{w} - l\mu^{A} \ddot{w}^{A} + f = 0, \\ &M_{\alpha\beta,\beta} - Q_{\alpha} = 0, \\ &Q^{A} + l^{2} \mu^{AB} \ddot{w}^{B} + l\mu^{A} \ddot{w} = 0. \end{aligned} \tag{6}$$

It is obvious that such a simplification involves exclusively microshape functions g^{A}, describing qualitatively an expected form of the disturbances in deflection of the plate caused by the microperiodicity of the microstructure or loadings. The example of such disturbances for one span plate band was shown on Fig. 2 where on the macrodeflections w the microdisturbances $g^{A} w^{A}$ are superimposed.

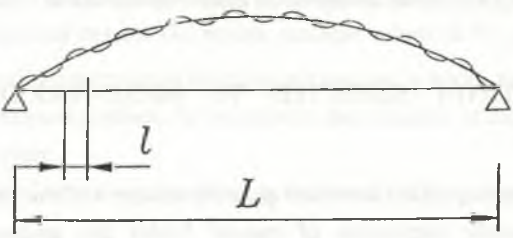

Fig. 2. Macro- and micro- vibrations of a simply supported l-periodic thin plate

Rys.2. Makro- i mikro-drgania swobodnie podpartej, l-periodycznej płyty cienkiej

III. EXAMPLE OF APPLICATIONS: VIBRATIONS ON THE BRIDGE TYPE PLATES

3.1. General equations

Let us consider a composite plate band of the thickness d and simply supported on edges $x_1 = 0$ and $x_1=L$. Along the axis x_1 there are distributed periodically concentrated masses with a period l ($l<<L$). The cell of periodicity is now one dimensional and bounded by the coordinates $x_1 = 0$, $x_1= l$. For the analysis of microdynamics of the plate there were assumed two microshape functions $h = h^1(x_1)$ and $g = g^1(x_1)$, where each of them has a form $l\, sin\frac{2\pi}{l}x_1$ or $l\, cos\frac{2\pi}{l}x_1$.

Denoting:

$$
\begin{aligned}
&G \equiv \langle G_{1111}\rangle, \quad G^{11} \equiv \left\langle G_{1111}(h_{,1})^2\right\rangle,\\
&C \equiv \langle C_{11}\rangle, \quad C^{11} \equiv \left\langle C_{11}(g_{,1})^2\right\rangle,\\
&m \equiv \langle \mu\rangle, \quad \mu^1 \equiv \langle \mu g\rangle l^{-1}, \quad \mu^{11} \equiv \left\langle \mu(g_{,1})^2\right\rangle l^{-2},\\
&j \equiv \langle J\rangle, \quad J^1 \equiv \langle Jh\rangle l^{-1}, \quad J^{11} \equiv \left\langle J(h_{,1})^2\right\rangle l^{-2},
\end{aligned}
$$

and defining in Eqs.(4) and (5) $W = w^1$, $\vartheta = \vartheta^1$, $\Theta = \vartheta_1^1$ we obtain a system of equations for macrodeflections w, macrorotations ϑ and correctors $W$and Θ:

$$
\begin{aligned}
&C(\vartheta_{,1} + w_{,11}) - m\ddot{w} - l\mu^1\ddot{W} + f = 0,\\
&G\vartheta_{,11} - C(\vartheta + w_{,1}) - j\ddot{\vartheta} - lJ^1\ddot{\Theta} = 0,\\
&G^{11}\Theta + J^{11}l^2\ddot{\Theta} + J^1 l\ddot{\vartheta} = 0,\\
&C^{11}W + \mu^{11}l^2\ddot{W} + \mu^1 l\ddot{w} = 0.
\end{aligned}
\tag{7}
$$

