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EQUIVALENT LINEARIZATION WITH CRITERIA IN PROBABILITY DENSITY FUNCTION SPACE FOR STOCHASTIC SYSTEMS UNDER PARAMETRIC EXCITATIONS

Summary. The concept of equivalent linearization in probability density space for dynamic systems under parametric Gaussian excitations is considered in this paper. New criteria of linearization and two approximate approaches are proposed.

RÓWNOWAŻNA LINEARYZACJA Z KRYTERIAMI W PRZESTRZENI FUNKCJI GĘSTOŚCI PRAWDOPODOBIEŃSTW DLA UKŁADÓW STOCHASTYCZNYCH Z WYMUSZENIAMI PARAMETRYCZNYMI

Streszczenie. W artykule przedstawiono koncepcję równoważnej linearyzacji z kryteriami w przestrzeni funkcji gęstości prawdopodobieństw dla stochastycznych układów dynamicznych z wymuszeniami parametrycznymi. Zaproponowano nowe kryteria linearyzacji i dwie techniki linearyzacji.

1. Introduction

Approximate analysis of the response of nonlinear stochastic dynamic systems with parametric and external excitations was developed recently. In particular, equivalent linearization techniques were proposed by Bruckner and Lin [1], Chang and Yang [2], Falsone [3], Kottalam, Lindberg and West [4], Wu [7]. In approaches proposed in [3] and [4] the equivalent system was assumed to be linear with additive stochastic excitations while in approaches proposed by other authors the equivalent system was assumed to be linear with parametric and external excitations. Since the complete information about random variable is contained in probability density function it is reasonable to consider a criterion depending on difference between probability densities of responses of nonlinear and linearized systems. This new philosophy for stochastic equivalent linearization was first proposed by author [5] for systems with Gaussian external excitations. New criteria of linearization were introduced and two approximate approaches were proposed. In the first one the direct minimization of a criterion is applied and the approximation of the probability density function by the Gram-Charlier expansion is used. In the second approach the linearization is made for the Fokker-Planck equations corresponding to the original nonlinear and linearized dynamic systems. The objective of this paper is to generalize these approaches to nonlinear systems with parametric and external excitations. It is shown that in contrast to the literature methods in the proposed criteria the both drift and diffusion coefficients (equivalent linearization coefficients) have to be jointly considered and determined. The detailed analysis for nonlinear oscillator is given to illustrate the results obtained. It is an extension of previous author's paper [6]. To compare characteristics of the responses obtained by proposed methods and other equivalent linearization techniques a literature example with response characteristics obtained by simulations has been chosen.

2. Problem statement

We consider a nonlinear stochastic model of dynamic system described by the Ito vector differential equation

$$dx(t) = \Phi(x,t)\,dt + \sum_{k=1}^{M} \sigma_k(x,t)\, d\xi_k(t), \tag{1}$$

where $x = [x_1, \dots, x_n]^T$ is vector state, $\Phi = [\Phi_1, \dots, \Phi_n]^T$ is a vector nonlinear function $\sigma_k = [\sigma_{k1}, \dots, \sigma_{kn}]^T$ are deterministic vectors, ξ_k are independent standard Wiener processes. We assume that the unique solution of equation (1) exists and equivalent linear systems in two cases have the form

a) with external and parametric excitations [1], [2]

$$dx(t) = [A(t)\,x(t) + C(t)]\,d(t) + \sum_{k=1}^{M} [D_k(t)\,x + G_k(t)]\,d\xi_k(t) \tag{2}$$

b) with external excitations [3], [4], [7]

$$dx(t) = [A(t)\,x(t) + C(t)]\,d(t) + \sum_{k=1}^{M} G_k(t)\,d\xi_k(t) \tag{3}$$

where $A = [a_{ij}]$, $D_k = [d_{ij}]$, $i, j = 1, \dots, n$, $k = 1, \dots, M$ are matrices and

$C = [C_1, \dots, C_n]^T$, $G_k = [G_{k1}, \dots, G_{kn}]^T$ are vectors of linearization coefficients.

