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THEORETICAL MECHANICS

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AXIAL DEFORMATIONS OF AN ELASTIC BAR IMMERSED IN THE NONLINEAR ELASTIC MEDIUM PART 2. NUMERICAL ANALYSIS

Summary. The article presents the application of the finite element method for the solution of variation equation in the theory of axial deformations bars immersed in the nonlinear elastic medium. The original programs of the finite element method have discussed on the basis of linear and quadratic approximation.

ROZCIĄGANY- ŚCISKANY PRĘT SPRĘŻYSTY ZANURZONY W NIELINIOWYM OŚRODKU SPRĘŻYSTYM CZĘŚĆ 2. ANALIZA NUMERYCZNA

Streszczenie. W pracy podano zastosowanie metody elementów skończonych do rozwiązywania równania wariacyjnego teorii prętów ściskanych - rozciąganych zanurzonych w nieliniowym ośrodku sprężystym. Opracowano oryginalne programy metody elementów skończonych, wykorzystując aproksymację liniową i kwadratową.

РАСТЯЖЕНИЕ И СЖАТИЕ УПРУГОГО БРУСА ПОГРУЖЕННОГО В НЕЛИНЕЙНОЙ УПРУГОЙ СРЕДЕ ЧАСТЬ 2. ЧИСЛЕННЫЙ АНАЛИЗ

Резюме. В работе использовано метод конечных элементов для решения вариационного уравнения теории растягиваных и сжимаемых брусов погруженных в нелинейной упругой среде. Разработано оргинальные программы, для кусочно - линейных и -нелинейных полиномов, с использованием метода конечных элементов.

1. INTRODUCTION

In the article we discuss the variational equation (with the application of mechanical terminology without the mathematical analysis introduced later on) which is related to one of the boundary problems described in the work [4]. We apply the method of finite elements to the variational equation in the form mentioned in [1] and also [2], [3], [5].

In the analysis of the problem we adopt the linear and quadratic approximation of the researched field of displacement and then we compare the results thus obtained with the exact solutions which are known in the researched example. The estimation in an appropriate form is not made. However, the obtained numerical results prove a great exactness of calculation both for the field of displacement and, most essentially, the field of stress. The nonlinear description of the problem has been obtained which is a generalization of ealier analysed linear problems. The original programs based on the finite element method have been constructed. The numerical example has been solved for the first iteration and the results obtained have been compared with analytic solution.

2. VARIATIONAL EQUATION OF THE PROBLEM

We consider the boundary value problem of axial deformations of an elastic bar immersed in the elastic medium where on the boundaries $x=0$, $x=L$, cf. [4], the displacements are given:

$$-\frac{d}{dx}\left[EF(x)\frac{du(x)}{dx}\right]+k[u(x)]u(x)=n(x),\ x\in(0,L),$$
$$u(x)\big|_{x=0}=u_0,\ u(x)\big|_{x=L}=u_L. \tag{2.1}$$

The boundary value problem (2.1) will be reduced to the problem with the homogenous boundary conditions using the displacement field defined as:

$$\bar{u}(x):=u(x)-\tilde{u}(x), \tag{2.2}$$

where

$$\tilde{u}(x):=\frac{x}{L}(u_L-u_0)+u_0,$$
$$\frac{d\tilde{u}(x)}{dx}=\frac{1}{L}(u_L-u_0). \tag{2.2'}$$

Substituting (2.2') to the equation (2.1) we obtain

$$-\frac{d}{dx}\left[EF(x)\frac{d\overline{u}(x)}{dx}\right]+k\left[\overline{u}(x)+\tilde{u}(x)\right]\left[\overline{u}(x)+\tilde{u}(x)\right]=\overline{n}(x),\ x\in(0,L),$$
$$\overline{u}(x)\big|_{x=0}=0,\ \overline{u}(x)\big|_{x=L}=0, \tag{2.3}$$

where

$$\overline{n}(x)=n(x)+\frac{d}{dx}\left[\frac{EF(x)}{L}(u_L-u_0)\right], \tag{2.3'}$$

and if $EF(x)=const.$, then (2.3') becomes

$$\overline{n}(x)\big|_{EF(x)=const}=n(x). \tag{2.3''}$$

Thus we obtain the boundary value problem (2.3), in which the boundary conditions are homogenous. Formulating of (2.3) is more convenient in the research applying the formal apparatus, cf. p.5 of this work.

