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UNIQUENESS IN THE LINEAR THEORY OF VISCOELASTIC THERMO-DIFFUSION

Summary. In this paper we proved the theorem about uniqueness of the solution of the boundary-initial value problem for the linear system of the partial differential-integral equations which describe the process of thermodiffusion in the three-dimensional anisotropic, homogeneous, viscoelastic medium. The proof is based on the use of the Laplace transform.

JEDNOZNACZNOŚĆ ROZWIĄZAŃ ZADAŃ LEPKOSPRĘŻYSTEJ TERMO-DYFUZJI

Streszczenie. W pracy dowodzi się twierdzenia o istnieniu rozwiązania problemu początkowo-brzegowego w liniowej teorii termodyfuzji w ciele lepkosprężystym. Dowód przeprowadza się w oparciu o transformatę Laplace'a.

РЕШЕНИЕ ЗАДАЧ ВЯЗКОУПРУГОЙ ТЕРМОДИФФУЗИИ

Резюме. В работе доказывается утверждение единственности решения смешанной граничной задачи в линейной теории термодиффузии в вязкоупругом теле. Доказательство базируется на изпользовании интегрального преобразования Лапласа.

1. INTRODUCTION

Theory of thermodiffusion describes the interactions between processes of mass transport, heat flows and deformations of the body.

These coupled thermal, mechanical and diffusion flows phenomena are typical for capillary-porous media. One can also observe them in the majority of technological processes. The material rebuilding structure is also concerned there.

The phenomenon of thermodiffusion was described in [4,5,6] in an elastic body and in [3,7] in a viscoelastic medium.

In this paper we analyse the system differential-integral equations describing the processes of thermodiffusion in viscoelastic solids. We proved the theorem about uniqueness to the boundary-initial problem of the viscoelastic thermodiffusion using the method presented in [1].

2. THE EQUATIONS OF THE PROBLEM

We consider a body that a time t=0 occupies the region B of Euclidean three-dimensional space and is bounded by the regular boundary S. By $\mathbf{x}$ we denote the typical point of B, and by (x_1, x_2, x_3) the coordinates of $\mathbf{x}$ with respect to a fixed Cartesian coordinate system. We shall employ the usual summation and differentiation convention. summation over repeated subscripts is implied and subscripts preceded by a comma denote partial differentiation with respect to the space variables. We denote time derivatives by a superposed dot. The fundamental equations in the linear theory of viscoelastic thermodiffusion are [3]:

- the equations of motion

$$\sigma_{ij},_j + X_i = \rho \ddot{u}_i \tag{2.1}$$

- the constitutive relations

$$\sigma_{ij} = E_{ijkl} * d\varepsilon_{kl} - \varphi_{ij} * dQ - \Phi_{ij} * dM \tag{2.2}$$

- the equation of diffusion

$$K_{ij}M,_{ji}-\frac{d}{dt}\left[n*dM+l*dQ+\Phi_{ij}*d\varepsilon_{ij}\right]=-R \quad (2.3)$$

- the equation of heat transfer

$$k_{ij}Q,_{ji}-T_o\frac{d}{dt}\left[m*dQ+l*dM+\varphi_{ij}*d\varepsilon_{ij}\right]=-r \quad (2.4)$$

- the geometrical equations

$$\varepsilon_{ij}=\frac{1}{2}\left(u_{i},_{j}+u_{j},_{i}\right) \quad (2.5)$$

In this relation $\mathbf{u}=\mathbf{u}(\mathbf{x},t)$ is the displacement vector of the medium, $Q=Q(\mathbf{x},t)=T(\mathbf{x},t)-T_o$ is the temperature difference, $M=M(\mathbf{x},t)$ - the chemical potential, $\mathbf{X}=\mathbf{X}(\mathbf{x},t)$ - the body force vector, $r=r(\mathbf{x},t)$ - the intensity of heat source, $R=R(\mathbf{x},t)$ - rate of internal mass generation, $\sigma_{ij}=\sigma_{ij}(\mathbf{x},t)$, $\varepsilon_{ij}=\varepsilon_{ij}(\mathbf{x},t)$ - stress and strain tensor respectively, ρ - denotes the mass density, k_{ij}, K_{ij} - constants dependent on thermal and diffusive properties of the material, $\mathbf{n}$ -the unit outward normal to S and $*$ - the Stielties convolution.

The functions $E_{ijkl}=E_{ijkl}(t)$, $\varphi_{ij}=\varphi_{ij}(t)$, $\Phi_{ij}=\Phi_{ij}(t)$, $n=n(t)$, $l=l(t)$, $m=m(t)$ are the relaxation functions detrmining physical properties of the material. They satisfy the following relations [1,3]:

$$E_{ijkl}=E_{jikl}=E_{klij}$$

$$\varphi_{ij}=\varphi_{ji},\quad \Phi_{ij}=\Phi_{ji}\quad \text{for each t in } (-\infty,\infty) \quad (2.6)$$

$$E_{ijkl}(t)=0,\ \varphi_{ij}(t)=0,\ \Phi_{ij}=0,\quad n(t)=m(t)=l(t)=0\quad \text{on } (-\infty,0) \quad (2.7)$$

Let S_α $(\alpha=\sigma,\mu,q,Q,j,M)$ be subset of S so that $S_u\cup S_\sigma=S_q\cup S_Q=S_j\cup S_M=S$, $S_u\cap S_\sigma=S_g\cap S_Q=S_j\cap S_M=\varnothing$.