Assuming that the plate is subjected to the time-dependent loading $f = f_o\, sin(kx_1)cos(\omega t)$, $f_o \neq 0$, $k = 2\pi / L$, a solution of Eqs.(7) can be presented in the form: $w = w_o\, sin(kx_1)cos(\omega t)$, $W = W_o\, cos(kx_1)cos(\omega t)$, $\vartheta = \vartheta_o\, cos(kx_1)cos(\omega t)$, $\Theta = \Theta_o\, sin(kx_1)cos(\omega t)$, where $w_o, W_o, \vartheta_o, \Theta_o$ are constant vibrations amplitude. Substituting this solution to Eq.(7), linear system of algebraic equations for amplitudes $w_o, W_o, \vartheta_o, \Theta_o$ was obtained:

$$\begin{bmatrix} k^2G - m\omega^2 & kC & 0 & -\mu^1 l\omega^2 \\ kC & k^2G + C - j\omega^2 & -J^1 l\omega^2 & 0 \\ 0 & -J^1 l\omega^2 & G^{11} - J^{11} l^2\omega^2 & 0 \\ -\mu^1 l\omega^2 & 0 & 0 & C^{11} - \mu^{11} l^2\omega^2 \end{bmatrix} \begin{bmatrix} w_o \\ \vartheta_o \\ \Theta_o \\ W_o \end{bmatrix} = \begin{bmatrix} f_o \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{8}$$

Eliminating W_o and Θ_o

$$W_o = \frac{\mu^1 l\omega^2}{C^{11} - \mu^{11} l^2\omega^2} w_o, \quad \Theta_o = \frac{J^1 l\omega^2}{G^{11} - J^{11} l^2\omega^2} \vartheta_o$$

and denoting:

$$m_\omega = m + \frac{(\mu^1)^2 l^2\omega^2}{C^{11} - \mu^{11} l^2\omega^2}, \quad j_\omega = j + \frac{(J^1)^2 l^2\omega^2}{G^{11} - J^{11} l^2\omega^2},$$

we arrive at result the system of equations for w_o and ϑ_o:

$$\begin{bmatrix} k^2C - m_\omega\omega^2 & kC \\ kC & k^2G + C - j_\omega\omega^2 \end{bmatrix} \begin{bmatrix} w_o \\ \vartheta_o \end{bmatrix} = \begin{bmatrix} f_o \\ 0 \end{bmatrix} \tag{9}$$

Determinant $D(\omega)$ of Eq.(9) has a form:

$$D(\omega) = \omega^4 m_\omega j_\omega - \omega^2\left[k^2C\, j_\omega + (k^2G + C)m_\omega\right] + k^4GC.$$

From $D(\lambda) = 0$ resonance frequencies λ of the plate can be calculated. If the scale effect on the plate dynamics is neglected, i.e. if we put $l \searrow 0$, then we obtain $m_\omega = m$, $j_\omega = j$.

Denoting

$$\Delta = \left[k^2C\, j + (k^2G + C)m\right]^2 - 4k^4GCmj,$$

it can be proved that the macro-resonance frequencies λ^I, λ^{II} are equal to:

$$\begin{aligned} (\lambda^I)^2 &= \frac{1}{2mj}\left[k^2Cj + (k^2G + C)m - \sqrt{\Delta}\right] + O(l^2), \\ (\lambda^{II})^2 &= \frac{1}{2mj}\left[k^2Cj + (k^2G + C)m + \sqrt{\Delta}\right] + O(l^2), \end{aligned} \tag{10}$$

where $\Delta > 0$.

It can be also proved, [12], that the micro-resonance frequencies $\lambda^{III}, \lambda^{IV}$, are:

$$\begin{aligned}
\left(\lambda^{III}\right)^2 &= \frac{1}{m\mu^{11} - \left(\mu^1\right)^2}\left[\frac{C^{11}m}{l^2} + k^2 C\frac{\left(\mu^1\right)^2}{m}\right] + O\left(l^2\right),\\
\left(\lambda^{IV}\right)^2 &= \frac{1}{jJ^{11} - \left(J^1\right)^2}\left[\frac{G^{11}j}{l^2} + \left(k^2 G + C\right)\frac{\left(J^1\right)^2}{j}\right] + O\left(l^2\right),
\end{aligned} \tag{11}$$

where

$$m\mu^{11} - \left(\mu^1\right)^2 > 0 \text{ and } jJ^{11} - \left(J^1\right)^2 > 0.$$

