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THEORY OF ELASTIC THICK SHELLS. LAGRAGIAN VARIATIONAL EQUATIONS

Summary.The foregoing article presents a method allowing to formulate approximate solutions of boundary problems of thick nonhomogeneous orthotropic shells. The problem was formulated in general terms, both with regard to the description of the shell surface and to the method of problem solution.

TEORIA POWŁOK GRUBYCH, SPRĘŻYSTYCH. RÓWNANIA WARIACYJNE LAGRANGE'A

Streszczenie. W niniejszym artykule przedstawiono sposób formułowania przybliżonego rozwiązania zadań brzegowych teorii powłok grubych niejednorodnych, ortotropowych. Zadanie sformułowano w sposób ogólny, tak w opisie obszaru powłoki, jak i metody rozwiązywania zadania.

ТЕОРИЯ ТОЛСТЫХ УПРУГИХ ОБОЛОЧЕК. ВАРИАЦИОННЫЕ УРАВНЕНИЯ ЛАГРАНЖА

Резюме. В работе представлен метод приближенного решения краевых задач теории толстых упругих, неоднородных ортотропных оболочек. Задача сформулирована в общем плане, как в описаниу области оболочки так и метода решения задачи.

INTRODUCTION

Problems of theories of thick shells have already been undertaken in classical works of G.LAMÉ, the description whereof can be found e.g. in the elaboration of J.LIPKA [32]. Research in this field was continued in the 1930s and 1950s, by F.KRAUSE [30], B.G.GALERKIN [12], A.I.LURIE [33], V.K.PROKOPOV [39] among others. Short account of the obtained results can be found in the book of A.I.LURIE [33] quoted above.

Due to the importance of the theory of thick shells in other technical problems, attantion was paid to the so called technical theories of thick shells, where the hypothesis of LOVE-KIRCHHOFF was not applied on a wide basis. This course of research was followed by F.KRAUS [30] quoted above and E.TREFFTZ [49]. General theory of thick shells was presented by V.Z.VLASOV [53,54] and A.N.VOLKOV [56]. Analytical solution methods of boundary problems of the theory of thick shells can be found in the works of V.V.VLASOV [55] and A.N.VOLKOV [56,57] among others.

Current methods of analysis, or more precisely, methods for building of the equations of thick shells, are based on the application of iterative boundary problems. Such iterative problems are built using the projection method, and hence, diffrent basic functions. The works of I.N.VIEKULA [51] use Legendre polynomials as basic functions, with tapering functions sellected in such a way as to satisfy all formal requirements concerning functional analysis. This course of research was undertaken by many authors; we will present here the results obtained by V.I.GULAJEV and his co-workers, see [22]. These authors obtained also many numerical results, particulary in the field of cylindrical and spherical thick shells. In the field concerning nonlinear problems, for the constitutive relations of isotropic elastic body worked T.JĘKOT, see [26,27]. The method of VIEKULA was also used by T.MEUNARGIA [34] and H.STRUMPF, J.MAKOVSKI [46]; the latter authors analyzed great strains. The application of Legendre polynomials in formulation of iterative two-dimentional equations of the theory of thick shells can be found in the works of A.A.AMOSOV [1] and I.Ju.KHOMA [5]. The analogous problems concerning the formulation of shell theory equations from the spatial description of an elastic body, with the application of asymptotic expansions, were undertaken in the works of P.G.CIARLET, B.MIARA [7] and G.E.O.WIDERA, H.FAN [52].

A special group of research refers to the shells of TIMOSHENKO type; in these problems the influence of tangent stress in the sections perpendicular to the middle surface of the shell is taken into consideration. That kind of stress should be connected with nondilational strains. There are numerous works devoted to this problem, but we will present the theory offered by K.Z.GALIMOV [14], where the problem is discussed in a very general manner. This course of research was continued by: J.M.GREGORIENKO (and his team), see works [15.16.17.18.19.20.21], where the problems of layered shells were introduced, with regard to anisotropy.

The research of J.N.REDDY [41,42] refers to thick anisotropic layered shells, where, in the formulation of exact theory equations, third-order polynomials describing the displacements tangent in the shell are applied.

I.K.GODUSHAUR [23] investigates the cylindrical thick-wall shells made from reinforced concrete and subjected to the action of pressure (of arches) and temperature fields. S.FARAJ, R.ARCHER [10] also investigate cylindrical shells, using appropriately selected basic functions. Analogous research- reduction of three-dimentional problems to two-dimentional problems in spherical and cylindrical shells - was undertaken by N.D.PANKRATOVA [36,37].

