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THEORETICAL MECHANICS

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ON CERTAIN MEASURES OF ROTATIONAL INERTIA

Summary. Certain measures of rotational inertia are derived, wich are different from commonly known matrix of inertia. The aplication of the new measures of inertia in the construction of the equation of motion simplifies the course of calculations and makes it possible to abandon the notion of angular velocity.

O PEWNYCH MIARACH BEZWŁADNOŚCI W RUCHU OBROTOWYM

Streszczenie. W pracy zdefiniowano pewne miary bezwładności w ruchu obrotowym, różne od powszechnie znanej macierzy bezwładności. Pokazano, że ich wykorzystanie przy tworzeniu równań ruchu upraszcza tok obliczeń, czyniąc je możliwym bez użycia prędkości kątowej.

DE CERTAINES MESURES D'INERTIE DU MOUVEMENT DE ROTATION

Résumé. Ce travail definit certaines mesures d'inertie du mouvement de rotation différentes de la matrice d'inertie généralement connue.
Le cours de compte est réduit par la mise en pratique de nouvelles mesures d'inertie dans la construction des équations de mouvement ce qui permet aussi de résigner la conception de la vitesse angulaire.

General motion of a rigid solid is a superposition of translation and rotation. Each of the component motions has three degrees of freedom. The kinematics of rotation non-deformable elements is characterized by a vector of angular velocity. Such description determines the particular measures of rotational inertia of a rigid body in the form of the well know matrix of inertia.

The motion of a non-deformable body can also be described without using angular velocity e.g., by determining the motion of three characteristic points of solid which are not lying on one straight line. Unfortunately, these characteristics are not independent as they should fulfill the equations of constraints, taking into consideration the constant distance of these points during the motion. Introduction the Cartesian system of co-ordinates associated with the selected characteristic points (and thus rigidly tied with the whole solid), in a natural and logical way leads to the definition of other measures of rotational inertia. These measures, though similar to the traditional matrix of inertia are essentially different from it, and certain relations are, thanks to them, much simplified.

The relations between Cartesian co-ordinates in inertial system, and in a system rigidly connected with the moving solid, are as follows:

$$x_j = u_j + \sum_{k=1}^{3} A_{j,k} x'_k \ , \ j,k = 1,2,3. \tag{1}$$

where:

x_j - j-th co-ordinate of point in inertial system;

x'_k - k-th co-ordinate of the same point in a system rigidly connected with the moving

u_j - j-th co-ordinate of origin of the mobile co-ordinate system

$A_{j,k}$ -j-th component (in a inertial system) of the k-th versor of the mobile system.

Since the versors in both co-ordinate systems form orthogonal systems, the matrix $[A_{j,k}]$ fulfills the relationships:

$$\sum_{j=1}^{3} A_{i,j} A_{k,j} = \delta_{i,k} \tag{2}$$

where: i,k = 1,2,3;

$\delta_{i,k}$ - Kronecker delta.

The fact of motion of a rigid solid means here that the component of the vector $[u_j]$ and matrix $[A_{j,k}]$ - are the functions of time. The first describe translation and are independent, and the latter, rotation (independent are only three of the nine elements of the matrix of rotation as the relationships (2) give six independent equations).

The velocity field expressed by co-ordinate in the mobile system will be obtained differentiating the formula (1):

$$v_j = \frac{dx_j}{dt} = \frac{du_j}{dt} + \sum_{k=1}^{3} \frac{dA_{j,k}}{dt} x'_k \tag{3}$$

where: v_j - j-th velocity component in a constant system.

The same field expressed by co-ordinates of inertial system has the form (relation inverse to (1) put into (3)):

$$v_j = \frac{du_j}{dt} - \sum_{i=1}^{3} \omega_{j,i}(x_i - u_i) \tag{4}$$

where: $\omega_{j,i} = \sum_{k=1}^{3} \frac{dA_{j,k}}{dt} A_{i,k}$, i,j = 1,2,3.

From equality (2) it results that the matrix $\omega_{j,i}$ - is antisymmetric.

This permits the introduction of the vector of angular velocity with the components (1):

$$\omega_i = \frac{1}{2} \sum_{j,k=1}^{3} \varepsilon_{i,j,k} \omega_{j,k} \tag{5}$$

where: i,j,k = 1,2,3;

$\varepsilon_{i,j,k}$ - Ricci's permutation symbol;

$$\varepsilon_{i,j,k} = \begin{cases} 0 & \text{- if at least two indices are equal;} \\ 1 & \text{- when the sequence } \{i,j,k\} \text{ is the even permutation} \\ & \text{of the sequence } 1,2,3; \\ -1 & \text{- in the remaining cases.} \end{cases}$$

In all, the velocity field for a rigid solid assumes the know form :

$$v_i = \frac{du_i}{dt} + \sum_{j,k=1}^{3} \varepsilon_{i,j,k} \omega_j (x_k - u_k) \tag{6}$$

An identical dependence takes place in a non-stationary system.