In all proposed in the literature criteria of equivalent linearization the mean-square errors $E\ [\varepsilon_i^2]$ for drift and diffusion parts of considered systems are separated, for instance, for the linearized systems with external excitations they have the form

$$\varepsilon_1 = \Phi(x,t) - A(t)x - C \tag{4}$$

$$\varepsilon_{2k} = \sigma_k(x,t) - G_k(t) \quad \text{or} \quad \varepsilon_{2k}^* = \sigma_k^{[2]}(x,t) - G_k^{[2]}(t), \tag{5}$$

where $A^{[2]} = A \otimes A$, $\otimes$ denotes Kronacker product.

The objective of the probability density equivalent linearization in the case (a) or (b) is to find the elements a_{ij}, C_i, d_{ij} and G_i or a_{ij}, C_i and G_i, respectively which minimize the criterion,

$$I_{1_p} = \int_{-\infty}^{+\infty} w(x)\Psi(g_N(x) - g_{Lp}(x))dx \quad \text{or} \quad I_{1_*} = \int_{-\infty}^{+\infty} w(x)\Psi(g_N(x) - g_L(x))dx \tag{6}$$

respectively; where Ψ is a convex function, $w(x)$ is a weight function, $g_N(x)$, $g_{Lp}(x)$ and $g_L(x)$ are probability density functions of stationary solutions of nonlinear system (1) and linearized systems (2) and (3), respectively. It means that the discussed equivalent linearization method is made in the space of probability density functions. In the case of linearized system the probability density of the solution of system (3) is known and can be expressed as follows

$$g_L(x) = [(2\pi)^n |K_L|]^{-\frac{1}{2}} \exp\{-\frac{1}{2}(x-m)^T K_L^{-1}\ (x-m)\}, \tag{7}$$

where $m = m(t) = E[x(t)]$ and $K_L = K_L(t) = E[x(t)\ \ x(t)^T] - m(t)\ \ m(t)^T$ are the mean value and the covariance matrix of the solution $x = x(t)$ of system (3), respectively, $|K_L|$ denotes the determinant of the matrix K_L.

3. Direct optimization method.

Since the probability density of the linearized system $g_L(x)$ is a function of coefficients of linearization a_{ij} and C_{ij} therefore in the case when the function $\Psi(x)$ is differentiable the necessary conditions of minimization one can find, for instance, from conditions .

$$\frac{\partial I_{1_e}}{\partial a_{ij}} = 2\int_{-\infty}^{+\infty} w(x) \frac{\partial \Psi(g_N, g_L)}{\partial g_L} \frac{\partial g_L(x)}{\partial a_{ij}} dx = 0 \ , \ \frac{\partial I_{1_e}}{\partial C_i} = 2\int_{-\infty}^{+\infty} w(x) \frac{\partial \Psi(g_N, g_L)}{\partial g_L} \frac{\partial g_L(x)}{\partial C_i} dx = 0 \tag{8}$$

In this paper we consider a differentiable function defined by $w(x) = 1$ and $\Psi(x) = x^2$, i.e.

$$I_{2_e} = \int_{-\infty}^{+\infty} (g_N(x) - g_L(x))^2 \, dx \tag{9}$$

Since the criterion I_{2_e} is known in the mathematical literature of probabilistic metrics as square metric we call the corresponding linearization technique by square metric equivalent linearization. The necessary conditions (8) for criterion (9) will be shown in details on an example in Sections 6.

