ZESZYTY NAUKOWE POLITECHNIKI ŚLĄSKIEJ 1998
Seria: BUDOWNICTWO z. 85 Nr kol. 1404

Jerzy SKRZYPCZYK

A NOTE ON INTERVAL FREDHOLM INTEGRAL EQUATIONS

Summary. In the paper basic concepts of the analysis of the solutions to the interval Fredholm integral equations are considered, where a free term is taken to be an interval square-integrable function and non-interval kernel is square-integrable in [a,b]x[a,b]. All arithmetic operations and integration are in interval sense. The existence and uniqueness of the interval-valued solution are investigated. For including a set of solutions of the interval integral equation we apply interval calculus. At the end the theory is illustrated by a simple analytical example.

UWAGI O PRZEDZIAŁOWYCH RÓWNANIACH CAŁKOWYCH FREDHOLMA

Streszczenie. W pracy badane jest istnienie i jednoznaczność zbioru rozwiązań przedziałowego równania całkowego Fredholma II rodzaju z niejednorodnością, która jest funkcją przedziałową, całkowalną z kwadratem, natomiast jądro równania całkowego jest całkowalne z kwadratem na zbiorze [a,b]x[a,b]. W pierwszej kolejności badane jest zagadnienie zbieżności ciągu aproksymującego do dokładnego zbioru rozwiązań, a następnie problem jednoznaczności zbioru przedziałowego, zawierającego zbiór rozwiązań. W celu wyznaczenia tej aproksymacji zastosowano analizę przedziałową. Teoria zilustrowana jest prostym przykładem analitycznym.

1. Introduction

The theory of Fredholm integral equations is very well developed and has a great bibliography [11,12,18,19,26]. Such equations play a great role in investigation of many technical problems in which boundary problems are of the greatest importance, cf. [4]. One of the most popular methods of investigation of existence and uniqueness of solutions of integral equations is the method of successive approximations. In presented paper we apply this method to discuss interval solutions of non-homogeneous interval integral equations. In recent years the analysis for problems of integral equations with random parameters has also been discussed intensively, cf. [9,22,25]. However, in many particular cases probabilistic analysis is very difficult from mathematical point of view, as well as sometimes the necessary knowledge about probabilistic characteristic of parameters is very poor. On the over hand, especially in engineering sciences, because of manufacturing errors, values of structural parameters of many materials are uncertain and the character of that uncertainty is interval, i.e., unknown and bounded.

If the structural parameters are interval the equations describing the system become of interval character. Following, the corresponding solutions are of set character. Since detailed calculations of the shape of that set are very difficult, approximate methods are needed.

For engineers it is often sufficient to estimate the upper and lower bounds of the solutions of system equations under considerations i.e. to find upper and inner interval approximations of solution sets respectively. We are interested in real solutions only, but the complex case is handled by the method used, since it is more appropriate from mathematical point of view.

Interval integral equations were investigating earlier in some papers [7,8,10,13,21]. Similar method was applied for contracting integral fuzzy operators in paper [24].

Section 2 is devoted to notations and terminology and in Section 3 we discuss the existence of solutions of interval Fredholm integral equations of the second kind with interval free term and with contractive integral operator, as well as the uniqueness in some sense of exact interval solutions. In section 4 we study the simple example which can be solved analytically. Presented methods give new results to study uncertain boundary problems with interval or fuzzy parameters, which are very important in an engineering practice, cf. [5,6,22,23]

2. Elementary Concepts and Results

In this report, the following concepts and notations will be used. R^n $\left(R^{n\times m}\right)$ was reserved for the set of n-dimensional vectors (nxm matrices), R the set of reals.

Let the symbols $P(R^n)$, $P(R^{n\times m})$ denote the power sets of R^n and $R^{n\times m}$ respectively. Let $I(R)$ denote the set of all closed bounded intervals $\bar{z}=\left[z^-,z^+\right]$ on the real line R, where z^- and z^+ denote the end points of $\bar{z}$. We call further elements of $I(R)$ interval numbers. In the similar way we introduce $I(R^n)$ - the space of interval vectors, and $I\left(R^{n\times m}\right)$ - the space of interval matrices.