We begin to formulate the variational equation for (2.1).

We introduce the function space (testing space):

$$V=\{v(x)\ :x\in(0,L), v(0)=u_0, v(L)=u_L, ...\}, \tag{2.4}$$

from which we choose the elements for the construction of further formulated solution. The dots placed in the parenthesis in (2.4), replace the conditions of formal nature which guarantee the existence and uniqueness of solution.

What follows is a simple description without the formal precision.

Let $v(x)$ be a function

$$(0,L)\ni x\mapsto v(x), \tag{2.5}$$

from the testing space V. The equation of the problem (2.1) is multiplied on both sides by the function (2.5), and then we integrate in the interval $(0,L)$ and make elementary transformations which results in

$$a(u,v)=l(v),\ \forall v\in V, \tag{2.6}$$

where in (2.6) the bilinear form

$$a(u,v):=\int_0^L EF(x)\frac{du}{dx}\frac{dv}{dx}dx+\int_0^L k[u(x)]u(x)v(x)dx, \tag{2.6'}$$

and linear

$$l(v) := \langle n, v \rangle + [N(x)v(x)]_0^L, \ \langle n, v \rangle := \int_0^L n(x)v(x)dx. \tag{2.6"}$$

are introduced.
The normal force occurring above

$$N(x) = EF(x)\frac{du(x)}{dx}, \tag{2.7}$$

attains at the boundary points $x = (0+)$, $x = (L-)$ the values

$$N(x)\big|_{x=0+} = -P_0, \ N(x)\big|_{x=L-} = P_L, \tag{2.8}$$

which represent, respectively, the values of external forces normal to boundary cross-section.

The function $(0, L) \ni x \mapsto u(x)$ satisfying (2.6) is called a generalized solution and this equation is a variational equation of a problem considered.

3. AN APPLICATION OF THE FINITE ELEMENT METHOD

3.1. Variational equation

The variational equation (2.6), with the marked (2.6') - (2.7), is modified when we assume new functions and notations:

$$\begin{aligned} &\varepsilon(x) = \frac{du(x)}{dx}, \ \sigma(x) = E\varepsilon(x) \equiv E\frac{du(x)}{dx}, \\ &N(x) = F(x)\sigma(x) \equiv EF(x)\varepsilon(x), \\ &v(x) \equiv \delta u(x), \ \frac{\mathrm{dv}}{\mathrm{dx}} \equiv \frac{d}{dx}(\delta u(x)) \equiv \delta\left(\frac{du}{dx}\right) \equiv \delta\varepsilon(x), \end{aligned} \tag{3.1}$$

Considering the accepted notation we obtain:

$$\int_0^L F(x)\delta\varepsilon(x)\sigma(x)dx + \int_0^L k[u(x)]\delta u(x)u(x)dx$$
$$= \int_0^L \delta u(x)u(x)dx + [\delta u(x)N(x)]\Big|_0^L \ , \ \forall \delta u \subset V. \tag{3.2}$$

Let interval $(0, L)$ be divided by the points

$$x_1 = 0,\ x_2,\ \ldots,\ x_{e-1},\ x_e,\ x_{e+1},\ \ldots,\ x_{N_e},\ x_{N_e+1} = L \tag{3.3}$$

into the intervals

$$(x_e,\ x_{e+1}),\ e = 1,\ 2,\ \ldots,\ N_e, \tag{3.3'}$$

with the lengths

$$h_e := x_{e+1} - x_e; \tag{3.3''}$$

in every interval (3.3') an unknown function $u(x)$ will be represented by a polynomian (linear $-\alpha = 1$, quadratic $-\alpha = 2$) symbolically noted as

$$(x_e, x_{e+1}) \ni x \mapsto u(x) \to u^{(e)}(x) \equiv w_\alpha^{(e)}(x),\ \alpha = 1,2\ ,$$
$$V(0, L) \to V_h(0, L). \tag{3.4}$$