Neglecting the rotational inertia i.e. assuming that $j = J^1 = J^{11} = 0$ we obtain from (6) the following system of equations for w, W and ϑ:

$$\begin{aligned}
&C\left(\vartheta_{,1} + w_{,11}\right) - m\ddot{w} - l\mu^1\ddot{W} + f = 0,\\
&G\vartheta_{,11} - C\left(\vartheta + w_{,1}\right) = 0,\\
&C^{11}W + \mu^{11}l^2\ddot{W} + \mu^1 l\ddot{w} = 0.
\end{aligned} \tag{12}$$

Assuming that $f = f_o \sin(kx_1)\cos(\omega t)$, $w = w_o \sin(kx_1)\cos(\omega t)$, $W = W_o \cos(kx_1)\cos(\omega t)$, $\vartheta = \vartheta_o \cos(kx_1)\cos(\omega t)$, solution of the system (12) can be written down in the presented above form, obtaining system of the linear algebraic equations for the amplitudes w_o, ϑ_o, W_o:

$$\begin{bmatrix} k^2C - m\omega^2 & kC & -\mu^1 l\omega^2 \\ kC & k^2G + C & 0 \\ -\mu^1 l\omega^2 & 0 & C^{11} - \mu^{11}l^2\omega^2 \end{bmatrix}\begin{bmatrix} w_o \\ \vartheta_o \\ W_o \end{bmatrix} = \begin{bmatrix} f_o \\ 0 \\ 0 \end{bmatrix} \tag{13}$$

Eliminating W_o from this system of equations we get:

$$W_o = \frac{\mu^1 l^2\omega^2}{C^{11} - \mu^{11}l^2\omega^2} w_o$$

and preserving presented above definition of the quantity m_ω, a system of equations for w_o and ϑ_o takes the form:

$$\begin{bmatrix} k^2C - m_\omega\omega^2 & kC \\ kC & k^2G + C \end{bmatrix}\begin{bmatrix} w_o \\ \vartheta_o \end{bmatrix} = \begin{bmatrix} f_o \\ 0 \end{bmatrix} \tag{14}$$

Determinant $D(\omega)$ of Eq (14) is equal to:

$$D(\omega) = k^4GC - m_\omega\left(k^2G + C\right)\omega^2.$$

From $D(\lambda) = 0$ the resonance frequencies λ were calculated. The macro-resonance frequency λ^I neglecting of the scale effect (terms containing l drop out), leads to $m_\omega = m$, is equal to:

$$\left(\lambda^I\right)^2 = \frac{k^4GC}{m\left(k^2G + C\right)} + O\left(l^2\right). \tag{15}$$

The micro-resonance frequency λ^{III} is:

$$\left(\lambda^{III}\right)^2 = \frac{1}{m\mu^{II} - \left(\mu^I\right)^2}\left[\frac{C^{II}m}{l^2} + \frac{k^4GC}{m\left(k^2G + C\right)}\left(\mu^I\right)^2\right] + O\left(l^2\right). \tag{16}$$

3.2. Homogenous and isotropic plate band

A homogenous and isotropic plate band with a thickness d, a mass density ρ, Young modulus E and Poisson coefficient ν, is loaded periodically along the x_1-axis by system of the concentrated masses of the value M, having the inertial moment I related to x_1-axis per unit length of this band (cf. Fig 3). For the purpose of calculation reduced loading mass densities are defined:

$$\rho_M = \frac{M}{ld} \quad , \qquad \rho_I = \frac{12I}{ld^3} \quad .$$

For the isotropic material described by Young modulus E and Poisson coefficient ν we obtain:

$$D_{1111} = \frac{E}{1-\nu} \quad , \qquad C_{1313} = \frac{E}{1+\nu} \quad .$$

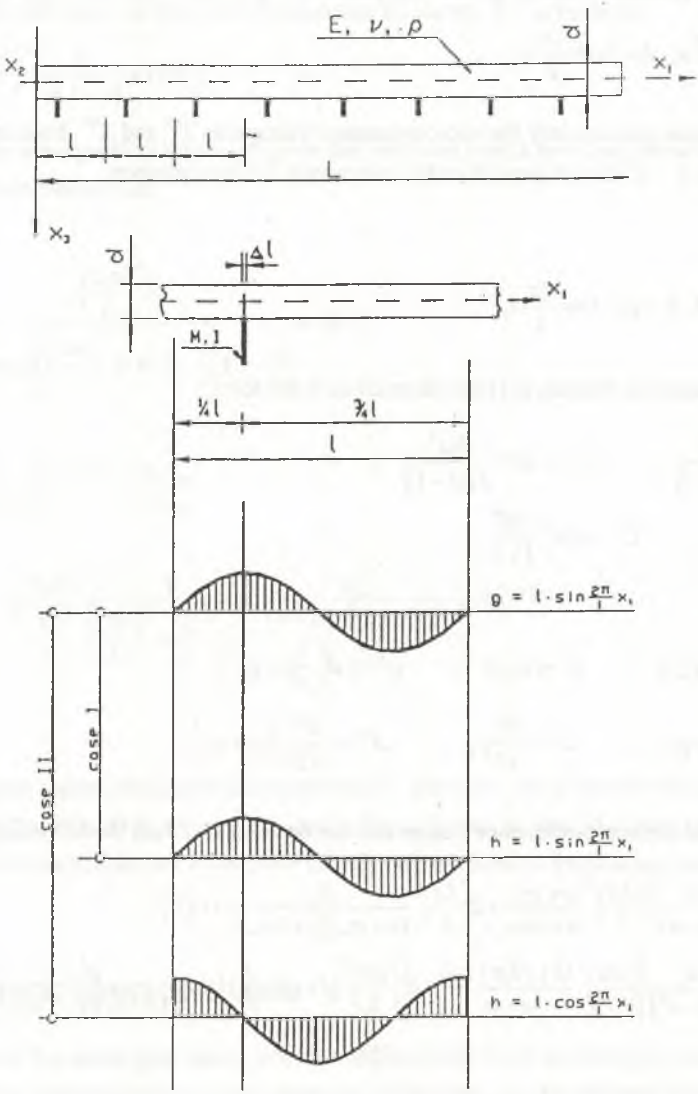

Fig. 3. A scheme of the periodic mass distribution and diagrams of the microshape functions inside a periodicity cell

Rys. 3. Schemat periodycznego obciążenia inercyjnego i wykresy funkcji mikrokształtu wewnątrz komórki periodyczności

Calculations of the resonance frequencies λ were carried out for two cases of the microshape function h and g. In the first variant $h = g = l\, sin\frac{2\pi}{l}x_1$, and in the second one $g = l\, sin\frac{2\pi}{l}x_1\,, h = l\, cos\frac{2\pi}{l}x_1$.

Calculations concern only the micro-resonance frequencies λ^{III} and λ^{IV} because the macro-frequencies λ^{I} , λ^{II} do not depend on the microshape function selection.

1. Case I, $h = g = l\, sin\frac{2\pi}{l}x_1$

The constants in formula (11) can be assumed in the form:

$$G = \frac{Ed^3}{12(1-\nu)}\quad,\qquad G^{II} = 2\pi^2\frac{Ed^3}{12(1-\nu)}\quad,$$
$$C = \frac{Ed}{1+\nu}\quad,\qquad C^{II} = 2\pi^2\frac{Ed}{1+\nu}\quad,$$

$$m = d(\rho + \rho_M)\quad,\qquad \mu^{I} = \rho_M d\quad,\qquad \mu^{II} = d\left(\frac{1}{2}\rho + \rho_M\right)\quad,$$
$$j = \frac{d^3}{12}(\rho + \rho_I)\quad,\qquad J^{I} = \frac{d^3}{12}\rho_I\quad,\qquad J^{II} = \frac{d^3}{12}\left(\frac{1}{2}\rho + \rho_I\right)\quad.$$