The application of variational methods to thick shells can be found in the works of S.V.SHULINA [48], A.DI SCIUVA [8], Ja.I.BURAK [4] (in the latter work, the Reissner functional was applied), J.B.KOSMATKA [29], here the Hamilton functional was applied. The application of specific approximations of Galerkin type, with the reduction of spatial problems to the plane ones, was applied in the work of I.N.FIGUIREDO, L.TRABUCHO[11] where the shallow shells were analyzed. The theory of rotational shells, important for its application, was developed by P.Ja.NOSATENKO [35].

The application of the finite elements method in the analysis of specific cases of shell theories can be found in the works of: J.B.KOSMATKA [29] -element of five degree freedom in each node; G.P.SOKOLOV[45] - rotational shells; R.B.RIKARDS [43] - rotational shells, convergence analysis, programmes; A.M.KNASHI, A.M.SOLER [28] - general problems; H.P.HUTTENMAIER, M.EPSTAIN [25] - multilayered thick shells; G.R.HEPLER, J.S.HANSEN [24] - thick shells.

Many problems concerning thick technical shells were discussed in the monographs of: L.P.CHORUSHUN[6]; W.I.GULAJEV (and co-authors)[22], V.L.PIELEKH[38], A.O.RASSKAZOV (and co-authors)[40], S.V.SHULINA[48], see also the review work of Ja.N.NIEMISH and I.Ju.KHOMA[34a].

The discussed theory is becoming more and more popular since the calculations of thick shells can be successfully done on digital computers, providing the approximate method of boundery problem solutions is worked out beforehand. Apart from traditional examples of stress condition analysis in thick shells, there are problems of stress concentration, or the problems of concentrated load, the description whereof cannot be done with the use of the thin shell theory, since in such an analysis it is necessary to take into considration a much more complex stress condition, which can be accounted for in the theory of thick shells.

I.BASIC EQUATIONS OF EXACT THEORY OF THICK SHELLS

1. GEOMETRY OF THE SHELL. STATE OF STRAIN

1.1. Curvilinear coordinates

Let the domain V, which is the subset of the three-dimentional Euclidean space $\varepsilon^3\left(V \subset \varepsilon^3\right)$, see Cz.WOŹNIAK[58], be the domain of shell configuration as a material deformable body (continuum), at time t_0. The boundary of this domain, see Fig.1b, defined as δV will be the sum of disjoint subdomains $\delta\tilde{V}$, $\delta\hat{V}$, i.e. such subdomains, that:

$$\partial\tilde{V} \cup \partial\hat{V} = \partial V \ , \quad \overline{\partial\hat{V} \cup \partial\tilde{V}} = \partial V \ , \tag{1.1}$$

in the domains $\tilde{V}$, $\hat{V}$ respectively, dislocations and loads are prescribed. Let the domain V, see Fig.1, be parameterized by curvilinear coordinates