The variational equation (3.2) appears as

$$I_1 + I_2 + I_3 = [\delta u(x)N(x)]\Big|_0^L, \tag{3.5}$$

where

$$\{I_1 \quad I_2 \quad I_3\} := \sum_{e=1}^{N_e} \int_{x_e}^{x_{e+1}} \{F^{(e)}(x)\delta\varepsilon^{(e)}(x)\sigma^{(e)}(x)$$
$$k^{(e)}[u^e(x)]\delta u^{(e)}(x)u^{(e)}(x) \quad \delta u^{(e)}(x)n(x)\}dx. \tag{3.5'}$$

3.2. Linear approximation

Applying the procedure discussed in [4] for the equation (3.5) we obtain the matrix form of the above equation that holds true in this case for the linear approximations:

$$\{\delta U\}^T\left([K(U)]\{U\}-\{N\}\right)=0,\ \forall\{\delta U\}\in V_h\equiv R^{N_e+1}, \tag{3.6}$$

where

$$[K]:=\begin{bmatrix} \ddots & & & 0 \\ & +k_{11}^{(e-1)} & k_{12}^{(e-1)} & \\ & k_{21}^{(e-1)} & k_{22}^{(e-1)}+k_{11}^{(e)} & \\ & & k_{21}^{(e)} & k_{22}^{(e)}+ \\ 0 & & & \ddots \end{bmatrix}_{(N_e+1)\times(N_e+1)}$$

$$\left[\left[k^{(e)}\right]\right]\equiv\left[\begin{bmatrix} k_{11}^{(e)} & k_{12}^{(e)} \\ k_{21}^{(e)} & k_{22}^{(e)} \end{bmatrix}\right]\equiv\left[\begin{bmatrix} \dfrac{EF^{(e)}}{h_e}+\dfrac{\tilde{k}_{11}^{(e)}h_e}{3} & -\dfrac{EF^{(e)}}{h_e}+\dfrac{\tilde{k}_{12}^{(e)}h_e}{6} \\ -\dfrac{EF^{(e)}}{h_e}+\dfrac{\tilde{k}_{21}^{(e)}h_e}{6} & \dfrac{EF^{(e)}}{h_e}+\dfrac{\tilde{k}_{22}^{(e)}h_e}{3} \end{bmatrix}\right], \tag{3.7}$$

$$\{U\}=\{u_1\quad u_2\quad \dots\quad u_{e-1}\quad u_e\quad u_{e+1}\quad \dots\quad u_{N_e}\quad u_{N_e+1}\}^T,$$

$$\{\delta U\}=\{\delta u_1\quad \delta u_2\quad \dots\quad \delta u_{e-1}\quad \delta u_e\quad \delta u_{e+1}\quad \dots\quad \delta u_{N_e}\quad \delta u_{N_e+1}\}^T,$$

$$\begin{aligned}\{N\}=\{&N_1^{(1)}+P_1\quad N_2^{(1)}+N_1^{(2)}\quad N_2^{(2)}+\ \dots\\ &+N_1^{(e-1)}\quad N_2^{(e-1)}+N_1^{(e)}\quad N_2^{(e)}+\ \dots\\ &+N_1^{(N_e)}\quad N_2^{(N_e)}+N_1^{(N_e+1)}\quad N_2^{(N_e+1)}+P_{N_e+1}\}^T,\end{aligned}$$

$$\{N_1^{(e)}\quad N_2^{(e)}\}:=\int_0^{h_e}\frac{n(\xi)}{h_e}\{h_e-\xi\quad \xi\}d\xi,$$