4. The Fokker - Planck equation approach

When the probability density function of nonlinear system is unknown and for some reason the direct optimization technique can not be applied we propose instead of state equations (1) and (2) or (3) to consider the corresponding reduced Fokker-Planck equations

$$\frac{\partial g_N}{\partial t} = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i} [\Phi_i(x,t) g_N] + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j} [b_{Nij} g_N] = 0 \tag{10}$$

and

in the case (a)

$$\frac{\partial g_{Lp}}{\partial t} = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i} [(A_i^T x + C_i) g_{Lp}] + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j} [b_{Lpijj} g_{Lp}] = 0 \tag{11}$$

in the case (b)

$$\frac{\partial g_L}{\partial t} = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i} [(A_i^T x + C_i) g_L] + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j} [b_{Lij} g_L] = 0, \tag{12}$$

where A_i^T is i-th row of matrix A, $B_N = [b_{Nij}]$, $B_L = [b_{Lij}]$, $B_{Lp} = [b_{Lpij}]$ are the diffusion matrices

$$b_{Nij} = \sum_{k=1}^{M} \sigma_{ki} \sigma_{kj} \, , \; b_{Lij} = \sum_{k=1}^{M} G_{ki} G_{kj} \;\; , \; b_{Lpij} = \sum_{k=1}^{M} (D_{ki}^T x + G_{ki})(D_{kj}^T x + G_{kj}) \, , \tag{13}$$

where D_{ki}^T is i-th row of matrix D_k.

Then the following errors of aproximation are proposed

$$I_{4_e} = \int_{-\infty}^{+\infty} [\varepsilon_1^2(x) + \varepsilon_2^2(x)] \; dx, \tag{14}$$

where

$$\varepsilon_1 = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} [(\Phi_i - A_i^T x - C_i) g_L] \tag{15}$$

and in the case (a)

$$\varepsilon_{2Lp} = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2}{\partial x_i \partial_j} [(b_{Nij} - b_{Lpij}) g_{Lp}] \tag{16}$$

and in the case (b)

$$\varepsilon_{2L} = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2}{\partial x_i \partial_j} [(b_{Nij} - b_{Lij}) g_L] \tag{17}$$

5. Application to single degree of freedom systems

Consider a nonlinear oscillator described by:

$$\begin{aligned} dx_1 &= x_2 dt \\ dx_2 &= -f(x_1, x_2) \; dt \; + \; \sum_{k=1}^{M} \sigma_{k2}(x_1, x_2, t) \, d\xi_k \end{aligned} \tag{18}$$

The corresponding linearized systems in two considered cases have the forms

a) with external and parametric excitations

$$\begin{aligned} dx_1 &= x_2 dt \\ dx_2 &= [-k_2 x_2 - k_1 x_1] dt + \delta_1 x_1 \sum_{k=1}^{M} \beta_k(t) \, d\xi_k + \delta_2 x_2 \sum_{k=1}^{M} \gamma_k(t) \, d\xi_k + \delta_3 \sum_{k=1}^{M} \nu_k(t) \, d\xi_k \end{aligned} \tag{19}$$

b) with external excitations

$$\begin{aligned} dx_1 &= x_2 dt \\ dx_2 &= [-k_2 x_2 - k_1 x_1] dt + \delta_3 \sum_{k=1}^{M} \nu_k(t) \, d\xi_k \end{aligned} \tag{20}$$

where k_1, k_2, $\delta_1, \delta_2, \delta_3$ are positive constans, $\beta_k(t), \gamma_k(t)$ and $\nu_k(t)$ are continnous function.

The application of the direct optimization method to the determination of linearization coefficients requires of calculation of approximate probability density functions, for instance, using the Gram-Charlier expansion. In the case (a) the square criterion has the form:

$$I_{2_p} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (g_N(x_1,x_2) - g_{Lp}(x_1,x_2))^2 dx_1 dx_2 , \tag{21}$$

where $g_N(x_1,x_2)$ and $g_{Lp}(x_1,x_2)$ are defined by the Gram-Charlier expansion i.e.

$$g_N(x) = g_{GC}(x) = g_G(x)\left[1 + \sum_{k=3}^{N} \sum_{\sigma(v)=k} \frac{C_{v_1v_2} H_{v_1v_2}(x)}{v_1! v_2!}\right], \tag{22}$$

where

$$g_G(x) = \frac{1}{2\pi\sqrt{k_{11}k_{22} - k_{12}^2}} \exp\left\{-\frac{k_{11}(x_2)^2 - 2k_{12}x_1x_2 + k_{22}(x_1)^2}{2(k_{11}k_{22} - k_{12}^2)}\right\} \tag{23}$$

$C_{v_1v_2} = E[G_{v_1v_2}(x)]$ are quasimoments $v_1, v_2 = 0,1,\dots,N$, $v_1 + v_2 = 3,4,\dots,N$, $H_{v_1v_2}(x)$ and $G_{v_1v_2}(x)$ are Hermite's polynomials.