The elementary operations on elements from $I(R^n)$ and $I\left(R^{n\times m}\right)$ are described in monographs [1,3, 15-17,20].

The symbols $P_c(R^n)$, $P_c(R^{n\times m})$, $P_c(C)$, $P_c(C^n)$, $P_c(C^{n\times m})$ denote the families of all non-empty compact convex subsets of corresponding spaces.

Let T be one of the sets: R^n, $R^{n\times m}$. The operations in the power set $P(T)$ are as usually defined by

$$A*B:=\{a*b \mid a\in A, b\in B\},\quad A,B\in P(T),\quad *\in\{+,-,\cdot,/\} \tag{1}$$

with well known restrictions for "/". Naturally for matrices the symbols denote corresponding matrix operations [1,3,16,17].

Further informations on interval analysis the reader can be found in papers [1,3, 15-17,20]. Details of operations over interval numbers are described in the second paper of the same author ,see this journal.

Recall that the value $d(\bar{a},\bar{b}):=\left|a^- - b^-\right|\vee\left|a^+ - b^+\right|$ is called the distance between interval numbers $\bar{a}$ and $\bar{b}$. It is easy to see that, if $\bar{a}=a$ and $\bar{b}=b$ are real numbers, then $d(\bar{a},\bar{b})=|a-b|$. For further information see refs. [1,13,14,16,17,20]. It is the Hausdorff metric specified for I(R).

Recall that the Hausdorff metric is defined as

$$H(A,B):=\inf\{\varepsilon: A\subseteq N(B,\varepsilon), B\subseteq N(A,\varepsilon)\}, \tag{2}$$

where

$$A,B\in P_c(T) \text{ and } N(A,\varepsilon)=\left\{x\in T: \|x-y\|<\varepsilon \text{ for some } y\in A\right\} \tag{3}$$

We have: $H(A+C,B+C)=H(A,B)$, $\forall\, A,B,C\in P_c(T)$, $\lambda N(A,\varepsilon)=N(\lambda A,|\lambda|\varepsilon)$ $\forall\,\lambda\in C^1$.
where $\bar{x}(t),\bar{y}(t)\in C(J;I(T))$ and d is the Hausdorff metric defined in $I(T)$.

Further we denote by $L^2(J;I(T))$ the set of all measurable interval-valued mappings $\bar{x}(\cdot)$ from J to I(T), where $J \subseteq R$ and $x^-(.), x^+(.) \in L^2(J;T)$. We metricize $L^2(J;I(T))$ by setting

$$D_2(\bar{x}(t),\bar{y}(t)) := \left(\int_J d^2(\bar{x}(t),\bar{y}(t))dt \right)^{1/2}, \tag{4}$$

where $\bar{x}(t),\bar{y}(t) \in L^2(J;I(T))$. Similar arguments as in [13] apply to show that also $L^2(J;I(T))$ is complete.

An interval-valued real (complex) function is a special closed-valued set valued function $\bar{f}: R \to I(T)$. It is usually written as $\bar{f}(x) = [f^-(x), f^+(x)]$, where

$$f^-(x) = \inf \bar{f}(x), \quad f^+(x) = \sup \bar{f}(x). \tag{5}$$

Then $\bar{f}$ is measurable iff f^- and f^+ are measurable [2,7,13].

3. Method of Successive Approximations for Interval Fredholm Integral Equations

A linear interval Fredholm integral equation of the second kind with the kernel $K(\cdot,\cdot)$ and the interval-type free term $\bar{g}(\cdot)$ is defined as the family of linear nonhomogeneous Fredholm integral equations

$$f(x) = g(x) + \int_a^b K(x,y)f(y)dy, \quad g(x) \in \bar{g}(x), \quad a \le x \le b \tag{6}$$

Thus we consider a system of equations in which the functions take unknown values ranging in certain intervals.