We arrive at the measures of inertia when determining kinetic energy :

$$E = \frac{1}{2} \int_V \sum_{i=1}^{3} v_i^2 \mu dV \tag{7}$$

where :

E - kinetic energy

μ - function describing the distribution of mass density in a body;

V - volume of rigid solid.

The simplest way is to carry out the integration in formula (7) when the velocity field is expressed by the co-ordinates of a non-stationary system connected with the moving body. In this case the boundaries of integration and the function of mass density μ are the easiest to define. It is also convenient to assume the beginning of the non-stationary system of co-ordinates in the centre of the mass as the integrals are then zeroed:

$$\int_V x_i \mu dV = 0 \qquad i = 1,2,3$$

Calculating the kinetic energy with the above assumption we shall obtain:

1. when integration the velocity field determined by the formula (6) ;

$$E=\frac{1}{2}\left[m\sum_{i=1}^{3}\left(\frac{du_i}{dt}\right)^2+\sum_{i,j=1}^{3}I_{i,j}\omega_i'\omega_j'\right] \tag{8}$$

where :

$m = \int_V \mu dV$ - mass solid;

$$I_{i,j} = \int_V \left[\delta_{i,j}\sum_{k=1}^{3}x_k'^2 - x_i'x_j'\right]\mu dV,\ i,j = 1,2,3; \tag{8a}$$

$I_{i,j}$ - components of the matrix of inertia (traditional measures of rotational inertia);

ω_i' - components of angular velocity in non-stationary system;

$\omega_i' = \sum_{k=1}^{3} A_{k,i}\omega_k$

2. when in the formula (7) we insert the relation (3);

$$E=\frac{1}{2}\left[m\sum_{i=1}^{3}\left(\frac{du_i}{dt}\right)^2+\sum_{i,j,k=1}^{3}\frac{dA_{i,j}}{dt}\frac{dA_{i,k}}{dt}I_{j,k}'\right] \tag{9}$$

where :

$$I_{j,k}' = \int_V x_j'x_k' dV,\ j,k = 1,2,3; \tag{9a}$$

$I_{j,k}'$ - other than the traditional ones, measures of rotational inertia of rigid body.

Noticeable is the similarity between both measures. Here are their relationships :

$$I_{j,k}' = \begin{cases} -I_{j,k} & \text{for } j \neq k \\ \frac{1}{2}tr[I_{j,k}] - I_{j,k} & \text{for } j = k \end{cases} \tag{10}$$

where : $tr[I_{j,k}] = \sum_{j=1}^{3} I_{j,j}$.

$$I'_{j,k} = \begin{cases} -I_{j,k} & \text{for } j \neq k \\ tr[I_{j,k}] - I_{j,k} & \text{for } j = k \end{cases} \tag{11}$$

where: j,k = 1,2,3.

From the formulas (10) and (11) it results that if one the matrices of inertia is diagonal, the other is also diagonal and thus they both define the same principal axes.

It can be checked that with the rotation of non-stationary system of co-ordinates both measures of inertia are transformed identically. It is somewhat different with the translations :

$$x'_{i,1} = u_i + x'_i \tag{12}$$

If the axes of the primary system (in which the co-ordinates are denoted x'_i) are central axes, the equivalent of Steiner's formula has here the form

$$I'_{1,i,j} = u_i u_j m + I'_{i,j} \tag{13}$$

where : $I_{1,i,j}$ - non-traditional measures of inertia in a shifted system.

When we make a shifting along the first axis by a section a that is :

$u_1 = a$ $\qquad u_2 = u_3 = 0$

we shall obtain:

$$\left.\begin{array}{l} I_{1,1,1} = ma^2 + I_{1,1} \\ I_{1,i,i} = I_{i,i} \qquad i = 2,3 \end{array}\right\} \tag{13b}$$

It can be seen that inversly to the standard matrix of inertia, parallel shifting of the axis does not change the respective component of the defined above measures of inertia, which is changed, however, when the starting point on the axis is changed. The diagonal terms of the new matrix of inertia can here be interpreted for the i-th axis as mass moments of inertia (moments of the second order) towards the plane $x_i = 0$.