However, in the case of $g_{Lp}(x_1,x_2)$ no closure techniques are required.

In the case (b) the probability density function is known in the exact form and one can find necessary conditions of minimum of I_2 defined by (7) and (9) in the analytical form:

$$\frac{\partial I_{2_e}}{\partial k_1} = 2\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (g_N(x) - g_L(x))\left(\frac{1}{2k_1} - k_2\frac{x_1^2}{q^2}\right) g_L(x)\, dx_1 dx_2 = 0 \tag{24}$$

The application of the Fokker - Planck equations leads to the following systems of partial differential equations:

for nonlinear system

$$\frac{\partial g_N}{\partial t} = x_2\frac{\partial g_N}{\partial x_1} + \frac{\partial}{\partial x_2}\left\{\left[-f(x_1,x_2) + \frac{1}{2}\sum_{k=1}^{M} \sigma_{k2}\frac{\partial \sigma_{k2}}{\partial x_2}\right] g_N\right\} - \frac{1}{2}\sum_{k=1}^{M}\frac{\partial^2}{\partial x_2^2}[\sigma_{k2}^2 g_N] = 0 \tag{25}$$

for linearized system

a) with external and parametric excitations

$$\begin{aligned}\frac{\partial g_{Lp}}{\partial t} &= -x_2\frac{\partial g_{Lp}}{\partial x_1} + \frac{\partial}{\partial x_2}\left\{\left[-k_1x_1 - k_2x_2 + \frac{1}{2}\delta_2^2\sum_{k=1}^{M}\gamma_k^2 x_2\right] g_{Lp}\right\} \\ &\quad - \frac{1}{2}\sum_{k=1}^{M}\frac{\partial^2}{\partial x_2^2}\left\{[\delta_1^2x_1^2\beta_k^2 + \delta_2^2x_2^2\gamma_k^2 + \delta_3^2 v_k^2] g_{Lp}\right\} = 0\end{aligned} \tag{26}$$

b) with external excitations

$$\frac{\partial g_L}{\partial t} = -x_2\frac{\partial g_L}{\partial x_1} + \frac{\partial}{\partial x_2}\{[-k_1x_1 - k_2x_2] g_L\} + \frac{1}{2}\sum_{i=1}^{M} v_k^2\frac{\partial^2 g_L}{\partial x_2^2} = 0 \tag{27}$$

Then the following errors of aproximation are proposed

The corresponding criteria of linearization are:

$$I = I_1 + I_2, \tag{28}$$

where:

a) for external and parametric excitations case

$$I_1 = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left\{\frac{\partial}{\partial x_2}\left[\left(-f(x_1,x_2)+k_1x_1+k_2x_2+\frac{1}{2}\sum_{k=1}^{M}\left(\sigma_{k2}\frac{\partial\sigma_{k2}}{\partial x_2}-\delta_2^2\gamma_k^2\right)\right)g_{Lp}\right]\right\}^2 dx_1dx_2 \tag{29}$$

$$I_2 = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left\{\frac{1}{2}\frac{\partial}{\partial x_2^2}\left[\sum_{k=1}^{M}\left(\sigma_{k2}^2(x_1,x_2)-\delta_1^2\beta_k^2(t)x_1^2-\delta_2^2\gamma_k^2(t)x_2^2-\delta_3^2v_k^2(t)\right)g_{Lp}\right]\right\}^2 dx_1dx_2 \tag{30}$$