Further a linear interval Fredholm integral equation of the second kind is written in the form

$$\bar{f}(x) = \bar{g}(x) + \int_a^b K(x,y)\bar{f}(y)dy, \quad a \le x \le b \tag{7}$$

Define an interval integral Fredholm - type operator as follows

$$(A\bar{f})(x) = \bar{g}(x) + \lambda \int_a^b K(x,y)\bar{f}(y)dy, \quad a \le x \le b \tag{8}$$

where the integration is in the interval sense, cf. [2,7].

We formulate an approximation procedure, namely

$$
\begin{aligned}
&\bar{f}_1(x) = \bar{g}(x)\\
&\bar{f}_2(x) = \left(A\bar{f}_1\right)(x)\\
&\bar{f}_3(x) = \left(A\bar{f}_2\right)(x)\\
&\dots\dots\dots\dots\dots\dots\\
&\bar{f}_n(x) = \left(A\bar{f}_{n-1}\right)(x)
\end{aligned}
\tag{9}
$$

Further we have

$$
\begin{aligned}
&\left|(A\bar{f})(x)\right| \le \left|\bar{g}(x)\right| + \int_a^b \left|K(x,y)\right|\left|\bar{f}(y)\right|dy \le\\
&\le \left|\bar{g}(x)\right| + \left(\int_a^b \left|K(x,y)\right|^2 dy\right)^{1/2}\left(\int_a^b \left|\bar{f}(y)\right|^2 dy\right)^{1/2} \le\\
&\le \left|\bar{g}(x)\right| + \kappa(x)D_2\left(\bar{f}(\cdot),\bar{0}\right) \quad \forall\ a \le x \le b
\end{aligned}
\tag{10}
$$

where $\kappa(x) = \left(\int_a^b \left|K(x,y)\right|^2 dy\right)^{1/2}$ and following the known properties of Hausdorff metric function cf. [1,7,13,14,16,17]. Thus

$$
D_2\left(A\bar{f},\bar{0}\right) \le D_2\left(\bar{g},\bar{0}\right) + \|\kappa\| D_2\left(\bar{f},\bar{0}\right) \tag{11}
$$

and from eq. (10) follows that the operator A: $L^2(a,b;I(R)) \to L^2(a,b;I(R))$. Similarly

$$
\begin{aligned}
&d\left(A\bar{f},\bar{g}\right) \le d\left(\bar{g}(x) + \int_a^b K(x,y)\bar{f}(y)dy, \bar{g}(x)\right) \le\\
&\le d\left(\int_a^b K(x,y)\bar{f}(y)dy, \bar{0}\right) \le \int_a^b d\left(K(x,y)\bar{f}(y),\bar{0}\right)dy \le\\
&\le \int_a^b \left|K(x,y)\right|\left|\bar{f}(y)\right|dy \le \left(\int_a^b \left|K(x,y)\right|^2 dy\right)^{1/2}\left(\int_a^b \left|\bar{f}(y)\right|^2 dy\right)^{1/2} \le\\
&\le \kappa(x)D_2\left(\bar{f}(y),\bar{0}\right)
\end{aligned}
\tag{12}
$$

and in consequence

$$
D_2\left(\bar{f}_2,\bar{g}\right) = D_2\left(A\bar{g},\bar{g}\right) \le \|\kappa\| D_2\left(\bar{g},\bar{0}\right) \tag{13}
$$

Following this procedure we obtain

$$
\bar{f}_3(x) = (A^2\bar{g})(x) = \bar{g}(x) + \int_a^b K(x,y)(A\bar{g})(y)dy, \quad a \le x \le b \tag{14}
$$