Therefore the micro-resonance frequencies can be calculated from the following formulas:

$$\begin{aligned}
\left(\lambda^{III}\right)^2 &= \frac{E}{\rho(1+\nu)}\left[\left(\frac{2\pi}{l}\right)^2\frac{\rho+\rho_M}{\rho+3\rho_M} + 2\left(\frac{2\pi}{L}\right)^2\frac{\rho_M^2}{(\rho+\rho_M)(\rho+3\rho_M)}\right] + O(l^2)\ ,\\
\left(\lambda^{IV}\right)^2 &= \frac{E}{\rho(1-\nu^2)}\left\{\left(\frac{2\pi}{l}\right)^2\frac{(1+\nu)(\rho+\rho_I)}{\rho+\rho_I} + 2\left[\left(\frac{2\pi}{L}\right)^2(1+\nu) + \frac{1}{d^2}(1-\nu)\right]\frac{\rho_I^2}{(\rho+\rho_I)(\rho+3\rho_I)}\right\} + O(l^2).
\end{aligned}\tag{17}$$

2. Case II $g = l\, sin\frac{2\pi}{l}x_1\,, h = l\, cos\frac{2\pi}{l}x_1$.

In this case $J^{I} = 0$, $J^{II} = \frac{1}{2}\rho d^{3}$, and the remaining constants calculated for the first case undergo no change. So for the different microshape function h the resonance frequency λ^{III} is identical as for the case I, in the contrary resonance frequency λ^{IV} is equal to:

$$\left(\lambda^{IV}\right)^{2} = \left(\frac{2\pi}{l}\right)^{2}\frac{E}{\rho(1-\nu)} + O\left(l^{2}\right). \tag{18}$$

Solving the same problem with negligence the rotational inertia terms we obtain two values of the resonance frequencies:

(i) *macro* –

$$\left(\lambda^{I}\right)^{2} = \frac{\left(\frac{2\pi}{L}\right)^{4} E}{(\rho+\rho_{M})\left[\left(\frac{2\pi}{L}\right)^{2}(1+\nu) + \frac{12}{d^{2}}(1-\nu)\right]} + O\left(l^{2}\right)$$

(ii) *micro* – (19)

$$\left(\lambda^{III}\right)^{2} = \frac{E}{\rho(1+\nu)}\left[\left(\frac{2\pi}{l}\right)^{2}\frac{\rho+\rho_{M}}{\rho+3\rho_{M}} + 2\left(\frac{2\pi}{L}\right)^{4}\frac{1}{\left(\frac{2\pi}{L}\right)^{2} + \frac{12}{d^{2}}\frac{1-\nu}{1+\nu}}\cdot\frac{\rho_{M}^{2}}{(\rho+\rho_{M})(\rho+3\rho_{M})}\right] + O\left(l^{2}\right).$$

The diagrams representing interrelation between λ and k (i.e. the dispersion lines), given in a general form by Eqs. (10), (11), are shown in Fig. 4. It can be seen that the values of free micro-vibration frequencies are of an order of high macro-vibration frequencies, provided that $l/d \cong 1$.

IV. ASYMPTOTIC HOMOGENISATION

Equations of the asymptotic theory (effective stiffness) which do not take into consideration influence of the microstructure size on dynamics of the plate, can be obtained from Eqs. (7) neglecting terms of an order of microstructure parameter rate l. In that case:

$$m_{eff} = m, \qquad j_{eff} = j,$$

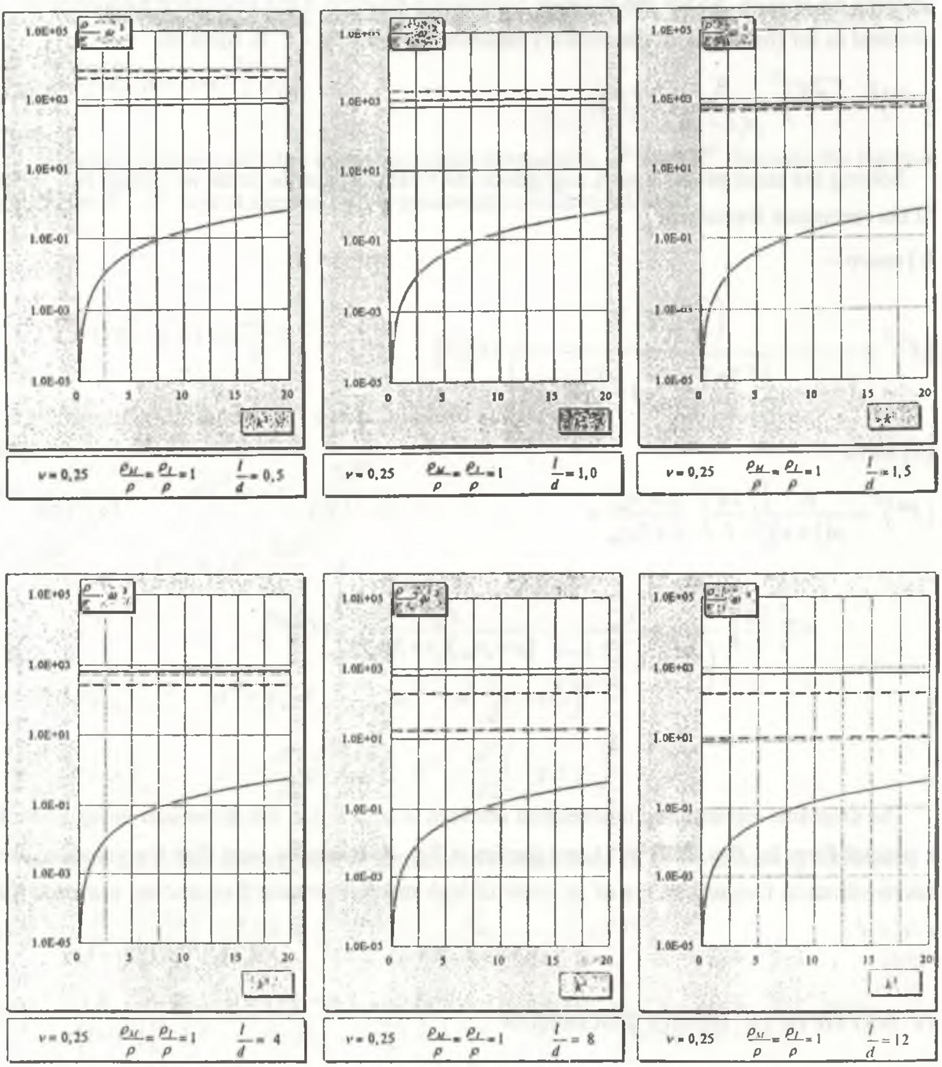

Fig. 4. Diagrams of the dispersion lines for different values of *l:d*, related to the macro- (continuos lines) and micro- (broken lines) resonance vibrations frequencies.

Rys. 4. Wykresy krzywych dyspersyjnych w zależności od l:d, odpowiednio, makro- (linie ciągłe) i mikro- (linie przerywane) częstości drgań rezonansowych which according to the presented in section III way of proceeding and assumed notations leads to the following equations for the amplitudes w_o and ϑ_o :

$$\begin{bmatrix} k^2C - m_{eff}\omega^2 & kC \\ kC & k^2G + C - j_{eff}\omega^2 \end{bmatrix} \begin{bmatrix} w_o \\ \vartheta_o \end{bmatrix} = \begin{bmatrix} f_o \\ 0 \end{bmatrix}. \tag{20}$$

Eqs. (20) of the theory of effective stiffness are identical with Eqs. (9) of the refined theory for the macro resonance frequencies for the dynamic coefficients m_ω and j_ω modulo terms $O(l^2)$. Determinant of Eqs. (20) has the form:

$$D_o(\omega) = \omega^4 m_{eff} j_{eff} - \omega^2\left[k^2C\, j_{eff} + \left(k^2G + C\right)m_{eff}\right] + k^4GC.$$