$$x^i, \quad i=1,2,3, \tag{1.2}$$

whereof the domain of a map, see Fig.1c, is the subdomain

$$\Omega = \Omega_0 \times (-h^3_{(-)}, h^3_{(+)}). \tag{1.3}$$

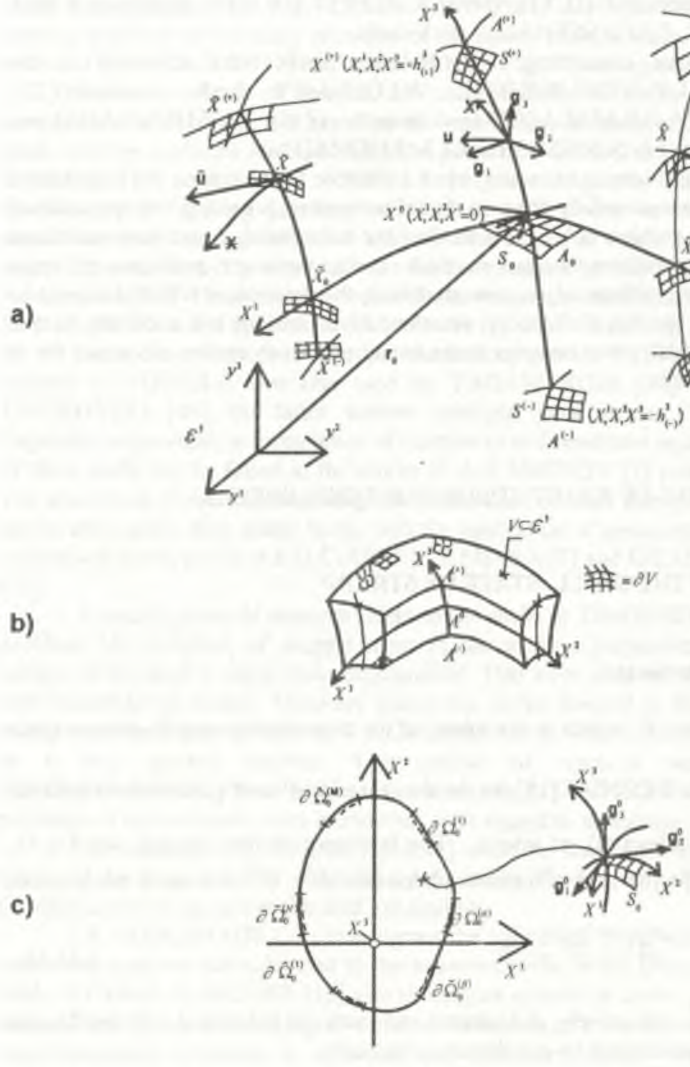

Fig. 1

In (1.3) Ω_0 stands for the set defining the curvilinear coordinates $x^1, x^2, x^3 = 0$ and $h^3_{(-)}$, $h^3_{(+)}$ stand respectively for points whereof the coordinates x^3 are equal to

$$x^3 = -h^3_{(-)}(x^1, x^2),\quad x^3 = +h^3_{(+)}(x^1, x^2),\quad (x^1, x^2) \in \Omega_0 \tag{1.4}$$

If we run a normal at point A_0, perpendicular to the basis $\mathbf{g}_1, \mathbf{g}_2$, it will cut the surface $S^{(+)}, S^{(-)}$, see Fig.1b, in points $\overline{A}^{(+)}, \overline{A}^{(-)}$; defining the lengths $A_0\overline{A}^{(+)}, A_0\overline{A}^{(-)}$ respectively by $h^{(+)}, h^{(-)}$, we assume that the sum $(h^{(+)} + h^{(-)}) = 2h$ will stand for the thickness of the shell. If

$$2h \equiv \bar{h}(x^1, x^2) \leftarrow (x^1, x^2) \in \Omega_0, \tag{1.5}$$

the shells will be of variable thickness.

We will analyse the case of small displacements and also small strains; on the basis of the above the strain tensor, see A.C.ERINGEN[9], is defined as follows:

$$\varepsilon_{ij} = \frac{1}{2}(u_{i;j} + u_{j;i}) = \varepsilon_{ij}(x^k),\ (x^k) \in \Omega,\ \Omega = \Omega_0 \times (-h^3_{(-)}, h^3_{(+)}), \tag{1.6}$$

where symbol (;) stands for the covariant derivative of displacement vector field (u_i), and the comma stands for the partial derivative, with regard to a respective curvilinear coordinate. We distinguish in the shell the middle surface S_0 being a set of points projected from the domain of a map Ω_0 of variables (x^1, x^2)

$$\Omega_0 \to S_0, \tag{1.7}$$

creating a subset of three-dimentional space. The set of points placed at the distance $h^3_{(+)}$, $h^3_{(-)}$ from the middle surface (more precisely, for $x^3 = h^3_{(-)}$, $x^3 = h^3_{(+)}$) constitutes the upper $S^{(+)}$,(lower $S^{(-)}$) boundary surface of the shell. The loads (known functions) will be mainly defined on sets $S^{(+)}$, $S^{(-)}$; in particular cases they may equal zero.

Side boundary of the shell

$$\Gamma \equiv \partial\Omega_0 \times (-h^3_{(-)}, h^3_{(+)}). \tag{1.8}$$

will be divided into two subsets: $\partial\hat{\Gamma}$ i $\partial\bar{\Gamma}$, such subsets on which loads nad displacements will be respectively prescribed.

1.2. Curvilinear ortogonal coordinates

If curvilinear ortogonal coordinates occur in the description, the following relations are satisfied:

$$g_{ij} = \begin{cases} 0, & \text{dla} \quad i \neq j \\ g_{\underline{ii}} \equiv A_i^2 & \quad\quad\; i = j \end{cases}, \tag{1.9}$$

where, in (1.9), A_i stands for Lamé coefficients. Hence, introducing physical coordinates of displacement vectors and strain tensors to (1.6):

$$\begin{aligned} \breve{u}_i &\equiv u_{(i)} := \frac{u_i}{A_{\underline{i}}} \\ \breve{\varepsilon}_{ij} &\equiv \varepsilon_{(i)(j)} := \frac{\varepsilon_{ij}}{A_{\underline{i}} A_{\underline{j}}} \end{aligned}, \tag{1.10}$$

and matrices of operators $\left[L_{\alpha j} \right]$; $\alpha = 1,2,\ldots,6$; $j = 1,2,3$:

$$L_{11}(*) = \frac{1}{A_1}\frac{\partial(*)}{\partial x^1} + \frac{1}{A_1}\frac{\partial A_1}{\partial x^1}\frac{(*)}{A_1}, \quad L_{12}(*) = \frac{1}{A_1}\frac{\partial A_1}{\partial x^1}\frac{(*)}{A_2},$$

$$L_{13}(*) = \frac{1}{A_1}\frac{\partial A_1}{\partial x^1}\frac{(*)}{A_3}, \quad L_{21}(*) = \frac{1}{A_2}\frac{\partial A_2}{\partial x^2}\frac{(*)}{A_1},$$

$$L_{22}(*) = \frac{1}{A_2}\frac{\partial(*)}{\partial x^2} + \frac{1}{A_2}\frac{\partial A_2}{\partial x^2}\frac{(*)}{A_2}, \quad L_{23}(*) = \frac{1}{A_2}\frac{\partial A_2}{\partial x^2}\frac{(*)}{A_3},$$

$$L_{31}(*) = \frac{1}{A_3}\frac{\partial A_3}{\partial x^3}\frac{(*)}{A_1}, \quad L_{32}(*) = \frac{1}{A_3}\frac{\partial A_3}{\partial x^3}\frac{(*)}{A_2},$$

$$L_{33}(*) = \frac{1}{A_3}\frac{\partial(*)}{\partial x^3} + \frac{1}{A_3}\frac{\partial A_3}{\partial x^3}\frac{(*)}{A_3}, \quad L_{41}(*) = \frac{1}{2}\frac{A_1}{A_2}\frac{\partial}{\partial x^2}\left(\frac{(*)}{A_1}\right),$$

$$L_{42}(*) = \frac{1}{2}\frac{A_2}{A_1}\frac{\partial}{\partial x^1}\left(\frac{(*)}{A_2}\right), \quad L_{43}(*) = 0,$$

$$L_{51}(*)=\frac{1}{2}\frac{A_1}{A_3}\frac{\partial}{\partial x^3}\left(\frac{(*)}{A_1}\right),\ L_{52}(*)=0,\ L_{53}(*)=\frac{1}{2}\frac{A_3}{A_1}\frac{\partial}{\partial x^1}\left(\frac{(*)}{A_3}\right),$$

$$L_{61}(*)=0,\ L_{62}(*)=\frac{1}{2}\frac{A_2}{A_3}\frac{\partial}{\partial x^3}\left(\frac{(*)}{A_2}\right),\ L_{63}(*)=\frac{1}{2}\frac{A_3}{A_2}\frac{\partial}{\partial x^2}\left(\frac{(*)}{A_3}\right), \quad (1.11)$$

instead of (1.6), we will obtain the equation:

$$\{\breve{\varepsilon}\}=[L]\{\breve{u}\}, \quad (1.12)$$

wherein the following notations were assumed:

$$\{\breve{\varepsilon}\}=\{\breve{\varepsilon}_{11}\breve{\varepsilon}_{22}\breve{\varepsilon}_{33}\breve{\varepsilon}_{12}\breve{\varepsilon}_{13}\breve{\varepsilon}_{23}\}^T,$$

$$\{\breve{u}\}=\{\breve{u}_1\breve{u}_2\breve{u}_3\}^T, \quad (1.12')$$

$$[L]=\left[L_{\alpha j}\right];\alpha=1,2,3,4,5,6;j=1,2,3;$$

further on, we will use the matrix of operators transfermed to (1.12'):

$$[L]^T=\left[L_{i\beta}\right];i=1,2,3;\beta=1,2,...,6. \quad (1.12'')$$

2. STATE OF STRESS

2.1. Description in curvilinear coordinates

In most general description of the stress state we introduce tensor field of stresses (contravariant tensor of 2nd order):

$$\sigma^{ij}=\sigma^{ij}\left(x^k\right),\ \left(x^k\right)=\Omega_o\times(-h^3_{(-)},h^3_{(+)}), \quad (2.1)$$

being the subject of analysis, i.e. it will have to be defined in each specific boundary problem. If the field of body forces is defined by:

$$X^i=X^i\left(x^k\right),\ \left(x^k\right)=\Omega_o\times(-h^3_{(-)},h^3_{(+)}), \quad (2.1')$$

than, the equations of local equilibrium of the curvilinear element (neighbourhood of the partical - initial configuration, small strain) for static problems assume the form:

$$\sigma^{ij}{}_{;j} + X^{i}\left(x^{k}\right) = 0, \quad \left(x^{k}\right) = \Omega_{0} \times (-h^{3}_{(-)}, h^{3}_{(+)}) . \tag{2.2}$$