From eqs. (12) and (13) we conclude

$$
\begin{aligned}
&d\left(\bar{f}_3,\bar{g}\right) = d\left(A^2\bar{g},\bar{g}\right) \le d\left(\bar{g}(x) + \int_a^b K(x,y)\left(A\bar{g}\right)(y)dy, \bar{g}(x)\right) \le\\
&\le d\left(\int_a^b K(x,y)\left(A\bar{g}\right)(y)dy, \bar{0}\right) \le \kappa(x)D_2\left(A\bar{g},\bar{g}\right) \le \kappa(x)\|\kappa\| D_2\left(\bar{g},\bar{0}\right)
\end{aligned}
\tag{15}
$$

and

$$D_2(\bar{f}_3,\bar{g}) = D_2(A^2\bar{g},\bar{g}) \le \|\kappa\|^2 D_2(\bar{g},\bar{0}) \tag{16}$$

Following the above procedure we obtain

$$d(\bar{f}_n,\bar{g}) = d(A^{n-1}\bar{g},\bar{g}) \le \kappa(x)\|\kappa\|^{n-2} D_2(\bar{g},\bar{0}) \tag{17}$$

and

$$D_2(\bar{f}_n,\bar{g}) = D_2(A^{n-1}\bar{g},\bar{g}) \le \|\kappa\|^{n-1} D_2(\bar{g},\bar{0}) \tag{18}$$

Now we prove that the sequence $\{\bar{f}_n\}$ is the Cauchy sequence

$$\begin{aligned} d(\bar{f}_n,\bar{f}_m) &= d(A\bar{f}_{n-1},A\bar{f}_{m-1}) = d\Big(\bar{g}(x)+\int_a^b K(x,y)\bar{f}_{n-1}(y)dy,\bar{g}(x)+\int_a^b K(x,y)\bar{f}_{m-1}(y)dy\Big) = \\ &= d\Big(\int_a^b K(x,y)\bar{f}_{n-1}(y)dy,\int_a^b K(x,y)\bar{f}_{m-1}(y)dy\Big) \le \int_a^b d(K(x,y)\bar{f}_{n-1}(y),K(x,y)\bar{f}_{m-1}(y))dy \le \\ &\le \int_a^b |K(x,y)| d(\bar{f}_{n-1}(y),\bar{f}_{m-1}(y))dy \le \kappa(x) D_2(\bar{f}_{n-1},\bar{f}_{m-1}) \end{aligned} \tag{19}$$

Assume for simplicity that m>n

$$\begin{aligned} D_2(\bar{f}_n,\bar{f}_m) &\le \|\kappa\| D_2(\bar{f}_{n-1},\bar{f}_{m-1}) \le \\ &\le \|\kappa\|^2 D_2(\bar{f}_{n-2},\bar{f}_{m-2}) \le \;\ldots\ldots \\ \ldots\; &\le \|\kappa\|^{n-1} D_2(\bar{f}_1,\bar{f}_{m-n+1}) = \|\kappa\|^{n-1} D_2(\bar{g},\bar{f}_{m-n+1}) \le \|\kappa\|^{m-1} D_2(\bar{g},\bar{0}) \end{aligned} \tag{20}$$

If $\|\kappa\|<1$ then $\{\bar{f}_n\}$ is the Cauchy-type sequence. Let further $\bar{f}(\cdot) := \lim_{n\to\infty} \bar{f}_n(\cdot)$. It is easy to prove that f(.) is the solution of eq. (7). Denote $\bar{h}(x) := \bar{g}(x) + (A\bar{f})(x)$. We have

$$\begin{aligned} d(\bar{f}_n,\bar{h}) &= d(A\bar{f}_{n-1},\bar{h}) = d\Big(\bar{g}(x)+\int_a^b K(x,y)\bar{f}_{n-1}(y)dy,\bar{g}(x)+\int_a^b K(x,y)\bar{f}(y)dy\Big) = \\ &= d\Big(\int_a^b K(x,y)\bar{f}_{n-1}(y)dy,\int_a^b K(x,y)\bar{f}(y)dy\Big) \le \int_a^b d(K(x,y)\bar{f}_{n-1}(y),K(x,y)\bar{f}(y))dy \le \\ &\le \int_a^b |K(x,y)| d(\bar{f}_{n-1}(y),\bar{f}(y))dy \le \kappa(x) D_2(\bar{f}_{n-1},\bar{f}) \end{aligned} \tag{21}$$

and

$$D_2(\bar{f}_n,\bar{h}) \le \|\kappa\| D_2(\bar{f}_{n-1},\bar{f}) \tag{22}$$

Since $D_2(\bar{f}_{n-1},\bar{f}) \xrightarrow{n\to\infty} 0$ we have $\bar{h}(\cdot) = \bar{f}(\cdot)$ almost everywhere in [a,b].Thus $\bar{f}(\cdot)$ is the solution of interval integral equation (7).