When forming equations of motion for a rigid solid it is necessary to know the relation (1). This results from the fact that in formula (8), the components of the matrix of inertia and of angular velocities must be expressed in the same system of co-ordinates. If this is a matrix of inertia, if this is a non-stationary system, then the angular velocities must be transposed.

Usually, in the formula (1), the translation v tor and the elements of the matrix of rotation are the function of the generalized co-ordinates assumed for the description of the motion of the rigid body and only through them are dependent on time.

The formulas (4) and (5) show how to determine the velocity components in a stationery system when the matrix $[A_{i,j}]$ is know. The inverse relations are necessary especially when the motion of a rigid body is defined as the angular velocities are independent.

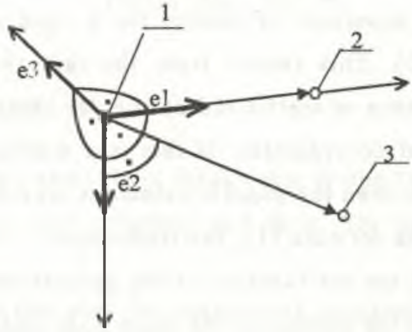
The elements of the matrix of rotation, when the angular velocities are know, can be determined as the solution of a system of differential equations :

$$\frac{d A_{i,j}}{dt} = - \sum_{k=1}^{3} \omega_{i,k}(t) A_{k,j} \tag{14}$$

where: i, j = 1,2,3.

The solution of the system of equations (14) will fulfill the restrictions (2) over the whole axis of time if this condition is met at the initial instant. The motion of a rigid body is fully determined by the motion of its three characteristic points which are not lying on one straight line. Then the solution of the equations (14) can be constructed in the following way :

the first versor of non-stationary system lies on the straight line connecting points 1 and 2 - it has sense from point 1 to point 2; the seconds is perpendicular to the plane on which lie all the characteristics points; the third one is an orthogonal complement of the orientation consistent with the orientation of the co-ordinates system (see the figure).



$$A_{i,1} = \frac{a_i}{\left(\sum_{i=1}^{3} a_i^2\right)^{0,5}} \qquad A_{i,2} = \frac{c_i}{\left(\sum_{i=1}^{3} c_i^2\right)^{0,5}} \qquad A_{i,3} = \sum_{j,k=1}^{3} \varepsilon_{i,j,k} A_{j,i} A_{k,2} \qquad (15)$$

where:

$a_i = r_{i,2} - r_{i,1}$, $b_i = r_{i,3} - r_{i,1}$, $i,k = 1,2,3$;

$r_{i,k}$ - i-th co-ordinate of the k-th characteristics point;

$$c_i = \sum_{j,k=1}^{3} \varepsilon_{i,j,k} a_j b_k .$$

To sum up- the application of the measures of inertia defined in the paper when determining the equations of motion in the form of Lagrange's equations of the second kind simplifies the determination of kinetic energy (it is then possible to abandon completely the notion of angular velocity). The algorithm presented here found use in computer analysis of mechanisms (where the user could operate with traditional measures of inertia). A reduction of the calculation time and the domain of memory needed was then obtained (through elimination of angular velocities). The saving in both cases, for the test examples were of several per cent.

The way through which one arrives at new measures of inertia is banally simple and natural. The elements of the new matrix of inertia also have more symmetry than the traditional measures. In certain case their use offers definite benefits and the author has already met with a reproach that he is using new measures of inertia.

It was this that became an inspiration to write this paper.

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Streszczenie

W pracy wyprowadza się, wychodząc z prostych zależności geometrycznych, pewne miary bezwładności w ruchu obrotowym różne od powszechnie znanej macierzy bezwładności. Nowe miary bezwładności definiują w bryle sztywnej ten sam układ osi głównych, jak miary tradycyjne. Podano też wzory na transformację składników nowej macierzy bezwładności przy obrotach i translacjach, formułując analogon twierdzenia Steinera. Pokazano też, że wyznaczenie energii kinetycznej staje się znacznie prostsze przy użyciu nietradycyjnej macierzy bezwładności - można wtedy w ogóle zrezygnować z pojęcia prędkości kątowej.

Nowe miary bezwładności zastosowano w programie komputerowej analizy mechanizmów. Uzyskano tu kilkunastoprocentowe oszczędności zarówno czasu obliczeń, jak i obszaru zajętej pamięci komputera.

Podkreślono też banalnie prosty sposób dojścia do konstrukcji nowej macierzy bezwładności, godny polecenia dla celów dydaktycznych, jak i symetrię otrzymanych wzorów.