where, as before, the symbol (;) stands for the covariant derivative of tensor field of stress (physical object twice contravariant).
Stress boundary conditions are defined by the equation:

$$\sigma^{ij} n_{j} = \hat{X}^{i}\left(x^{k}\right), \quad \left(x^{k}\right) \in \hat{\Gamma} \cup \hat{\Gamma}^{(+)} \cup \hat{\Gamma}^{(-)}, \tag{2.3}$$

where, respectively, $\hat{\Gamma}, \hat{\Gamma}^{(+)}, \hat{\Gamma}^{(-)}$ stand for subsets of a map of coordinates determining the side, upper and lower surfaces. The symbol n_j occuring above is a covariant vector of the normal, external to the boundary surface of the shell, and

$$\hat{X}^{i} = \hat{X}^{i}\left(x^{k}\right), \quad \left(x^{k}\right) \in \hat{\Gamma} \cup \hat{\Gamma}^{(+)} \cup \hat{\Gamma}^{(-)} , \tag{2.4}$$

demonstrates the field of boundary loads of the shell.

2.2. System of curvilinear ortogonal coordinates

After the introduction of physical coordinates to the discussed problem, see A.C.ERINGEN[9], the following is obtained in the curvilinear ortogonal coordinates:

$$\sum_{l=1}^{3} \left\{ \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^{l}} \left[\sigma^{(l)}{}_{(k)} \frac{\sqrt{g}}{\sqrt{g_{\underline{ll}}}} \right] + \frac{1}{\sqrt{g_{\underline{kk}}}\sqrt{g_{\underline{ll}}}} \frac{\partial \sqrt{g_{\underline{kk}}}}{\partial x^{l}} \sigma^{(l)}{}_{(k)} \right.$$

$$\left. - \frac{1}{\sqrt{g_{\underline{kk}}}\sqrt{g_{\underline{ll}}}} \frac{\partial \sqrt{g_{\underline{kk}}}}{\partial x^{l}} \sigma^{(l)}{}_{(k)} \right\} + X^{(k)}\left(x^{n}\right) = 0 , \quad \left(x^{n}\right) = \Omega . \tag{2.5}$$

Assuming in the above equation respectively k=1,2,3, we obtain three equilibrium equations, which assume the following form in the matrix notation:

$$[M(*)]\{\bar{\sigma}\} + \{\bar{X}\} = \{0\}, \tag{2.6}$$

where:

$$\{\breve{\sigma}\}=\{\breve{\sigma}_{11}\breve{\sigma}_{22}\breve{\sigma}_{33}\breve{\sigma}_{12}\breve{\sigma}_{13}\breve{\sigma}_{23}\}^{T},$$

$$\{\breve{X}\}=\{\breve{X}_1\breve{X}_2\breve{X}_3\}^{T}, \tag{2.6'}$$

and the following operators are the $[M(*)]$ matrix operators:

$$M_{11}(*)=\frac{1}{A_1A_2A_3}\frac{\partial}{\partial x^1}(A_2A_3(*)), \qquad M_{12}(*)=-\frac{(*)}{A_1A_2}\frac{\partial A_2}{\partial x^1},$$

$$M_{13}(*)=-\frac{(*)}{A_1A_3}\frac{\partial A_3}{\partial x^1}, \quad M_{14}(*)=\frac{1}{A_1A_2A_3}\frac{\partial}{\partial x^2}(A_1A_2(*))+\frac{(*)}{A_1A_2}\frac{\partial A_1}{\partial x^2},$$

$$M_{15}(*)=\frac{1}{A_1A_2A_3}\frac{\partial}{\partial x^3}(A_1A_2(*))+\frac{(*)}{A_1A_3}\frac{\partial A_1}{\partial x^3}, \quad M_{16}(*)=0,$$

$$M_{21}(*)=-\frac{(*)}{A_1A_3}\frac{\partial A_1}{\partial x^2}, \quad M_{22}(*)=\frac{1}{A_1A_2A_3}\frac{\partial}{\partial x^2}(A_1A_3(*)),$$

$$M_{23}(*)=-\frac{(*)}{A_2A_3}\frac{\partial A_3}{\partial x^2}, \quad M_{24}(*)=\frac{1}{A_1A_2A_3}\frac{\partial}{\partial x^1}(A_1A_3(*))+\frac{(*)}{A_1A_2}\frac{\partial A_2}{\partial x^1},$$

$$M_{25}(*)=0, \quad M_{26}(*)=\frac{1}{A_1A_2A_3}\frac{\partial}{\partial x^3}(A_1A_3(*))+\frac{(*)}{A_2A_3}\frac{\partial A_2}{\partial x^3},$$