We can prove the uniqueness of that solution. Assume we have two different interval solutions $\bar{f}_1(\cdot)$ and $\bar{f}_2(\cdot)$ of eq. (7). Then

$$d(\bar{f}_1,\bar{f}_2) = d\left(\bar{g}(x) + \int_a^b K(x,y)\bar{f}_1(y)dy, \bar{g}(x) + \int_a^b K(x,y)\bar{f}_2(y)dy\right) =$$
$$= d\left(\int_a^b K(x,y)\bar{f}_1(y)dy, \int_a^b K(x,y)\bar{f}_2(y)dy\right) \le \int_a^b d\left(K(x,y)\bar{f}_1(y), K(x,y)\bar{f}_2(y)\right)dy \le \quad (23)$$
$$\le \int_a^b |K(x,y)| d\left(\bar{f}_1(y),\bar{f}_2(y)\right)dy \le \kappa(x) D_2(\bar{f}_1,\bar{f}_2)$$

From eq. (23) we have $D_2(\bar{f}_1,\bar{f}_2) \le \|\kappa\| D_2(\bar{f}_1,\bar{f}_2)$. Since for different $\bar{f}_1(\cdot)$ and $\bar{f}_2(\cdot)$ $D_2(\bar{f}_1,\bar{f}_2) \neq 0$, thus $1 \le \|\kappa\|$, which result leads to contradiction . So we get $D_2(\bar{f}_1,\bar{f}_2) = 0$ i.e. $\bar{f}_1(\cdot) = \bar{f}_2(\cdot)$ almost everywhere in [a,b].□

4. Example

Solve the interval Fredholm integral equation of the second kind

$$\bar{f}(x) = \left[\frac{2}{3}x, \frac{5}{6}x\right] + \int_0^1 \frac{1}{2} xy\bar{f}(y)dy, \quad 0 \le x \le 1 \quad (24)$$

We calculate terms of approximating interval sequence as follows

$$\bar{f}_1(x) = \left[\frac{2}{3}x, \frac{5}{6}x\right],$$
$$\bar{f}_2(x) = \left[\frac{2}{3}x, \frac{5}{6}x\right] + \int_0^1 \frac{1}{2} xy\left[\frac{2}{3}y, \frac{5}{6}y\right]dy = \left[\frac{7}{9}x, \frac{35}{36}x\right], \quad (25)$$
$$\bar{f}_3(x) = \left[\frac{2}{3}x, \frac{5}{6}x\right] + \int_0^1 \frac{1}{2} xy\left[\frac{7}{9}y, \frac{35}{36}y\right]dy = \left[\frac{43}{54}x, \frac{215}{216}x\right],$$

....

It can be proved that $\bar{f}_n \xrightarrow{n\to\infty} \left[\frac{4}{5}x, x\right]$, which is the interval solution of eq.(24).

REFERENCES

1. Alefeld G. & Herzberger, J.: Introduction to Interval Computations, Academic Press, New York, 1983.
2. Aumann R.J.: Integrals of Set-Valued Functions, J.Math.Anal.Appl., 1965, 12, 1-12.