For the system of field equations (2.1)-(2.5) we consider the following boundary conditions:

$$u_i=\tilde{u}_i \qquad \text{on}\quad S_u\times I \quad (2.8)$$

$$\sigma_{ij}n_j = \tilde{p}_i \quad \text{on} \quad S_\sigma \times I \tag{2.9}$$

$$k_{ij}Q,_i n_j = \tilde{q} \quad \text{on} \quad S_q \times I, \quad Q = \tilde{Q} \quad \text{on } S_Q \times I \tag{2.10}$$

$$K_{ij}M,_i n_j = \tilde{j} \quad \text{on} \quad S_j \times I, \quad M = \tilde{M} \quad \text{on } S_M \times I \tag{2.11}$$

where $\tilde{u}_i$, $\tilde{p}_i$, $\tilde{q}$, $\tilde{Q}$, $\tilde{j}$`$\tilde{M}$ are prescibed and $I=[0,\infty)$.

To the above we adjoin the initial conditions

$$\sigma_{ij}(\mathbf{x},t)=0, \;\; Q(\mathbf{x},t)=M(\mathbf{x},t)=0, \;\; u_i(\mathbf{x},t) = 0 \quad \text{for } t < 0$$

$$u_i(\mathbf{x},0) = \hat{u}_i(\mathbf{x}), \;\; \dot{u}_i(\mathbf{x},0) = \hat{v}_i(\mathbf{x}) \quad \mathbf{x} \in \overline{B} \tag{2.12}$$

where $\hat{u}_i$ and $\hat{v}_i$ are given.

3. UNIQUENESS THEOREM

In this section we prove that the following theorem is true.

Theorem.

We assume that:

- there exists the Laplace transform of all variables of the field and the equations with the boundary-initial conditions (2.8) - (2.12);

- initial value of the relaxation functions satisfy the following relations

$$E_{ijkl}(0)\gamma_{ij}\gamma_{kl} > 0 \tag{3.1}$$

$$m(0)n(0) > l^2(0) \tag{3.2}$$

where γ_{ij} is the arbitrary, symmetric tensor of rank two;

- coefficients k_{ij} and K_{ij} satisfy the conditions

$$k_{ij}r_ir_j \geq 0, \quad K_{ij}r_ir_j \geq 0, \quad \text{for each} \quad r_i \in R\,. \tag{3.3}$$

Then, under the above mentioned conditions, the problem (2.1)-(2.5) (2.8)-(2.12) has the unique solutions.

In order to prove it, we assume that there exist two different solutions $\left(u_i^{(1)},\varepsilon_{ij}^{(1)},\sigma_{ij}^{(1)},Q^{(1)},M^{(1)}\right)$ and $\left(u_i^{(2)},\varepsilon_{ij}^{(2)},\sigma_{ij}^{(2)},Q^{(2)},M^{(2)}\right)$ of that problem.

Their differences

$$\begin{aligned} &u_i^{\circ} = u_i^{(1)} - u_i^{(2)},\ \ \varepsilon_{ij}^{\circ} = \varepsilon_{ij}^{(1)} - \varepsilon_{ij}^{(2)},\ \ \sigma_{ij}^{\circ} = \sigma_{ij}^{(1)} - \sigma_{ij}^{(2)}, \\ &Q^{\circ} = Q^{(1)} - Q^{(2)},\ \ M^{\circ} = M^{(1)} - M^{(2)} \end{aligned} \tag{3.4}$$

satisfy the homogenous system of equations with homogenous initial and boundary conditions.

Let $L[f(\mathbf{x},t)] = \bar{f}(\mathbf{x},p)$ denote the Laplace transform with respect to time of the function $f(\mathbf{x},t)$.

Let us calculate the Laplace transform of the equations and boundary conditions.