$$P_1\equiv P_0,\ P_{N_e+1}\equiv P_L.$$

$$\left[\begin{bmatrix} \tilde{k}_{11}^{(e)} & \tilde{k}_{12}^{(e)} \\ \tilde{k}_{21}^{(e)} & \tilde{k}_{22}^{(e)} \end{bmatrix}\right]=3\int_0^1\tilde{k}^{(e)}\left[u^{(e)}(\xi)\right]\left[\begin{bmatrix} (1-\xi)^2 & 2\xi(1-\xi) \\ 2\xi(1-\xi) & \xi^2 \end{bmatrix}\right]d\xi.$$

The equations (3.6) lead to the matrix equations of the finite element method whose construction and solution is realized on the computer.

3.3. Quadratic approximation (quadratic polynomials)

Following the procedure in p. 3.2 we analogically obtain the equation of the finite element method

$$\{\delta U\}^T([K(U)]\{U\}-\{N\})=0,\ \forall\{\delta U\}\in V_h\equiv R^{2N_e+1}, \tag{3.8}$$

where

$$\{\delta U\}=\left\{\delta u_1\quad \delta u_{\frac{3}{2}}\quad \delta u_2\quad \dots\quad \delta u_{e-\frac{1}{2}}\quad \delta u_e\quad \delta u_{e+\frac{1}{2}}\quad \dots\right.$$
$$\left.\dots\quad \delta u_{N_e+\frac{1}{2}}\quad \delta u_{N_e+1}\right\},$$

$$\{U\}=\left\{u_1\quad u_{\frac{3}{2}}\quad u_2\quad \dots\quad u_{e-\frac{1}{2}}\quad u_e\quad u_{e+\frac{1}{2}}\quad \dots\right.$$
$$\left.\dots\quad u_{N_e+\frac{1}{2}}\quad u_{N_e+1}\right\}^T,$$

$$\{N\}=\left\{N_1^{(1)}+P_1\quad N_2^{(1)}\quad N_3^{(1)}+N_1^{(2)}\quad \dots\right.$$
$$N_3^{(e-1)}+N_1^{(e)}\quad N_2^{(e)}\quad N_3^{(e)}+N_1^{(e+1)}\quad \dots$$
$$\left.\dots\quad N_2^{(N_e+1)}\quad N_3^{(N_e+1)}+P_{N_e+1}\right\}^T,$$

$$[K]=\underbrace{\begin{bmatrix} \ddots & & & & & & 0\\ & +k_{11}^{(e)} & k_{12}^{(e)} & k_{13}^{(e)} & & & \\ & k_{21}^{(e)} & k_{22}^{(e)} & k_{23}^{(e)} & & & \\ & k_{31}^{(e)} & k_{32}^{(e)} & k_{33}^{(e)}+k_{11}^{(e+1)} & k_{12}^{(e+1)} & k_{13}^{(e+1)} & \\ & & & k_{21}^{(e+1)} & k_{22}^{(e+1)} & k_{23}^{(e+1)} & \\ & & & k_{31}^{(e+1)} & k_{32}^{(e+1)} & k_{33}^{(e+1)} & \\ 0 & & & & & & \ddots \end{bmatrix}}_{(2N_e+1)\times(2N_e+1)}$$

$$\left[k^{(e)}\right] \equiv \begin{bmatrix} k_{11}^{(e)} & k_{12}^{(e)} & k_{13}^{(e)} \\ k_{21}^{(e)} & k_{22}^{(e)} & k_{23}^{(e)} \\ k_{31}^{(e)} & k_{32}^{(e)} & k_{33}^{(e)} \end{bmatrix} \equiv \begin{bmatrix} k_{11}^{(e)} + \tilde{k}_{11}^{(e)} & k_{12}^{(e)} + \tilde{k}_{12}^{(e)} & k_{13}^{(e)} + \tilde{k}_{13}^{(e)} \\ \bullet & k_{22}^{(e)} + \tilde{k}_{22}^{(e)} & k_{23}^{(e)} + \tilde{k}_{23}^{(e)} \\ \bullet & \bullet & k_{33}^{(e)} + \tilde{k}_{33}^{(e)} \end{bmatrix}$$