From $D_o(\lambda_o) = 0$ we obtain the resonance frequencies. Denoting:

$$\Delta = \left[k^2C\, j_{eff} + \left(k^2G + C\right)m_{eff}\right]^2 - 4k^4GCm_{eff}\, j_{eff}$$

these frequencies are equal to:

$$\begin{aligned} \left(\lambda_o^I\right)^2 &= \frac{1}{2m_{eff}\, j_{eff}}\left[k^2C\, j_{eff} + \left(k^2G + C\right)m_{eff} - \sqrt{\Delta}\right], \\ \left(\lambda_o^{II}\right)^2 &= \frac{1}{2m_{eff}\, j_{eff}}\left[k^2C\, j_{eff} + \left(k^2G + C\right)m_{eff} + \sqrt{\Delta}\right]. \end{aligned} \tag{21}$$

It can be proved that $\Delta > 0$. Neglecting the rotational inertia terms we obtain, using a procedure similar to that of Sec. 3.1., the resonance frequency:

$$\left(\lambda_o^I\right)^2 = \frac{k^4GC}{m_{eff}\left(k^2G + C\right)}.$$

V. CONCLUSIONS

The example given in Section 4 illustrates possibilities of a 2-D theory of plates for an analysis of the composite plates microdynamics problems. It can be noted that refined theory equations have a relatively simply form. New unknown functions, correctors w^A and ϑ_α^A, are determined by the ordinary differential equations involving only time derivatives of w^A and ϑ_α^A. The same phenomena described by correctors are independent of the plate boundary conditions.

The asymptotic equations (effective stiffness formulations) are obtained from the equations of the refined theory by a formal negligence of the scale parameter l. Results from an asymptotic theory are certain approximations of the refined theory results, when we confine ourselves to macrodynamics problems.

Comparing the results obtained in the framework of the effective stiffness method with those of a refined theory it should be stated that:

(i) macro-resonance frequencies λ_o^I and λ_o^{II} obtained from the effective stiffness theory are comparable with frequencies λ^I and λ^{II}, respectively, calculated from the refined theory with an accuracy of $O(l^2)$:

$$(\lambda^I)^2 = (\lambda_o^I)^2 + O(l^2), \qquad (\lambda^{II})^2 = (\lambda_o^{II})^2 + O(l^2).$$

(ii) micro-resonance frequencies λ^{III}, λ^{IV}, cf. Eqs. (11), can not be calculated on the basis of the effective stiffness theory,

(iii) the effect of the composite plates' microstructure (i. e. the influence of a parameter l) on dynamics of the plate should be considered for the high frequencies of loadings when they are close to the resonance frequencies λ^{III}, λ^{IV}.

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Streszczenie

Przedmiotem tego opracowania są pewne zagadnienia dynamiki płyt kompozytowych w ramach liniowej teorii sprężystości.

Struktura analizowanej płyty nie jest dowolna lecz okresowa i można ją podzielić na powtarzające się tzw. komórki periodyczności. Punktem wyjścia jest nieasymptotyczna, rafinowana makrodynamika mikroperiodycznych struktur materiałowych, zaproponowana przez Cz. Woźniaka.

W artykule przedstawiono rozwiązanie zagadnienia, kiedy mikroperiodyczność płyty kompozytowej jest związana z rozkładem obciążających ją mas. Przykład dotyczy

zastosowania rafinowanej teorii płyt kompozytowych do obliczania częstości drgań własnych pasma płytowego. Rozwiązanie uzależniono od doboru funkcji mikrokształtu, a także podano wzory na częstości własne przy pominięciu członów obrotowych. Porównano otrzymane wyniki z metodą homogenizacji asymptotycznej (modułów efektywnych). Na wykresach zamieszczono krzywe dyspersyjne w zależności od skali mikrostruktury.