$$M_{31}(*)=-\frac{(*)}{A_1A_3}\frac{\partial A_1}{\partial x^3}, \quad M_{32}(*)=-\frac{(*)}{A_2A_3}\frac{\partial A_2}{\partial x^3},$$

$$M_{33}(*)=\frac{1}{A_1A_2A_3}\frac{\partial}{\partial x^3}(A_1A_3(*)),$$

$$M_{34}(*)=0, \quad M_{35}(*)=\frac{1}{A_1A_2A_3}\frac{\partial}{\partial x^1}(A_2A_3(*))+\frac{(*)}{A_1A_3}\frac{\partial A_3}{\partial x^2},$$

$$M_{36}(*)=\frac{1}{A_1A_2A_3}\frac{\partial}{\partial x^2}(A_1A_3(*)) \tag{2.7}$$

3. CONSTITUTIVE EQUATIONS

3.1. Nonhomogeneous, anisotropic medium

We assume that the medium is nonhomogeneus, anisotropic, and, in case of ortogonal coordinates, the directions of anisotropy overlap the lines of curvilinear coordinates (local orthotropy). In this case, constitutive equations of Hooke material have the form as follows:

$$\sigma^{ij} = E^{ijkl}\varepsilon_{kl}, \tag{3.1}$$

where E^{ijkl} is the tensor of elasticity.

3.2. Nonhomogenous, orthotropic medium

If the equations (3.1) are noted in physical coordinates:

$$\sigma^{(i)(j)} = E^{(i)(j)(k)(l)}\varepsilon_{(k)(l)}, \tag{3.2}$$

than, having assumed that (3.2) are now tensors of elasticity, the constitutive equations assume the form as follows:

$$\{\breve{\sigma}\} = [\breve{E}]\{\breve{\varepsilon}\}, \tag{3.2'}$$

where the column matrix - on the left of the equation (3.2') stands for the matrix of physical coordinates of stress tensor, see (2.6'), and the column vector, on the right of (3.2'), stands for the matrix of physical coordinates of strain tensor. In (3.2'), the square matrix $[\breve{E}]$ stands for the matrix of physical coordinates of the elasticity tensor.

4. VARIATIONAL EQUATIONS OF THE PROBLEM

4.1. General formulation

In this part, we will discuss the case of curvilinear, nonortogonal coordinates. In this general notation, we have the following equations:

a) static eq.

$$\sigma^{ij}{}_{;j} + X^i\left(x^k\right) = 0, \qquad \left(x^k\right) = \Omega_0 \times (-h^3_{(-)}, h^3_{(+)}),$$

$$\sigma^{ij} n_j = \hat{X}^i\left(x^k\right), \qquad \left(x^k\right) \in \hat{\Gamma} \cup \hat{\Gamma}^{(+)} \cup \hat{\Gamma}^{(-)}; \tag{4.1}$$

b) relations between the strain tensor and displacement vector:

$$\varepsilon_{ij} = \frac{1}{2}(u_{i;j} + u_{j;i}) = \varepsilon_{ij}(x^k), \quad (x^k) \in \Omega, \quad \Omega = \Omega_o \times (-h^3_{(-)}, h^3_{(+)}),$$

$$u_i = \tilde{u}_i, \; \left(x^k\right) \in \partial\tilde{\Gamma}; \tag{4.2}$$

c)Hooke's constitutive relations:

$$\sigma^{ij} = E^{ijkl} \varepsilon_{kl}, \quad \left(x^k\right) = \Omega_0 \times (-h^3_{(-)}, h^3_{(+)}). \tag{4.3}$$

We introduce arbitrary field of virtual displacements:

$$\bar{u}_i = \bar{u}_i\left(x^k\right), \qquad \left(x^k\right) = \Omega_0 \times (-h^3_{(-)}, h^3_{(+)}), \tag{4.4}$$

defined for the points of the domain $\partial\tilde{\Gamma}$. In (4.4) the dash over the letter core signals the virtual displacement which is usually defined by δu_i; hence:

$$\bar{u}_i \equiv \delta u_i \begin{cases} \neq 0, \left(x^k\right) \notin \partial\tilde{\Gamma} \\ = 0, \left(x^k\right) \in \partial\tilde{\Gamma} \end{cases} \tag{4.4'}$$

Variational equations are formulated in the traditional way, multiplying (4.1) by (4.4), then integrating over the domain V, and then, applying the Green formula, we obtain:

$$\int_{\partial\hat{V}} \sigma^{ij} n_j \bar{u}_i df - \int_V \sigma^{ij} \bar{\varepsilon}_{ij} dv + \int_V X^i \bar{u}_i dv = 0. \tag{4.5}$$