$$= \frac{EF^{(e)}}{3l} \begin{bmatrix} \frac{7}{\bar{h}_e} + \frac{2}{5}\bar{h}_e \bar{K}^{(e)} & \frac{8}{\bar{h}_e} + \frac{1}{5}\bar{h}_e \bar{K}^{(e)} & \frac{1}{\bar{h}_e} + \frac{1}{10}\bar{h}_e \bar{K}^{(e)} \\ \bullet & \frac{16}{\bar{h}_e} + \frac{8}{5}\bar{h}_e \bar{K}^{(e)} & \frac{8}{\bar{h}_e} + \frac{1}{5}\bar{h}_e \bar{K}^{(e)} \\ \bullet & \bullet & \frac{7}{\bar{h}_e} + \frac{2}{5}\bar{h}_e \bar{K}^{(e)} \end{bmatrix},$$

$$\left[\left|\tilde{k}^{(e)}\right|\right] = \int_0^1 \tilde{k}^{(e)}\left[u^{(e)}(\xi)\right] \begin{Bmatrix} (1-2\xi)(1-\xi) \\ 4\xi(1-\xi) \\ -\xi(1-\xi) \end{Bmatrix} \{(1-2\xi)(1-\xi) \quad 4\xi(1-\xi) \quad -\xi(1-\xi)\} d\xi$$

where in $(3.8')_6$ dots indicate the symmetric elements of matrix and

$$\bar{h}_e := \frac{h_e}{l}, \bar{K}^{(e)} := \frac{k^{(e)} l^2}{EF^{(e)}}, l := \frac{L}{2}, \tag{3.8''}$$

are nondimentional parameters.

4. NUMERICAL ANALYSIS OF THE PROBLEM

For the numerical analysis we assumed six elements $e = 1, 2, \ldots, 6$, which are cut out from the bar with the length $L = 2l$; a rigidity of axial deformation is $EF^{(e)} \equiv EF = const.$, $e = 1, 2, \ldots, 6$, $\bar{k}^{(e)} = 1$, $e = 1, 2, \ldots, 6$. The boundary conditions appear as

$$\hat{u}(\bar{x})\big|_{\bar{x}=0} = 100, \; \hat{u}(\bar{x})\big|_{\bar{x}=2} = 738.91, \tag{4.1}$$

where

$$\hat{u}(\bar{x}) := \frac{EF}{3}\frac{u(x)}{l}, \ \bar{x} = \frac{x}{l}, \ x \in (0,2l), \ \bar{x} \in (0,2). \qquad (4.1')$$

Then we assume that

$$E = 2 \cdot 10^7 \left[\frac{N}{cm^2}\right], \ F = 1\left[cm^2\right], \ l = 100[cm]. \qquad (4.1'')$$

On this basis the calculations were performed for displacements and stresses the values of which figure in Table 1.

Table 1

Node	1	2	3	4	5	6	7
Coordinate $\bar{x} =$	0.0	0.1	0.3	0.6	1.0	1.5	2.0
Exact solution $10^3 u(\bar{x}) =$	1.5	1.6578	2.0270	2.7331	4.0774	6.7225	11.0835
Approximate solution $10^3 u(\bar{x}) =$	1.5	16545	2.0160	2.7150	4.0485	6.6930	11.0835
Comparative error %	0.0	0.199	0.495	0.667	0.664	0.440	0.0
Exact solution $10^{-2}\sigma(\bar{x}) =$	3.0	3.30	4.05	5.47	8.15	13.75	22,17
Average exact solution $10^{-2}\sigma(\bar{x})\left[\frac{N}{cm^2}\right]$		3.15	3.67	4.76	6.81	10.80	17.80
Approximate solution $10^{-2}\sigma(\bar{x})\left[\frac{N}{cm^2}\right]$		3.08	3.62	4.66	6.66	10.58	17.60
Error in %		2.2	1.4	2.0	2.2	2.0	1.1
		7	11	17	22	27	25