3. Bauch H. & Jahn K.U. & Oelschlagel D. & Süsse H. & Wiebigke V.: Intervalmathematik, BSG B.G. Teubner Verlagsgeselschaft, Berlin, 1987.
4. Burczyński T.: Boundary Element Method, WNT,Warszawa, 1995. (in polish).
5. Burczyński T. & Skrzypczyk J.: The Fuzzy Boundary Element Method: A New Methodology, Sci. Fasc. of Silesian Tech. Univ., ser. Civil Eng., 1996, 83, Gliwice, 25-42
6. Burczyński T. & Skrzypczyk J.: Fuzzy Aspects of the Boundary Element Method, Engineering Analysis with Boundary Elements, 1997, 19, 209-216.
7. Caprani O., Madsen K., Rall L. B.: Integration of Interval Functions, SIAM J. Math. Anal., 1981, 12, 321 - 341.
8. Caprani O., Madsen K., Stauning O.: Enclosing Solutions of Integral Equations, Extended Abstracts, APIC'95, El Paso, A Supplement to the Reliable Computing.
9. Corduneanu C.: Integral Equations and Stability of Feedback Systems, Academic Press, New york and London, 1973.
10. Dobner H. J.: Einschließungsmethoden für Fredholmsche Integralgleichungen, ZAMM, 1990, 70, 592 - 594.
11. Fenyő S., Stolle H. W.: Theorie und Praxis der linearen Integralgleichungen 2, VEB Deutscher Verlag der Wissenschaften, Berlin 1983.
12. Jerri A. J.: Introduction to Integral Equations with Applications, Marcel Dekker, Inc., New York and Basel 1985.
13. Kaleva O.: Fuzzy Differential Equations, Fuzzy sets and Systems, 1987, 24, 301 - 317.
14. Mayer G.: Algebraische und metrische Strukturen in der Intervalrechnung und einige Anwendungen, Computing, 1970, 5, 144 - 162.
15. Moore R.E.: Interval Analysis, Prentice Hall, Englewood Cliffs, 1966.
16. Neumaier A.: New Techniques for the Analysis of Linear Interval Equations, Linear Algebra and Its Applications, 1984, 58, 273 - 325.
17. Neumaier A.: Interval Methods for Systems of Equations, Cambridge University Press, Cambridge, New York, Port Chester, Melbourne, Sydney, 1990.
18. Piskorek A.: Integral Equations, WNT, Warszawa, 1997. (in polish).
19. Pogorzelski W.: Integral Equations and Their Applications, PWN, Warszawa, 1970.
20. Rohn J.: Systems of Linear Interval Equations, Linear Algebra and Its Applications, 1989, 126, 39 - 78.

21. Seikkala S.: On the Fuzzy Initial Value Problem, Fuzzy sets and Systems, 1987, 24, 319 - 330.
22. Skrzypczyk J.: Dynamics of Complex Mechanical Systems with Random Parameters, Zesz. Nauk. Pol. Śl. Seria Mechanika 120, Gliwice 1994 (in polish).
23. Skrzypczyk J., Burczyński T.: Fuzzy Boundary Element Method in the Analysis of Uncertain Systems, J. of Theoretical and Appl. Mechanics, 1998, 36, 493-512.
24. Subrahmanyam P. V., Sudarsanam S. K.: A Note on Fuzzy Volterra Integral Equations, Fuzzy Sets and Systems, 1996, 81, 237 - 240.
25. Tsokos c. P., Padgett W., J.: Random Integral Equations with Applications to Stochastic Systems, Springer-Verlag, Berlin Heidelberg new York 1971.
26. Widom H.: Lectures on Integral Equations, Van Nostrand - Reinhold Company, New York, Toronto, London, Melbourne 1969.

Recenzent: Prof.dr hab.inż. Tadeusz Burczyński

Abstract

In the paper basic concepts of the structure of the solutions to the interval Fredholm integral equations

$$\bar{f}(x) = \bar{g}(x) + \int_a^b K(x,y)\bar{f}(y)dy, \quad a \le x \le b$$

are considered, where $\bar{g}(\cdot)$ is taken as an interval square-integrable function and non-interval kernel k(.,.) is square-integrable in [a,b]x[a,b]. All operations are in the interval sense. At first, the existence of the exact interval-set-valued solution is investigated. In addition the uniqueness of the solution set is obtained. For including a set of solutions of the interval integral equation we apply interval calculus. Finally the theory is illustrated by a simple analytical example.