Then, taking into account that $\partial V = \partial\hat{V} \cup \partial\tilde{V}$, and:

$$\bar{u}_{i}\Big|_{\partial\tilde{V}} = 0; \qquad \sigma^{ij} n_{j}\Big|_{\partial\hat{V}} = \hat{X}^i,$$

we obtain:

$$\int_V \sigma^{ij}\bar{\varepsilon}_{ij}dv - \int_V \bar{u}_i X^i dv - \int_{\partial\hat{V}} \bar{u}_i \hat{\mathcal{X}}^i df = 0\,,\ \forall \bar{u}_i \in H(V)\,. \tag{4.6}$$

Introducing the constitutive equations (4.3) to (4.6) we obtain:

$$\int_V \bar{\varepsilon}_{ij} E^{ijkl}\varepsilon_{kl}dv - \int_V \bar{u}_i X^i dv - \int_{\partial\hat{V}} \bar{u}_i \hat{\mathcal{X}}^i df = 0\,, \qquad \forall \bar{u}_i \in H(V). \tag{4.7}$$

The equation (4.7) demonstrates a basic variational equation, true for arbitrary elements $\bar{u}_i$ being part of the functional space $H(V)$, setting to zero on the subset $\partial\tilde{V}$.

As to the $H(V)$ space itself, we donnot make any assumptions, since we are not discussing here the problems of existance and uniquenes, general solutions represented by variational equations.

4.2. Systems of curvilinear, ortogonal coordinates

When solving the problem in curvilinear, ortogonal coordinates, we introduce physical coordinates of all elements occurring in the variational equation (4.8), and then, we introduce there the matrices of these elements. Having satisfied these assumptions, the variational equation has the following form:

$$\int_V \{\bar{\varepsilon}\}^T [E]\{\varepsilon\}dv - \int_V \{\bar{u}\}^T \{X\}dv + \int_{\partial\hat{V}} \{\bar{u}\}^T \{\hat{\mathcal{X}}\}df = 0. \tag{4.7'}$$

Then, assuming the relation between the matrix of strain tensor and the matrix of displacements, see (1.26), we obtain from (4.7'):

$$\int_V \{\breve{\bar{u}}\}^T [L(*)]^T [E][L]\{u\}dv - \int_V \{\breve{\bar{u}}\}^T \{X\}dv - \int_{\partial\tilde{V}} \{\bar{u}\}^T \{\hat{\mathcal{X}}\}df = 0. \tag{4.8}$$

If we introduce the matrix of operators to (4.9):

$$[A] := [L]^T [E][L], \tag{4.8'}$$

we will obtain the final form of the variational equation, very useful in the numerical analysis:

$$\int_V \{\breve{\bar{u}}\}^T [A]\{u\}dv - \int_V \{\breve{\bar{u}}\}^T \{X\}dv - \int_{\partial\tilde{V}} \{\bar{u}\}^T \{\hat{\mathcal{X}}\}df = 0, \forall \{\bar{u}\} \in V. \tag{4.9}$$

5. CONSTRUCTION OF THE APPROXIMATE SOLUTION

5.1. Description of the method

The formulation of the variational equation will be realized using the method of subdomains, and in particular using the algorithm based on the method of finite elements.

5.2. Algorithm of solution

Before starting the construction of the method algorithm, we divide the domain V into the sum of disjoint subdomains V_α satisfying the following:

$$V_\alpha \cap V_\beta = \varnothing;. \alpha \neq \beta; \alpha, \beta = 1,2..N \ ,$$

$$\bigcup_{\alpha=1}^{N} V_\alpha = V, \ \overline{V \cup \partial V} = \overline{V}, \tag{5.1}$$

where ∂V is the boundary of domain V, and $\overline{V}$ is the closed set. Similarly, we replace the domain boundary with subdomains on which the loads or the displacements are prescribed:

$$\partial V = \partial \hat{V} \cup \partial \tilde{V} \ , \ \partial \hat{V} = \bigcup_{A=1}^{k} \partial \hat{V}_A \ , \qquad \partial \hat{V}_A \cap \partial \hat{V}_B = \varnothing; A \neq B \ ,$$

$$\partial \tilde{V} = \bigcup_{\tilde{A}=1}^{\tilde{k}} \partial \tilde{V}_{\tilde{A}}, \ \ \partial \tilde{V}_{\tilde{A}} \cap \partial \tilde{V}_{\tilde{B}} = \varnothing; \tilde{A} \neq \tilde{B} \ . \tag{5.2}$$