We get

$$\bar{\sigma}_{ij,j} = p^2\rho\bar{u}_i^{\circ} \tag{3.5}$$

$$\frac{k_{ij}}{T_o}\bar{Q}_{,ij} - p^2\bar{m}\bar{M}^{\circ} - p^2\bar{l}\bar{M}^{\circ} - p^2\bar{\varphi}_{ij}\bar{\varepsilon}_{ij} = 0 \tag{3.6}$$

$$K_{ij}\bar{M}_{,ij}^{\circ} - p^2\bar{n}\bar{M}^{\circ} - p^2\bar{l}\bar{Q}^{\circ} - p^2\bar{\varphi}_{ij}\bar{\varepsilon}_{ij}^{\circ} = 0 \tag{3.7}$$

$$\bar{\sigma}_{ij}^{\circ} = p\bar{E}_{ijkl}\bar{\varepsilon}_{kl}^{\circ} - p\bar{\varphi}_{ij}Q^{\circ} - p\bar{\Phi}_{ij}\bar{M}^{\circ} \tag{3.8}$$

$$\int_S \bar{\sigma}_{ij}^{\circ} n_j \bar{u}_i^{\circ} dS = 0 \tag{3.9}$$

$$\int_S k_{ij}\bar{Q}^{\circ}\bar{Q}_{,i}^{\circ} n_j dS = 0 \tag{3.10}$$

$$\int_S K_{ij}\bar{M}^{\circ}\bar{M}_{,i}^{\circ} n_j dS = 0 \tag{3.11}$$

Applying Gauss theorem to the relation (3.9) and using the equations (3.5) and (3.8) we obtain

$$\int_B \left[p^2\rho\overline{u}_i^\circ\overline{u}_i^\circ + p\overline{E}_{ijkl}\overline{\varepsilon}_{kl}^\circ\overline{\varepsilon}_{ij}^\circ - p\overline{\varphi}_{ij}\overline{Q}^\circ\overline{\varepsilon}_{ij}^\circ - p\overline{\Phi}_{ij}\overline{M}^\circ\overline{\varepsilon}_{ij}^\circ\right]dB = 0 \tag{3.12}$$

Now we will take into account the equations (3.6) and (3.7) respectively by $\overline{Q}^\circ$ and $\overline{M}^\circ$, integrating over B and taking into considaration (3.10) and (3.7) we get

$$\int_B \left[\frac{k_{ij}}{T_o}\overline{Q},_i^\circ\overline{Q},_j^\circ + p^2\overline{m}\left(\overline{Q}^\circ\right)^2 + p^2\overline{l}\,\overline{M}^\circ\overline{Q}^\circ + p^2\overline{\varphi}_{ij}\overline{\varepsilon}_{ij}^\circ\overline{Q}^\circ\right]dB = 0 \tag{3.13}$$

$$\int_B \left[K_{ij}\overline{M},_i^\circ\overline{M},_j^\circ + p^2\overline{n}\left(\overline{M}^\circ\right)^2 + p^2\overline{l}\,\overline{M}^\circ\overline{Q}^\circ + p^2\overline{\Phi}_{ij}\overline{\varepsilon}_{ij}^\circ\overline{M}^\circ\right]dB = 0 \tag{3.14}$$

We substitute (3.13) and (3.14) into (3.12). We obtain

$$\begin{aligned}\int_B \Big\{ & p^2\rho\overline{u}_i^\circ\overline{u}_i^\circ + p\overline{E}_{ijkl}\overline{\varepsilon}_{ij}^\circ\overline{\varepsilon}_{kl}^\circ + \frac{k_{ij}}{T_o p}\overline{Q},_i^\circ\overline{Q},_j^\circ + \frac{K_{ij}}{p}\overline{M},_i^\circ\overline{M},_j^\circ + \\ & + p\left[\overline{m}\left(\overline{Q}^\circ\right)^2 + 2\overline{l}\,\overline{M}^\circ\overline{Q}^\circ + \overline{n}\left(\overline{M}^\circ\right)^2\right]\Big\}dB = 0\end{aligned} \tag{3.15}$$

Let us consider the case of real p such that $p \geq p_o > 0$. It is known [2], that

$$\begin{aligned}&\lim_{p\to\infty} p\overline{E}_{ijkl}(p) = E_{ijkl}(0)\\ &\lim_{p\to\infty} p\left[\overline{m}(p) + 2\overline{l}(p) + \overline{n}(p)\right] = m(0) + 2l(0) + n(0)\end{aligned} \tag{3.16}$$

In accordance with our assumption it follows

$$p\overline{E}_{ijkl}\overline{\varepsilon}_{ij}^\circ\overline{\varepsilon}_{kl}^\circ > 0 \tag{3.17}$$

$$p\left[\overline{m}\left(\overline{Q}^\circ\right)^2 + 2\overline{l}\,\overline{M}^\circ\overline{Q}^\circ + \overline{n}\left(\overline{M}^\circ\right)^2\right] > 0 \tag{3.18}$$

for sufficiently large p.

The relation (3.15) is a sum of nonnegative terms. It should be $\overline{u}_i^\circ = 0$, $\overline{Q}_i^\circ = 0$, $\overline{M}_i^\circ = 0$ for sufficiently large p.

Hence, by theorem [2] $u_i^\circ \equiv 0, Q_i^\circ \equiv 0, M_i^\circ = 0$.

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Streszczenie

W pracy analizuje się problem początkowo - brzegowy termodyfuzji w ciele lepkosprężystym. Dla przedstawionego problemu dowodzi się twierdzenia o istnieniu rozwiązania.

Dowód przeprowadza się w oparciu o transformację Laplace'a.