ZESZYTY NAUKOWE POLITECHNIKI ŚLĄSKIEJ 1995

Seria: BUDOWNICTWO z.83 Nr kol. 1314
THEORETICAL MECHANICS

Tadeusz BURCZYŃSKI
Jerzy SKRZYPCZYK

THE FUZZY BOUNDARY ELEMENT METHOD: A NEW METHODOLOGY

Summary. In the paper basic concepts of a new methodology of the fuzzy boundary element method are presented. This paper deals with fuzzy-set--valued mappings which are solutions of the fuzzy boundary integral equations. Exact fuzzy solutions of fuzzy boundary integral equations are introduced as well as aproximated ones. Computational fuzzy problems and applications are considered in details for boundary potential problems with fuzzy Dirichlet and Neumann type boundary conditions and fuzzy density source functions.

METODA ROZMYTYCH ELEMENTÓW BRZEGOWYCH: NOWE SFORMUŁOWANIE PROBLEMU

Streszczenie. W pracy przedstawione zostały podstawowe założenia nowej koncepcji w metodzie elementów brzegowych, nazwanej metodą rozmytych elementów brzegowych. Zdefiniowane zostały pojęcia funkcji rozmytych, które odgrywają rolę analogiczną do dokładnych i przybliżonych rozwiązań rozmytych brzegowych równań całkowych. Szczegółowe rozważania dotyczące problemów obliczeniowych i zastosowań przedstawiono na przykładzie rozmytego brzegowego równania całkowego wynikającego z zagadnienia brzegowego dla równania Poissona z niejednorodnymi, rozmytymi warunkami brzegowymi typu Dirichleta i Neumanna oraz z niezerową, rozmytą funkcją gęstości źródeł określoną w całym rozpatrywanym obszarze.

МЕТОД РАЗМЫТЫХ КРАЕВЫХ ЭЛЕМЕНТОВ: НОВЫЙ ПОДХОД

Резюме. В работе представлены основания нового подхода к методу размытых краевых элементов названного методом размытых краевых элементов. Определено размытые решения размытых краевых интегральных уравнений. Введены новые понятия точного и приближенного размытого решения размытого краевого интегрального уравнения. Вычислительные размытые проблемы и применения рассмотрены на примерах связанных с потенциальной краевой задачей с размытыми краевыми условиями типа Дирихлета и Неймана вместе с предположением, что функция плотности расположения источников является размытой функцей определенной в целой области.

1. INTRODUCTION

Because a very high safety level is required in civil engineering structures the uncertainty associated with the application of scientific calculations is very crudely and conservatively estimated using traditional methods. Current reliability theory enables a discussion of random parameter uncertainty but in civil engineering, system uncertainty and the possibility of human error is extremely important and must be included in any reliability calculations.

In civil engineering particularly the available theory never quite fits the actual problem and there is rarely the chance to test prototype as in other engineering industries. This is because civil engineering projects tend to be one of jobs whereas in the manufacturing industries production line techniques may be used to manufacture large quantities of the same product. Thus the uncertainty in applying theoretical solutions to practical civil engineering problems is large. The designer has obviously to take this into account because the standard of safety required by the general public concerning bridges, buildings and other structures is extremely high.

It is apparent from the previous discussion that there is a large amount of uncertainty surrounding the execution of any engineering project. The nature of this uncertainty can be disscused under three headings: human based uncertainty, system uncertainty and random uncertainty. The prediction of these three types of uncertainty is difficult and present methods, embodied in reliability theory, tend to concentrate on random uncertainty. There is, however, a fundamental difference between the nature of random uncertainty and that of human and system uncertainty. To analyse this type of uncertainty a mathematics which is directed at "vagueness" as distinct from randomness is required and this is the potential role of fuzzy sets.

The actual likelihood of a structure failing is a function of one or more of some factors. However, current reliability theory can only satisfactorily deal with that category where uncertainty is due mainly to random parameters. However, a more detailed analysis using fuzzy approximate reasoning may well be possible.

Finite element methods as well as boundary element methods (BEM) play a very important role in civil engineering since they are widely used in analysing structures. Stochastic boundary element method is an alternative numerical technique relative to the first type uncertainties i.e. random problems [11]. In presented paper some preliminaries are given of calculations appropriate rather to the second category which involve the use of fuzzy sets to estimate system uncertainty in BEM, called further fuzzy boundary element method (FBEM).

Notice that when a physical problem is transformed into the deterministic boundary problem, we usually cannot be sure that this modelling is perfect. The boundary problem may not be known exactly and some functions i.e. boundary conditions, external or internal excitations, solutions etc. may contain unknown parameters. Especially, if they are known through some measurements they necessarily are subjected to errors. The analysis of the efect of these errors leads to the study of the qualitative and quantitive behavior of the solution uncertainty.

If the nature of errors is random, then instead of a deterministic problem we can get a random boundary integral equation with stochastic functions and/or random coefficients, comp. [11]. But if the underlying structure is not probabilistic, e.g. because of subjective choices, then it may be appropriate to use fuzzy numbers instead of real random variables. A fuzzy number $\tilde{a}$ is called a fuzzy set of real numbers, i.e. there exists a function $\mu(\cdot|\tilde{a})$: $\mathbb{R} \longrightarrow [0,1]$ whose value $\mu(x|\tilde{a})$, $x \in \mathbb{R}$ is the grade of membership of x in $\tilde{a}$. This leads to a fuzzy boundary value problems and in consequence to fuzzy boundary integral equations (FBIE), comp. [12].

2. ELEMENTARY CONCEPTS AND RESULTS

In this paper, the following concepts and notations will be used. $\mathbb{R}^n$ was reserved for the set of n-dimensional reals, $(\mathbb{R}^n, |\cdot|)$-denotes the Euclidean space with metric $|\cdot|$, Γ is an arbitrary fixed n-dimensional manifold in Euclidean space, $\mathcal{A}$ is a σ-algebra formed by the subsets of Γ, $(\Gamma, \mathcal{A}, d\Gamma)$ is a classical complete and finite measure space (nonfuzzy), $\vee, \wedge$ will stand for "supremum" and "infimum" respectively.