b) for external excitations case

$$I_1 = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left\{\frac{\partial}{\partial x_2}\left[\left(-f(x_1,x_2)+\frac{1}{2}\sum_{k=1}^{M}\sigma_{k2}\frac{\partial\sigma_{k2}}{\partial x_2}+k_1x_1+k_2x_2\right)g_L\right]\right\}^2 dx_1dx_2 \tag{31}$$

$$I_2 = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left\{\frac{1}{2}\frac{\partial}{\partial x_2^2}\left[\sum_{k=1}^{M}\left(\sigma_{k2}^2(x_1,x_2)-q_k^2(t)\right)g_L\right]\right\}^2 dx_1dx_2 \tag{32}$$

6. Example

Consider a parametrically and externally excited nonlinear oscillator described by Chang and Yang [2].

$$dx = [Ax + \Phi(x)]\,dt + q_1\sigma(x)\,d\xi_1 + Qd\xi_0, \tag{33}$$

where

$$A = \begin{bmatrix} 0, & 1 \\ 0, & -2h \end{bmatrix}, \quad \Phi(x) = \begin{bmatrix} 0 \\ -\varepsilon x_1^3 \end{bmatrix}, \quad \sigma(x) = \begin{bmatrix} 0 \\ -x_1^3 \end{bmatrix}, \quad Q = \begin{bmatrix} 0 \\ q_2 \end{bmatrix} \tag{34}$$

The linearized system can be considered in the form (a)

$$dx = A_L x dt + q_1 D_{Lp} x d\xi_1 + Qd\xi_0, \tag{35}$$

where

$$A_L = \begin{bmatrix} 0 & 1 \\ -k_1 & -2h \end{bmatrix}, \quad D_{Lp} = \begin{bmatrix} 0 & 0 \\ -\delta_1 & 0 \end{bmatrix} \tag{36}$$

or in the form (b)

$$dx = A_L x dt + q_1 D_{L_e} d\xi_1 + Q d\xi_0, \tag{37}$$

where

$$A_L = \begin{bmatrix} 0 & 1 \\ -k_1 & -2h \end{bmatrix}, \quad D_{L_e} = \begin{bmatrix} 0 \\ -\delta_1 \end{bmatrix} \tag{38}$$

k_1 is a linearization coefficient.

The application of the direct optimization method to the determination of linearization coefficients requires of calculation of approximate probability density functions, for instance, for square criterion (21), where $g_N(x_1, x_2)$ and $g_{Lp}(x_1, x_2)$ are defined by the Gram-Charlier expansion .

The application of the Fokker-Planck equations approach leads to the following criteria

$$I = I_{1p} + I_{2p}, \tag{39}$$

where in the case (a)

$$I_{1p} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (-\varepsilon\ x_1^3 + k_1 x_1)^2 \left(\frac{\partial g_{Lp}}{\partial x_2}\right)^2 dx_1 dx_2 \quad , \quad I_{2p} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{1}{4} q_1^4 (x_1^6 - \delta_1^2 x_1^2)^2 \left(\frac{\partial^2 g_{Lp}}{\partial x_2^2}\right)^2 dx_1 dx_2 \tag{40}$$

in the case (b)

$$I_{1e} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (-\varepsilon\ x_1^3 + k_1 x_1)^2 \left(\frac{\partial g_L}{\partial x_2}\right)^2 dx_1 dx_2 \quad , \quad I_{2e} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{1}{4} q_1^4 (x_1^6 - \delta_1^2)^2 \left(\frac{\partial^2 g_L}{\partial x_2^2}\right)^2 dx_1 dx_2 \tag{41}$$

The comparison of relative errors of mean-square displacements is presented in Fig.1.