NOTES

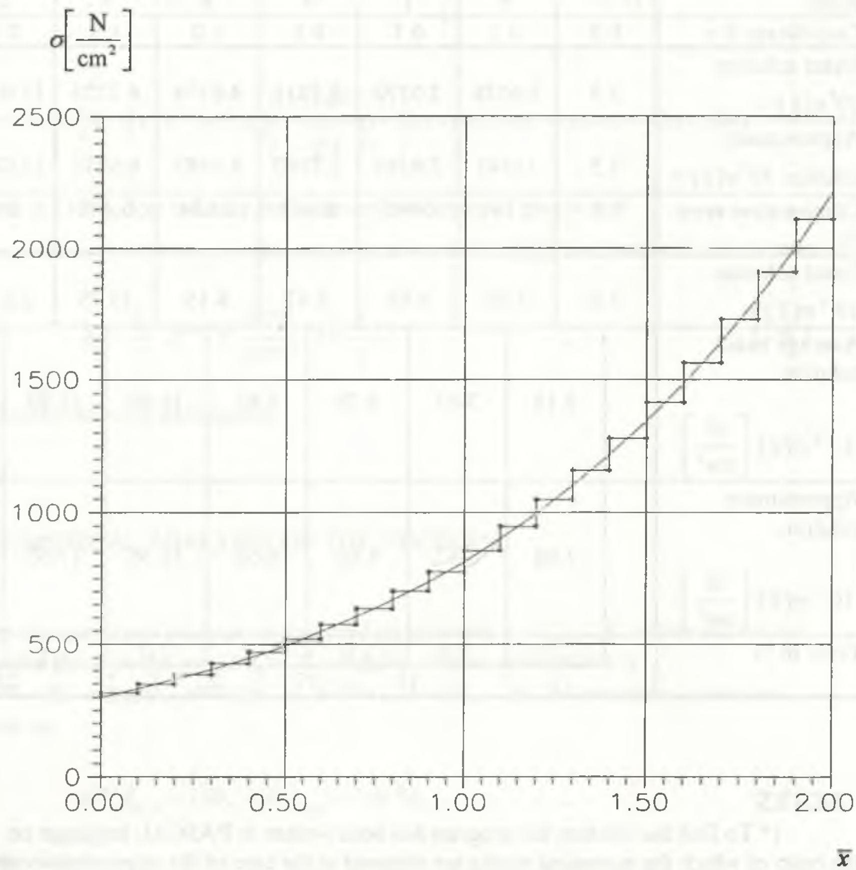
1° To find the solution the program has been written in PASCAL language on the basis of which the numerical results are obtained in the case of the approximation by piecewise linear functions (G.Porembski: Program MES01 applying the linear approximation in the problem of the bar immersed in stress medium, Chair of Theoretical Mechanics, Gliwice 1993).

2° As it follows from the results in Table 1 the error in stresses is big enough. As a result the subsequent calculations were made for the division of the bar into 20 elements which allowed for the diminishing of the error. The relevant diagram is presented in figure 1.

3° In further research the case of quadratic polynomial approximation has been considered by applying the method presented. The results obtained in this way, with the assumption of the equal length of the elements ($e = 1,2,\dots,6$), are shown in Table 2.

4° The calculation performed refer to the first iteration (linear solution). It is due to the shortage of the results of experimental research on the nonlinear problems, in this case $[K(U)] \equiv [K]$.

5° The satisfying convergence with the analytical solution has been obtained. For the solution of this problem the PASCAL program was used (D. Rozwadowski: Program MES applying the quadratic approximation in the problem for the bar immersed in stress medium. Chair of Theoretical Mechanics, Gliwice 1995, cf. [6]).