Let I($\mathbb{R}$) denote the set of all closed bounded intervals $\bar{z}=[z^-, z^+]$ on the real line $\mathbb{R}$, where z^- and z^+ denote the end points of $\bar{z}$. We call further elements of I($\mathbb{R}$) interval numbers. For further information we refer to [2,21].

By a fuzzy real vector we understand a fuzzy set $\tilde{a} \in \mathbb{R}^n$ i.e. a mapping $\mu(\cdot|\tilde{a}): \mathbb{R}^n \longrightarrow [0,1]$ associating with each real vector x its grade of membership $\mu(x|\tilde{a})$. The λ-cut set of a fuzzy real number $\tilde{a}$, $\lambda\in]0,1]$, denoted $\tilde{a}_\lambda$ is defined as $\tilde{a}_\lambda:=\{x:\ \mu(x|\tilde{a})\geq\lambda\}$ and is a closed interval, denoted by $\tilde{a}_\lambda=[a^-_\lambda, a^+_\lambda]$.

Let $\mathcal{F}^*(\mathbb{R}^n)$ denote a set of all fuzzy vectors. Elements of $\mathcal{F}^*(\mathbb{R})$ are called fuzzy numbers.

Two fuzzy numbers $\tilde{a}$ and $\tilde{b}$ are called equal, $\tilde{a}=\tilde{b}$, if $\mu(x|\tilde{a}) = \mu(x|\tilde{b})\ \forall\ x\in\mathbb{R}$. It follows that

$$\tilde{a} = \tilde{b} \ \Leftrightarrow\ \tilde{a}_\lambda = \tilde{b}_\lambda \quad \forall\ \lambda\in]0,1]. \tag{1}$$

For the arithmetic operations on fuzzy numbers and further informations we refer to [2,13,14,16,21,22,25]. Notice that $I(\mathbb{R})\subset\mathcal{F}^*(\mathbb{R})$.

If $f:\mathbb{R}^n\times\mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a function then according to Zadeh's extension principle we can extend f to $\mathcal{F}^*(\mathbb{R}^n)\times\mathcal{F}^*(\mathbb{R}^n) \longrightarrow \mathcal{F}^*(\mathbb{R}^n)$ by the equation

$$\tilde{f}(\tilde{u},\tilde{v})(z) = \sup_{z=f(x,y)}\mu(x|\tilde{u})\wedge\mu(y|\tilde{v}). \tag{2}$$

It is well known that

$$\tilde{f}_\lambda(\tilde{u},\tilde{v}) = \tilde{f}(\tilde{u}_\lambda,\tilde{v}_\lambda), \qquad \forall\ \tilde{u},\tilde{v}\in\mathcal{F}^*(\mathbb{R}^n), 0\leq\lambda\leq1 \tag{3}$$

and f continuous. A fuzzy-valued function is a mapping $\tilde{f}:\Gamma\longrightarrow\mathcal{F}^*(\mathbb{R})$.

An interval-valued function is a special closed-valued set valued function $\bar{f}:\Gamma\longrightarrow I(\mathbb{R})$. It is usually written as $\bar{f}(x)=[f^-(x),f^+(x)]$, where

$$f^-(x)=\inf \bar{f}(x), \qquad f^+(x)=\sup \bar{f}(x).$$

In the ordinary way (pointwise), we can define the operations, orders, convergences of interval-valued functions and fuzzy-valued functions [13,22]. For special problems of the fuzzy analysis we refer to [1,3-8,15-19,22,25].

3. FUZZY POTENTIAL BOUNDARY PROBLEMS

Many practical applications are governed by the Poisson equation. Consider that we are seeking to find the solution of a Poisson equation in a Ω (two or three dimensional) domain,

$$\nabla^2u(x) = \xi(x), \qquad x\in\Omega, \tag{4}$$

where ξ is a known source density function of position and with the following conditions on the Γ boundary of Ω:

(i) Dirichlet (essential) conditions of the type $u(x)=u_0(x)$, for $x\in\Gamma_1$;
(ii) Neumann (natural) conditions such as $q(x)=\partial u(x)/\partial n=q_0(x)$, for $x\in\Gamma_2$,
where n is the normal to the boundary, $\Gamma=\Gamma_1\cup\Gamma_2$ is the boundary decomposition, and functions u_0, q_0 are known. More complex boundary conditions such as combination of the above two, i.e.

$$\alpha u(x) + \beta q(x) = \gamma, \qquad x \in \Gamma_3, \quad \Gamma=\Gamma_1\cup\Gamma_2\cup\Gamma_3,$$

where α, β and γ are known parameters, can be easily included but they will not be considered now for simplicity's sake.

After substituting fundamental solutions U and $Q=\partial U/\partial n$ of the Laplace equation and grouping all boundary terms together (i.e. in $\Gamma=\Gamma_1\cup\Gamma_2$), one obtains a boundary integral equation of the form [9,10,23,24]

$$c(x)u(x) + \int_\Gamma Q(x,y)u(y)d\Gamma(y) + \int_\Omega U(x,y)\xi(y)d\Omega(y) = \int_\Gamma U(x,y)q(y)d\Gamma(y), \qquad x\in\Gamma. \tag{5}$$

For simplicity we now assume that equation (5) has a unique solution given in the form

$$u(x) = F(x,u_0,q_0,\xi), \qquad x\in\Gamma, \tag{6}$$

where F is a continuous operator of their arguments.