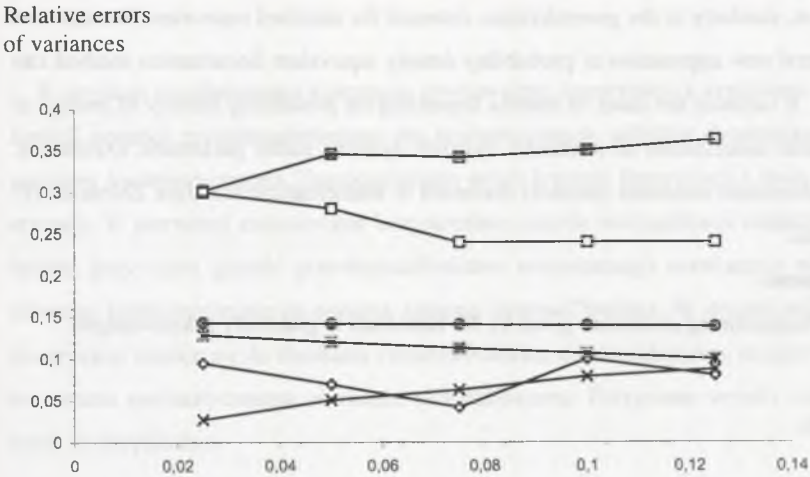

Fig.1. Comparison of relative errors of variances vs. intensity of additive noise q_0^2 for different equivalent linearization techniques; h = 0.5, ε = 5, q_1 = 5; CH - Chang method, FA - Falsone method, FPK - EE - Fokker-Planck-Kolomogorov method for linear model with external excitations, FPK - PE - Fokker-Planck-Kolomogorov method for linear model with parametric and external excitations, DIR - EE - Direct optimization method for linear model with external excitations, DIR - PE - Direct optimization method for linear model with paarametric and external excitations

Rys.1. Porównanie względnych błędów wariancji w funkcji intensywności szumu addtywnego q_0^2 dla różnych technik równoważnej linearyzacji; h = 0.5, ε = 5, q_1 = 5; CH -metoda Changa, FA -metoda Falsonea, FPK-EE - metoda Fokkera-Plancka-Kolomogorova dla układu liniowego z wymuszeniami addtywnymi, FPK - PE -metoda Fokkera-Plancka-Kolomogorova dla układu liniowego z wymuszeniami addtywnymi i parametrycznymi, DIR-EE- metoda bezpośredniej optymalizacji dla układu liniowego z wymuszeniami addtywnymi, DIR-PE-metoda bezpośredniej optymalizacji dla układu liniowego z wymuszeniami addtywnymi i parametrycznymi.

7. Concluding remarks

Equivalent linearization techniques with criteria in probability density function space applied to dynamic systems subjected to parametric and external Gaussian excitations have been considered. Two different approaches: direct optimization method and Fokker-Planck equation method have been examined on Duffing oscillator with nonlinear damping. From numerical results it follows that as well the direct method with square metric for linearized system with parametric excitations metric as Fokker-Planck equation method for linearized system with external excitation give smaller relative errors then equivalent linearization obtained by the best literature approaches.

We note that, similarly to the generalization obtained for standard equivalent linearization technique several new approaches of probability density equivalent linearization method can be considered. It includes the cases of criteria depending on probability density of energy of the response and linearization of stochastic dynamic systems under parametric excitations. Also other probabilistic measures (metrics) discussed in mathematical literature Zolotarev [8] can be analyzed.

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Streszczenie

W artykule przedstawiono koncepcję równoważnej linearyzacji z kryteriami w przestrzeni funkcji gęstości prawdopodobieństw dla stochastycznych układów dynamicznych z wymuszeniami parametrycznymi. Zaproponowano nowe kryteria linearyzacji i dwie techniki linearyzacji. W pierwszej zastosowano bezpośrednio metodę minimalizacji rozpatrywanego kryterium, przy czym gęstość prawdopodobieństwa stacjonarnego rozwiązania równania nieliniowego aproksymuje się za pomocą szeregu Grama-Charliera. W drugiej technice metodę linearyzacji stosuje się do równania Fokkera-Plancka, odpowiadającego rozpatrywanemu nieliniowemu stochastycznemu równaniu różniczkowemu. Otrzymane wyniki zostały zilustrowane na przykładzie.