$(0,2) \ni x \mapsto \sigma(x)$

Fig 1

Table 2

Node	1	2	3	4	5	6	7
Coordinate $\overline{x} =$	0.0	0.33	0.66	0.99	1.33	1.66	2.0
Exact solution $10^3 u(\overline{x}) =$	1.5	2.0934	2.9216	4.0774	5.6905	7.9418	11.0836
Approximate solution $10^3 u(\overline{x}) =$	1.5	2.0934	2.9216	4.0774	5.6905	7.9418	11.0836
Comparative error Δu [%]	0.0	0.0	0.0	0.0	0.0	0.0	0.0

Coordinate $\overline{x} =$	0.165	0.495	0.825	1.165	1.495	1.825
Average exact solution $10^{-2}\,\sigma(\overline{x}) \left[\frac{N}{cm^2}\right]$	3.561	4.969	6.935	9.678	13.507	18.857
Approximate solution $10^{-2}\,\sigma(\overline{x}) \left[\frac{N}{cm^2}\right]$	3.544	4.946	6.903	9.634	13.445	18.764
Comparative error $\Delta\sigma$ [%}	0.46	0.46	0.46	0.46	0.46	0.46

Conclusions

1. The application of linear finite elements, six in number, generating finite dimensional space (V_1^h) led to the following errors:

   a) for axials displacement $u(x)$: 0 - 0.7%,

   b) for axials stress $\sigma(x)$: 3 - 17%.

The errors mentioned above were calculated in reference to exact solutions which known in the case considerated (Table 1).

2. Because of big errors in numerical calculations for axial stresses the following solutions were applied to reduce the error:

   a) the number of elements was increased up to 20; in this case (still for linear finite elements) little errors were obtained for axial stresses $\sigma(x)$ (cf. Table 1),

   b) the number of 6 elements was adopted but quadratic finite element generating finite dimensional space. $(\overset{o}{V}{}_2^h)$ was applied. As a result the error of axial stress $\sigma(x)$, compared with the exact solution, was app. 0,5% (cf. Table 2). Such a solution is sufficient for applications. We think that this is the way of seeking numerical solutions also for other boundary value problems.

3. In the example considered the comparison of the result of finite element methods with the exact solutions was presented. Numerical results were obtained on the basis of one`s own original computer programs.
4. The analysis presented can be a starting point for further research in the field of:
   a) inclusion of physical or geometrical nonlinearity;
   b) dynamic problems;
   c) introduction of other boundary value conditions.

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Streszczenie

W pracy podano równania wariacyjne teorii prętów rozciąganych - ściskanych, zanurzonych w nieliniowym ośrodku sprężystym. Rozpatrzono przypadek przemieszczeniowych warunków brzegowych. Rozwiązanie równania wariacyjnego - po zastosowaniu metody elementów skończonych, uzyskano przy uwzględnieniu aproksymacji liniowe i kwadratowej. Na podstawie programów (G.Porembski,

J.Wątorski [4], D.Rozwadowski [6]) uwzględniono wyniki numeryczne, których zgodność z rozwiązaniami analitycznymi jest wystarczająca (Tablica 1,2, rys.1). W aproksymacji liniowej winna być duża liczba elementów, a w aproksymacji kwadratowej tych elementów może być mniej, w celu zapewnienia dobrego przybliżenia dla funkcji pola naprężeń. W pracy nie przedstawiono oszacowania błędu rozwiązania przybliżonego, gdyż problematyka ta należy do matematycznych problemów metod numerycznych. Tutaj postąpiono tradycyjnie, podobnie jak czyni się w wielu pracach stosujących metodę elementów skończonych. Uzyskane wyniki numeryczne porównywano z rozwiązaniami analitycznymi (dokładnymi). Oczywiście w ten sposób problem szacowania, a i zbieżności rozwiązań przybliżonych pozostaje otwarty dla tych zadań, których rozwiązania analityczne nie są znane.