We now assume that the values of some of the boundary quantities and the source density function are uncertain and we shall model this uncertainty using fuzzy variables. First let u_0, q_0 and ξ are fuzzy functions i.e.

$$\tilde{u}_0, \tilde{q}_0 : \Gamma \longrightarrow [0,1] \quad \text{and} \quad \tilde{\xi}: \Omega \longrightarrow [0,1].$$

We substitute $\tilde{u}_0, \tilde{q}_0$ and $\tilde{\xi}$ for u_0, q_0 and ξ into eq.(5) and then we wish to solve (5) for $\tilde{u}$ and $\tilde{q}$ which will now be fuzzy solutions of the boundary integral equation written as

$$c(x)\tilde{u}(x) + \int_\Gamma Q(x,y)\tilde{u}(y)d\Gamma(y) + \int_\Omega U(x,y)\tilde{\xi}(y)d\Omega(y) = \int_\Gamma U(x,y)\tilde{q}(y)d\Gamma(y) \tag{7}$$

In this equation all operations are in the fuzzy sense and the fuzzy integral is understood in the sense of fuzzy principal value, as defined in [27].

Practically, fuzzy solutions of those equations are not sought, but their approximations are.

4. FUZZY BOUNDARY INTEGRAL EQUATIONS - FUZZY SOLUTIONS

We now substitute fuzzy functions $\tilde{u}_0(x), \tilde{q}_0(x)$, $x\in\Gamma$ and $\tilde{\xi}(x)$, $x\in\Omega$ for the $u_0(x), q_0(x)$ and $\xi(x)$ into eq.(5) and look for the solution.

Define

$$\mathcal{U}_\lambda(x):=\Big\{u:\ c(x)u(x)+\int_\Gamma Q(x,y)u(y)d\Gamma(y)+\int_\Omega U(x,y)\xi(y)d\Omega(y)=\int_\Gamma U(x,y)q(y)d\Gamma(y),$$
$$u_0(x)\in\tilde{u}_{0\lambda}(x)\big|_{x\in\Gamma_1},\ q_0(x)\in\tilde{q}_{0\lambda}(x)\big|_{x\in\Gamma_2},\ \xi(x)\in\tilde{\xi}_\lambda(x)\ x\in\Omega\Big\}, \tag{8}$$

where $x\in\Gamma$, $0\le\lambda\le1$ and c is a known non-fuzzy function.

Our first type fuzzy solution $\tilde{u}_1(x)$, $x\in\Gamma$ is defined as follows

$$\mu\big(y|\tilde{u}_1(x)\big) := \sup\{\lambda:\ y\in\mathcal{U}_\lambda(x)\}, \qquad x\in\Gamma,\ y\in\mathbb{R}.$$

Thus $\tilde{u}_1(x), x\in\Gamma$ is a fuzzy function which solve the boundary integral eq. (5).

We may also base a fuzzy solution of eq. (5) on eq. (6). Let

$$\mathcal{V}_\lambda(x) := \Big\{F\big(x,u_0(x),q_0(x),\xi(x)\big):\ u_0(x)\in\tilde{u}_{0\lambda}(x)\big|_{x\in\Gamma_1},\ q_0(x)\in\tilde{q}_{0\lambda}(x)\big|_{x\in\Gamma_2},$$
$$\xi(x)\in\tilde{\xi}_\lambda(x)\ x\in\Omega\Big\}, \qquad \text{for } x\in\Gamma,\ 0\le\lambda\le1. \tag{9}$$

Then specify second type fuzzy solution $\tilde{u}_2(x)$, $x\in\Gamma$ as

$$\mu\big(y|\tilde{u}_2(x)\big) := \sup\{\lambda:\ y\in\mathcal{V}_\lambda(x)\}, \qquad x\in\Gamma,\ y\in\mathbb{R}.$$

THEOREM 4.1. $\tilde{u}_1(x) = \tilde{u}_2(x)$, $\forall\ x\in\Gamma$.

Proof. It is easy to show that $\mathcal{U}_\lambda(x) = \mathcal{V}_\lambda(x) = \mu\big(\cdot|\tilde{u}_{1\lambda}(x)\big) = \mu\big(\cdot|\tilde{u}_{2\lambda}(x)\big)$, $\forall\ 0\le\lambda\le1$, $x\in\Gamma$. ∎

Now we discuss how to solve eq. (7) for fuzzy function $\tilde{u}(x)$, $x\in\Gamma$. Eq. (7) is the fuzzy analogue of eq. (5). Let

$$\tilde{u}_{0\lambda}(x) = \big[u^-_{0\lambda}(x),u^+_{0\lambda}(x)\big], \qquad x\in\Gamma_1,$$
$$\tilde{q}_{0\lambda}(x) = \big[q^-_{0\lambda}(x),q^+_{0\lambda}(x)\big], \qquad x\in\Gamma_2,$$
$$\tilde{\xi}_\lambda(x) = \big[\xi^-_\lambda(x),\xi^+_\lambda(x)\big], \qquad x\in\Omega.$$

Let

$$\tilde{u}_\lambda(x) = \big[u^-_\lambda(x),u^+_\lambda(x)\big], \qquad x\in\Gamma,$$

where $0\le\lambda\le1$. Taking λ-cuts of eq. (7) we obtain

$$c(x)\big[u^-_\lambda(x),u^+_\lambda(x)\big] = \int_\Gamma Q(x,y)\big[u^-_\lambda(y),u^+_\lambda(y)\big]d\Gamma(y) +$$
$$+\int_\Omega U(x,y)\big[\xi^-_\lambda(y),\xi^+_\lambda(y)\big]d\Omega(y) = \int_\Gamma U(x,y)\big[q^-_\lambda(y),q^+_\lambda(y)\big]d\Gamma(y), \qquad x\in\Gamma. \tag{10}$$

Following th. 3.1 of [27] one can write

$$c(x)\left[u_\lambda^-(x), u_\lambda^+(x)\right] = \left[\int_\Gamma Q(x,y)u_\lambda^-(y)d\Gamma(y), \int_\Gamma Q(x,y)u_\lambda^+(y)d\Gamma(y)\right] +$$
$$+ \left[\int_\Omega U(x,y)\xi_\lambda^-(y)d\Omega(y), \int_\Omega U(x,y)\xi_\lambda^+(y)d\Omega(y)\right] =$$
$$= \left[\int_\Gamma U(x,y)q_\lambda^-(y)d\Gamma(y), \int_\Gamma U(x,y)q_\lambda^+(y)d\Gamma(y)\right], \quad x\in\Gamma, \qquad (11)$$

for all $0\le\lambda\le1$.