Then, we replace the variational equation (4.9) with the approximate variational equation defining the aproximate generalized solution (u_h):

$$\sum_{\alpha=1}^{N} \int_{V_\alpha} \{\bar{u}_h\}_\alpha^T [A]_\alpha \{u_h\}_\alpha dv - \sum_{\alpha=1}^{N} \int_{V_\alpha} \{\bar{u}_h\}_\alpha^T \{X\}_\alpha dv$$

$$- \sum_{A=1}^{k} \int_{\partial \hat{V}_A} \{\bar{u}_h\}_A^T \{\hat{\mathcal{X}}\}_A df = 0, \tag{5.3}$$

where u_h stands for approximate values of displacement field, as it was in the method of finite elements. Such fields, see the method given in [3], were defined, introducing the shape functions$[N]$ so, that

$$\{u_h\}_\alpha = [N]_\alpha \{U\}_\alpha,\ \{\bar{u}_h\}_\alpha^T = \{U\}_\alpha^T [N]_\alpha^T \quad \text{on } V,$$

$$\{u'_h\}_A = [N']_A \{U'\}_A, \{\bar{u}'_h\}_A = \{U'\}_A^T [N']_A^T \text{ on } \partial\hat{V}. \qquad (5.4)$$

Taking into consideration (5.4) in (5.3), we obtain:

$$\sum_{\alpha=1}^{N} \{\bar{U}\}_\alpha^T \left[\int_{V_\alpha} [N]_\alpha^T [A]_\alpha \{N\}_\alpha dv \right] \{U\}_\alpha - \sum_{\alpha=1}^{N} \{\bar{U}\}_\alpha^T \left[\int_{v_\alpha} \{N\}_\alpha^T \{X\}_\alpha dv \right]$$

$$- \sum_{A=1}^{k} \left\{ \{\bar{U}'\}_A^T \right\} \int_{\partial\hat{V}_A} \{N'\}_A^T \{\hat{\mathcal{X}}\}_A df = 0. \qquad (5.5)$$

Introducing the matrices:

$$\int_{V_\alpha} [N]_\alpha^T [A]_\alpha \{N\}_\alpha dv \equiv [K]_\alpha,$$

$$\int_{v_\alpha} \{N\}_\alpha^T \{X\}_\alpha dv \equiv [B]_\alpha,$$

$$\int_{\partial\hat{V}_A} \{N'\}_A^T \{\hat{\mathcal{X}}\}_A df \equiv [C']_A, \qquad (5.6)$$

and completing the matrices $\{\bar{U}'\}_\alpha^T$ to matrices $\{\bar{U}\}^T$, and similarly, the matrices $\{\bar{U}'\}_A^T$ to this matrix, which requires the modification of matrix $\{C'\}_A$ to the matrix $\{C\}_\alpha$, we obtain:

$$\sum_\alpha \{\bar{U}\}_\alpha \left([K]_\alpha \{U\}_\alpha - \{B\}_\alpha - \{C\}_\alpha \right) = 0. \qquad (5.7)$$

On the basis of (5.7), we build now a global matrix of rigidity and the column of absolute terms, so, we obtain the matrix equation of the method:

$$[K']\{U'\}=\{B'\}+\{C'\}\equiv\{D'\} \tag{5.8}$$

The following notation was applied in (5.8): $[K']$ global matrix of rigidity; $\{U'\}$- column of all nodal displacements, also the ones which are known; $\{D'\}$- known matrix of boundary and body loads. Taking into consideration the displacement boundary conditions in (5.8), i.e. the prescribed displacement vectors in nodal points of the domain $\partial\tilde{V}$, we obtain the final equation of the method:

$$[K]\{U\}=\{D\}, \tag{5.9}$$

where $[K]$ stands for the matrix of construction rigidity, and $\{U\}$ stands for unknown nodal displacements; the matrix $\{D\}$ bears the influence of body loads and boundary loads, as well as the values of displacements defined in the nodal points of the $\partial\tilde{V}$ domain.

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Streszczenie

W artykule przedstawiono opis teorii powłok grubych; zawiera on możliwie ogólny punkt widzenia tak w zakresie geometrii jak i równań konstytutywnych. Prowadzi się opis we współrzędnych krzywoliniowych, doprowadzając odpowiednie związki do zapisu macierzowego. Wychodząc z zasady prac przygotowanych buduje się równania wariacyjme metody, które odpowiadają warunkowi ekstremum dla funkcjonału Lagrange'a. Równania takie, po zastosowaniu algorytmu metody podobszarów (metody elementów skończonych) oraz po wprowadzeniu funkcji kształtu sprowadza się do równań macierzowych metody.