We solve eq. (11) for the $u_\lambda^-(x)$ and $u_\lambda^+(x)$ producing the fuzzy function $\tilde{u}_{3\lambda}(x)$ which we call the fuzzy solution $\tilde{u}_3(x)$, $x\in\Gamma$, defined by the relation

$$\mu\left(y|\tilde{u}_3(x)\right) = \sup\{\lambda:\ y\in\tilde{u}_{3\lambda}(x)\}, \qquad y\in\mathbb{R}.$$

Naturally we are now interested in the relationship between $\tilde{u}_1(x)$, $\tilde{u}_2(x)$ and $\tilde{u}_3(x)$.

THEOREM 4.2. $\tilde{u}_{1\lambda}(x) = \tilde{u}_{2\lambda}(x) \subseteq \tilde{u}_{3\lambda}(x)$, $\forall\ x\in\Gamma$, $\lambda\in]0,1]$.

Proof. Since interval operators in eq. (11) are inclusion monotonic we have immediately for all λ-cuts, $0\le\lambda\le1$

$$\tilde{u}_{1\lambda}(x) = \tilde{u}_{2\lambda}(x) \subseteq \tilde{u}_{3\lambda}(x), \quad \forall\ x\in\Gamma$$

and the conclusion follows.■

Notice that λ-cuts of $\tilde{u}_3(x)$ are rectangles in $\mathbb{R}$ (intervals), $\forall\ x\in\Gamma$.

5. FUZZY BOUNDARY ELEMENT METHOD - COMPUTATIONAL METHODOLOGY

Let us now consider how expressions (10-11) can be discretized to find the system of fuzzy algebraic equations from which the fuzzy boundary values can be found. Assume for simplicity that the body is two dimensional and its boundary is divided into N elements.

5.1. Constant fuzzy elements

The points where the unknown fuzzy values are considered are called as usual "nodes" and taken to be in the middle of the element for the so-called "constant fuzzy elements". Later on we will also discuss the case of "linear fuzzy elements", i.e. those elements for which the nodes are at the extremes or ends.

For the constant fuzzy elements considered here the boundary is assumed to be divided into N elements, let $\Gamma \cong \bigcup_{j=1}^{N} \Gamma_j$, where Γ_j is the boundary of the j-th element. The fuzzy (interval) values of $\tilde{u}_\lambda$ and $\tilde{q}_\lambda$ are assumed to be constant over each element and equal to the fuzzy value at the mid-element node. Equation (11) can be discretized for a given point "i" before applying any fuzzy boundary conditions, as follows

$$\frac{1}{2}\tilde{u}_\lambda(x_i) = \sum_{j=1}^{N}\int_{\Gamma_j} Q(x_i,y)\tilde{u}_\lambda(y)d\Gamma(y) + \int_{\Omega} U(x_i,y)\tilde{\xi}_\lambda(y)d\Omega(y) =$$

$$= \sum_{j=1}^{N}\int_{\Gamma_j} U(x_i,y)\tilde{q}_\lambda(y)d\Gamma(y), \qquad x_i \in \bigcup_{j=1}^{N}\Gamma_j \tag{12}$$

or with respect to interval ends of λ-cuts

$$\frac{1}{2}\left[u_\lambda^-(x_i), u_\lambda^+(x_i)\right] = \sum_{j=1}^{N}\int_{\Gamma_j} Q(x_i,y)\left[u_\lambda^-(y), u_\lambda^+(y)\right]d\Gamma(y) + \int_{\Omega} U(x_i,y)\left[\xi_\lambda^-(y), \xi_\lambda^+(y)\right]d\Omega(y) =$$

$$= \sum_{j=1}^{N}\int_{\Gamma_j} U(x_i,y)\left[q_\lambda^-(y), q_\lambda^+(y)\right]d\Gamma(y), \qquad x_i \in \bigcup_{j=1}^{N}\Gamma_j \tag{13}$$

The point i-th is one of the boundary nodes. Note that for this type of fuzzy element (i.e. fuzzy constant) the boundary must be always "smooth" as the node is at the centre of the element, hence the multiplier of $\tilde{u}_\lambda(x_i)$ is 0.5.

5.2. Linear fuzzy elements

Up to now we have only considered the case of fuzzy constant elements, i.e. those with the values of the fuzzy variables assumed to be the same all over the element. Let us now consider a linear variation of fuzzy $\tilde{u}$ and $\tilde{q}$ for which case the nodes are considered to be at the ends of the element.

The governing fuzzy integral statement can now be written as (7). After discretizing the boundary into a series of N elements equation (10) can be written as

$$c_i\tilde{u}_\lambda(x_i) = \sum_{j=1}^{N}\int_{\Gamma_j} Q(x_i,y)\tilde{u}_\lambda(y)d\Gamma(y) + \int_{\Omega} U(x_i,y)\tilde{\xi}_\lambda(y)d\Omega(y) =$$

$$= \sum_{j=1}^{N}\int_{\Gamma_j} U(x_i,y)\tilde{q}_\lambda(y)d\Gamma(y), \qquad x_i \in \bigcup_{j=1}^{N}\Gamma_j. \tag{14}$$

Notice that the 0.5 coefficient of $\tilde{u}_\lambda(x_i)$ in eq. (12) has been replaced by an coefficient c_i. This is because $c_i=0.5$ applies only for a smooth boundary. Since the boundary is non-fuzzy the value of c_i for any other boundary can be calculated as in any deterministic case i.e. $c_i=\theta/2\pi$, where θ is the internal angle of the corner in radians [9,10].

The values of $\tilde{u}_\lambda$ and $\tilde{q}_\lambda$ at any point on the element can be defined in terms of their nodal fuzzy values and two linear interpolating functions ϕ_1 and ϕ_2, which are given in terms of the homogeneous coordinate ξ, i.e.

$$\tilde{u}_\lambda(\xi) = \phi_1\tilde{u}_{1\lambda} + \phi_2\tilde{u}_{2\lambda},$$
$$\tilde{q}_\lambda(\xi) = \phi_1\tilde{q}_{1\lambda} + \phi_2\tilde{q}_{2\lambda},$$

where ξ is the dimensionless coordinate varying from -1 to +1 and the two interpolation functions are

$$\phi_1 = \frac{1}{2}(1-\xi), \qquad \phi_1 = \frac{1}{2}(1+\xi).$$

The integrals in eq. (14) are more difficult to calculate than those for the fuzzy constant element as the $\tilde{u}_\lambda$'s and $\tilde{q}_\lambda$'s vary fuzzy linearly over each Γ_j and hence it is not possible to take them out of the integrals.

Approximations based on higher order fuzzy elements i.e. quadratic, cubic etc. can be calculated in a similar way.

5.3. Methodology of fuzzy computations

If we now assume that the position of i-th point can vary from 1 to N one obtains a system of N fuzzy algebraic equations resulting from (13) or (14). This set of fuzzy equations can be expressed in matrix form as

$$\mathbf{H}_\lambda\tilde{\mathbf{U}}_\lambda = \mathbf{G}_\lambda\tilde{\mathbf{Q}}_\lambda + \tilde{\Xi}_\lambda, \tag{15}$$

where $\mathbf{H}_\lambda$ and $\mathbf{G}_\lambda$ are two N×N nonfuzzy matrices and $\tilde{\mathbf{U}}_\lambda, \tilde{\mathbf{Q}}_\lambda, \tilde{\Xi}_\lambda$ are fuzzy vectors of length N, $\forall\ \lambda\in]0,1]$. Notice that N_1 fuzzy values of $\tilde{u}_\lambda$ and N_2 fuzzy values of $\tilde{q}_\lambda$ are known on Γ_1 and Γ_2 respectively, hence there are only N fuzzy unknowns in the system of equations (15). One has to rearrange the system to obtain a standard system of fuzzy algebraic equations

$$\tilde{\mathbf{A}}_\lambda\tilde{\mathbf{X}}_\lambda = \tilde{\mathbf{F}}_\lambda, \qquad \lambda\in]0,1], \tag{16}$$

where $\tilde{\mathbf{X}}_\lambda$ is a fuzzy vector of unknowns $\tilde{u}_\lambda$'s and $\tilde{q}_\lambda$'s fuzzy boundary values.

Eq. (16) can now be solved and all the boundary values are then known. Let

$$\mathcal{X}_\lambda := \left\{\mathbf{X}\colon\ \mathbf{A}_\lambda\mathbf{X} = \mathbf{F}_\lambda,\ \mathbf{a}_{\lambda ij}\in\tilde{\mathbf{a}}_{\lambda ij},\ \mathbf{f}_{\lambda i}\in\tilde{\mathbf{f}}_{\lambda i},\ i,j=1,2,\dots,N\right\}, \qquad 0\le\lambda\le1, \tag{17}$$

where $\mathbf{A}_\lambda=\left[\mathbf{a}_{\lambda ij}\right]$, $\mathbf{F}=\left[\mathbf{f}_{\lambda i}\right]$. Define $\tilde{\mathbf{X}}_1$, a fuzzy subset of $\mathbb{R}^n$, by its membership function

$$\mu\left(x|\tilde{\mathbf{X}}_1\right) = \sup\{\lambda\colon\ x\in\mathcal{X}_\lambda\}, \qquad x\in\mathbb{R}^n. \tag{18}$$

We call $\tilde{X}_1$ an exact fuzzy solution of FBEM.

Consider the set of interval eqs.(16). Assume from now that no $A\in\tilde{A}_\lambda$ is singular $\forall\ 0\le\lambda\le1$. We wish to know the set of solutions $\mathcal{X}_\lambda$ and its relation to the equation (16) where the interval multiplication and addition is used to evaluate the left-hand side of eq.(16). We now try to solve eq.(16) for the $x^-_{i\lambda}$ and $x^+_{i\lambda}$, $i=1,2,\dots,N$, $0\le\lambda\le1$, and hope they produce the λ-cuts of fuzzy numbers $\tilde{x}_i$, $i=1,2,\dots,N$.

In any case, let us assume that this method does produce fuzzy numbers $\tilde{x}_i$, $i=1,2,\dots,N$. Define $\tilde{X}_2$, a fuzzy subset of $\mathbb{R}^N$, by its membership function

$$\mu\left(x\middle|\tilde{X}_2\right) = \min_{1\le i\le N}\left\{\mu\left(x_i\middle|\tilde{x}_i\right)\right\},\qquad x\in\mathbb{R}^n. \tag{19}$$

THEOREM 5.1 [2,21]. $\tilde{X}_{1\lambda}\subseteq\tilde{X}_{2\lambda}$, $\forall\ \lambda\in]0,1]$.

Proof. Since algebraic operations are inclusion monotonic it is easy to obtain that $\tilde{X}_{1\lambda}=\mathcal{X}_\lambda\subseteq\left[X^-_\lambda,\ X^+_\lambda\right]$, $\forall\ 0\le\lambda\le1$. ∎

Many authors [2,3-8,21] discussed methods for computing an interval vector $\tilde{X}_\lambda$ containing $\mathcal{X}_\lambda$. The exact calculation of $\mathcal{X}_\lambda$ is for multidimensional problems very difficult. An interval vector $\tilde{X}_\lambda=\left[X^-_\lambda,\ X^+_\lambda\right]$, $\forall\ 0\le\lambda\le1$ defines a region in an N-dimensional space bounded by the planes $x_{i\lambda}=x^-_{i\lambda}$ and $x_{i\lambda}=x^+_{i\lambda}$, $i=1,2,\dots,N$. Nevertheless, the smallest $\tilde{X}_\lambda$ is of interest. Notice that λ-cuts of $\tilde{X}_2$ are rectangles in $\mathbb{R}^N$. Since $\tilde{X}_{1\lambda}=\mathcal{X}_\lambda$ will usually not be a rectangle in $\mathbb{R}^N$ we would expect $\tilde{X}_{1\lambda}$ to be a proper subset of $\tilde{X}_{2\lambda}$. Hence, we would usually expect $\tilde{X}_1$ to be not equal $\tilde{X}_2$. We shall use $\tilde{X}_2$ as an approximate fuzzy solution of eq.(16).

6. EXAMPLE

The following example illustrates how the presented methods work. Analyse a simple potential problem. Consider the case of a square close domain of the type shown in fig.6.1, where the boundary has been discretized into 12 constant elements with 5 internal points, comp. deterministic case [9,p.67].

We assume that boundary conditions u_0 and q_0 in the considered potential problem are interval functions

$$\bar{u}_0(x) = \left[u^-_0(x),u^+_0(x)\right],\qquad x\in\Gamma_1,$$

$$\bar{q}_0(x) = \left[q^-_0(x),q^+_0(x)\right],\qquad x\in\Gamma_2$$

and $\bar{\xi}(x)\equiv0$, $\forall\ x\in\Omega$. Numerous values are given in table 6.1.

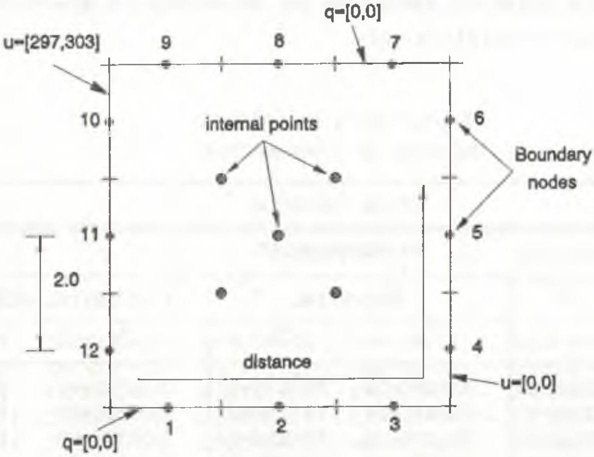

Fig. 6.1. Fuzzy potential problem

Rys. 6.1. Rozmyty problem potencjału

Table 6.1

Boundary values

NODE	TYPE	PRESCRIBED VALUE	
		low value	high value
1	q_0	.00000E+00	.00000E+00
2	q_0	.00000E+00	.00000E+00
3	q_0	.00000E+00	.00000E+00
4	u_0	.00000E+00	.00000E+00
5	u_0	.00000E+00	.00000E+00
6	u_0	.00000E+00	.00000E+00
7	q_0	.00000E+00	.00000E+00
8	q_0	.00000E+00	.00000E+00
9	q_0	.00000E+00	.00000E+00
10	u_0	.29700E+03	.30300E+03
11	u_0	.29700E+03	.30300E+03
12	u_0	.29700E+03	.30300E+03

Let the interval solution be denoted as $\bar{u}(x)=\left[u^{-}(x),u^{+}(x)\right]$, $x\in\Gamma$. Since all boundary values are interval functions it is enough to solve the problem (16) in the interval formulation only.

Table 6.2

Results of 1-st method

BOUNDARY NODES					
X	Y	POTENTIAL u_0^-	POTENTIAL u_0^+	POTENTIAL DERIVATIVE q_0^-	POTENTIAL DERIVATIVE q_0^+
.10000E+01	.00000E+00	.23689E+03	.26761E+03	.00000E+00	.00000E+00
.30000E+01	.00000E+00	.13289E+03	.16715E+03	.00000E+00	.00000E+00
.50000E+01	.00000E+00	.36520E+02	.58981E+02	.00000E+00	.00000E+00
.60000E+01	.10000E+01	.00000E+00	.00000E+00	-.14933E+03	.43404E+02
.60000E+01	.30000E+01	.00000E+00	.00000E+00	-.59269E+02	-.38273E+02
.60000E+01	.50000E+01	.00000E+00	.00000E+00	-.68243E+02	-.37681E+02
.50000E+01	.60000E+01	.30527E+02	.64974E+02	.00000E+00	.00000E+00
.30000E+01	.60000E+01	.13065E+03	.16939E+03	.00000E+00	.00000E+00
.10000E+01	.60000E+01	.23926E+03	.26524E+03	.00000E+00	.00000E+00
.00000E+00	.50000E+01	.29700E+03	.30300E+03	.21410E+02	.84529E+02
.00000E+00	.30000E+01	.29700E+03	.30300E+03	.37275E+02	.60198E+02
.00000E+00	.10000E+01	.29700E+03	.30300E+03	-.76468E+01	.11359E+03

INTERNAL POINTS			
X	Y	POTENTIAL u^-	POTENTIAL u^+
.20000E+01	.20000E+01	.10339E+03	.29717E+03
.20000E+01	.40000E+01	.94401F+02	.30616E+03
.30000E+01	.30000E+01	.49987E+02	.25003E+03
.40000E+01	.20000E+01	.82790E+01	.19120E+03
.40000E+01	.40000E+01	-.55476E+01	.20503E+03

To illustrate a specific character of interval calculations the algebraic system of fuzzy algebraic equations (16) was solved in two ways. The first method uses the fuzzy Gauss eliminating methodology with pivoting. The second method is based on the calculation of fuzzy inverse matrix $\tilde{A}$ to solve eq. (16). Notice the great divergence between related methods which follows well known facts from the theory of linear algebraic systems of interval equations. Corresponding results are presented in tables 6.2 and 6.3.

Table 6.3

Results of 2-nd method

BOUNDARY NODES					
X	Y	POTENTIAL u_0^-	u_0^+	POTENTIAL DERIVATIVE q_0^-	q_0^+
.10000E+01	.00000E+00	.24903E+03	.25547E+03	.00000E+00	.00000E+00
.30000E+01	.00000E+00	.14711E+03	.15293E+03	.00000E+00	.00000E+00
.50000E+01	.00000E+00	.45815E+02	.49686E+02	.00000E+00	.00000E+00
.60000E+01	.10000E+01	.00000E+00	.00000E+00	-.54732E+02	-.51191E+02
.60000E+01	.30000E+01	.00000E+00	.00000E+00	-.50565E+02	-.46977E+02
.60000E+01	.50000E+01	.00000E+00	.00000E+00	-.54732E+02	-.51191E+02
.50000E+01	.60000E+01	.45815E+02	.49686E+02	.00000E+00	.00000E+00
.30000E+01	.60000E+01	.14711E+03	.15293E+03	.00000E+00	.00000E+00
.10000E+01	.60000E+01	.24903E+03	.25547E+03	.00000E+00	.00000E+00
.00000E+00	.50000E+01	.29700E+03	.30300E+03	.49614E+02	.56324E+02
.00000E+00	.30000E+01	.29700E+03	.30300E+03	.44174E+02	.53300E+02
.00000E+00	.10000E+01	.29700E+03	.30300E+03	.49614E+02	.56324E+02

INTERNAL POINTS			
X	Y	POTENTIAL u^-	u^+
.20000E+01	.20000E+01	.19199E+03	.20857E+03
.20000E+01	.40000E+01	.19199E+03	.20857E+03
.30000E+01	.30000E+01	.14144E+03	.15858E+03
.40000E+01	.20000E+01	.90882E+02	.10860E+03
.40000E+01	.40000E+01	.90882E+02	.10860E+03

Table 6.4

Results for mid-point boundary conditions

BOUNDARY NODES			
X	Y	POTENTIAL	POTENTIAL DERIVATIVE
.10000E+01	.00000E+00	.25225E+03	.00000E+00
.30000E+01	.00000E+00	.15002E+03	.00000E+00
.50000E+01	.00000E+00	.47750E+02	.00000E+00
.60000E+01	.10000E+01	.00000E+00	-.52962E+02
.60000E+01	.30000E+01	.00000E+00	-.48771E+02
.60000E+01	.50000E+01	.00000E+00	-.52962E+02
.50000E+01	.60000E+01	.47750E+02	.00000E+00
.30000E+01	.60000E+01	.15002E+03	.00000E+00
.10000E+01	.60000E+01	.25225E+03	.00000E+00
.00000E+00	.50000E+01	.30000E+03	.52969E+02
.00000E+00	.30000E+01	.30000E+03	.48737E+02
.00000E+00	.10000E+01	.30000E+03	.52969E+02

INTERNAL POINTS		
X	Y	POTENTIAL
.20000E+01	.20000E+01	.20028E+03
.20000E+01	.40000E+01	.20028E+03
.30000E+01	.30000E+01	.15001E+03
.40000E+01	.20000E+01	.99740E+02
.40000E+01	.40000E+01	.99740E+02

To compare fuzzy results with deterministic ones [9-10] the potential problem under consideration was solved for mid-point boundary values i.e.

$$u_0(x) = \frac{1}{2}\left(u_0^-(x) + u_0^+(x)\right), \qquad x\in\Gamma_1,$$

$$q_0(x) = \frac{1}{2}\left(q_0^-(x) + q_0^+(x)\right), \qquad x\in\Gamma_2.$$

Numerical results are given in table 6.4.

To illustrate qualitative character of result fuzziness all values of potential and potential derivative are presented graphically as interval functions of domain circumference. Compare figs. 6.2 and 6.3.

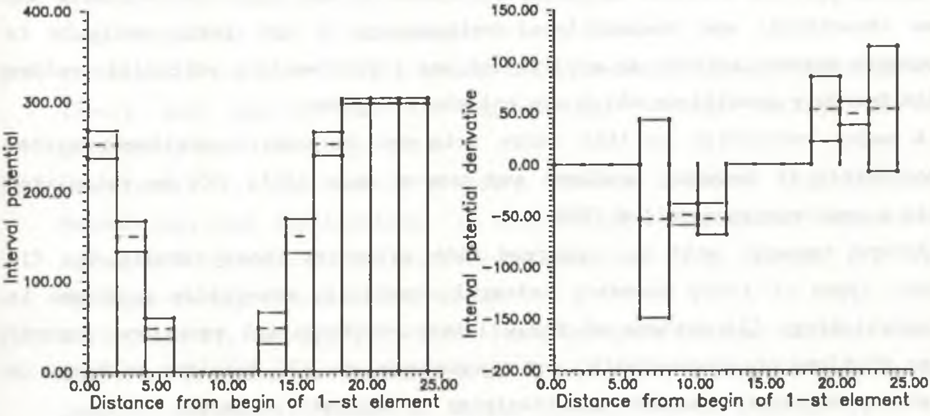

Fig.6.2. Fuzzy 1-st method solution (dotted line - mid-point solution, cont. line - interval solution)

Rys.6.2. Rozwiązanie rozmyte otrzymane 1. metodą (linia przerywana - rozwiązanie "centralne", linia ciągła - rozwiązanie rozmyte)

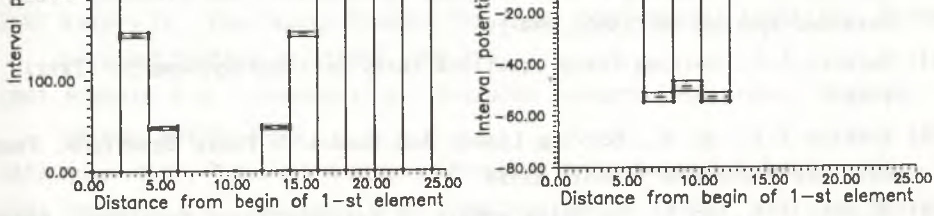

Fig.6.3. Fuzzy 2-nd method solution (dotted line - mid-point solution, cont. line - interval solution)

Rys.6.3. Rozwiązanie rozmyte otrzymane 2. metodą (linia przerywana - rozwiązanie "centralne", linia ciągła - rozwiązanie rozmyte)

7. CONCLUSIONS

This paper has a rather introductory character and was concerned with the new theoretical and computational methodology of the fuzzy analysis to boundary element method. An application was presented to a potential problem with boundary conditions which are not sharply given.

A major conclusion is that fuzzy sets can be used to estimate system uncertainty in boundary problems and random uncertainty can be calculated with a new technique called FBEM.

Future research will be concerned with extending these results to: (1) other types of fuzzy boundary integral equations, especially problems in elastostatics; (2) systems of fuzzy integro-differential equations describing problems of visco-elastic and elastodynamics; (3) detailed problems in higher order approximations and treatment of boundary corners.

SILESIAN TECHNICAL UNIVERSITY
FACULTY OF ENGINEERING MECHANICS, Konarskiego 18a, 44-100 Gliwice, POLAND
FACULTY OF CIVIL ENGINEERING, Krzywoustego 7, 44-100 Gliwice, POLAND

REFERENCES

[1] Banks H.T., Jacobs M.Q., A Differential Calculus for Multifunctions, J. Math. Anal. Appl. 29 (1970),246-272

[2] Bauch H., Jahn K.U., Oelschlägel d., Süsse H., Wiebigke V., Intervalmathematik, BSB B.G. Teubner Verlagsgeselschaft 1987

[3] Buckley J.J., Solving Fuzzy equations in Economics and Finance, Fuzzy Sets and Systems 48 (1992),289-296

[4] Buckley J.J., Solving Fuzzy equations,Fuzzy Sets and Systems 50 (1992), 1-14

[5] Buckley J.J., Qu Y., Solving Linear And Quadratic Fuzzy Equations, Fuzzy Sets and Systems 38 (1990), 43-59

[6] Buckley J.J., Qu Y., On Using α-Cuts To Evaluate Fuzzy Equations, Fuzzy Sets and Systems 38 (1990), 309-312

[7] Buckley J.J., Qu Y., Solving Fuzzy Equations: A New Solution Concept, Fuzzy Sets and Systems 39 (1991), 291-301

[8] Buckley J.J., Qu Y., Solving Systems of Linear Fuzzy Equations, Fuzzy Sets and Systems 43 (1991), 33-43

[9] Brebbia C.A., Dominguez J., Boundary Elements - An Introductory Course, Comp. Mechanics Publ., Southampton Boston 1989

[10] Brebbia C.A., Telles J.C.F., Wrobel L.C., Boundary Element Techniques - Theory and Applications in Engineering, Springer-Verlag, Berlin Heidelberg New-York Tokyo 1984

[11] Burczyński T., Stochastic Boundary Element Methods: Computational Methodology and Applications, in Probabilistic Structural Mechanics: Advances in Structural Reliability Methods, eds. T.D. Spanos, Y.T. Wu, Springer-Verlag, Berlin 1994, 42-55

[12] Burczyński T., Skrzypczyk J., The Fuzzy Boundary Element Method: A New Solution Concept, in Extended Abstracts of XII Polish Conference on Computer Methods In Mechanics, Warsaw-Zegrze, Poland, 9-13 May 1995

[13] Czogała E., Pedrycz W., Elementy i metody teorii zbiorów rozmytych, PWN, Warszawa 1985

[14] Dubois D., Prade H., Fuzzy Real Algebra: Some Results, Fuzzy Sets and Systems 2 (1979), 327-348

[15] Dubois D., Prade H., Towards Fuzzy Differential Calculus, Part 1: Integration of Fuzzy Mappings, Fuzzy Sets and Systems 8 (1982), 1-17

[16] Felbin C., Finite Dimensional Fuzzy Normed Linear Space, Fuzzy Sets and Systems 48 (1992), 239-248

[17] Guang-Quan Z., Fuzzy Continuous Function and Its Properties, Fuzzy Sets and Systems 43 (1991), 159-171

[18] Kaleva O., Fuzzy Differential Equations, Fuzzy Sets and Systems 24 (1987), 301-317

[19] Kaleva O., The Cauchy Problem For Fuzzy Differential Equations, Fuzzy Sets and Systems 35 (1990), 389-396

[20] Mikhlin S.G., Prössdorf S., Singular Integral Operators, Akademie - Verlag, Berlin 1986

[21] Moore R.E., Interval Analysis, Prentice Hall, Englewood Cliffs 1966

[22] Negoită C.V., Ralescu D.A., Applications of Fuzzy Sets to System Analysis, Ed. Technica, Birkhäuser Verlag, Basel und Stuttgart 1975

[23] Piskorek A., Równania całkowe, WN-T, Warszawa 1980

[24] Pogorzelski W., Równania całkowe i ich zastosowania t.IV, PWN, Warszawa 1970

[25] Ratschek H., Schröder G., Über die Ableitung von intervallwertigen Funktionen, Computing 7 (1971), 172-187

[26] Seikkala S., On The Fuzzy Initial Value Problem, Fuzzy Sets and Systems 24 (1987), 319-330

[27] Skrzypczyk J., On Fuzzy Singular Integration, this Journal

Streszczenie

W pracy zaproponowane zostały podstawowe założenia nowej koncepcji w metodzie elementów brzegowych, nazwanej metodą rozmytych elementów brzegowych. W pracy opisano funkcje rozmyte, które rozpatrywane są jako rozwiązania rozmytych brzegowych równań całkowych. Zdefiniowane zostały pojęcia funkcji rozmytych, które odgrywają rolę analogiczną do dokładnych i przybliżonych rozwiązań rozmytych brzegowych równań całkowych. Szczegółowe rozważania dotyczące metodyki formułowania zagadnień charakterystycznych dla nowej metody rozmytych elementów brzegowych oraz problemy obliczeniowe i zastosowania przedstawiono na przykładzie rozmytego brzegowego równania całkowego wynikającego z zadania brzegowego dla równania Poissona. Rozpatrzono zadanie z niejednorodnymi, rozmytymi warunkami brzegowymi typu Dirichleta i Neumanna oraz z niezerową funkcją gęstości źródeł, która jest rozmytą funkcją położenia określoną w całym rozpatrywanym obszarze.